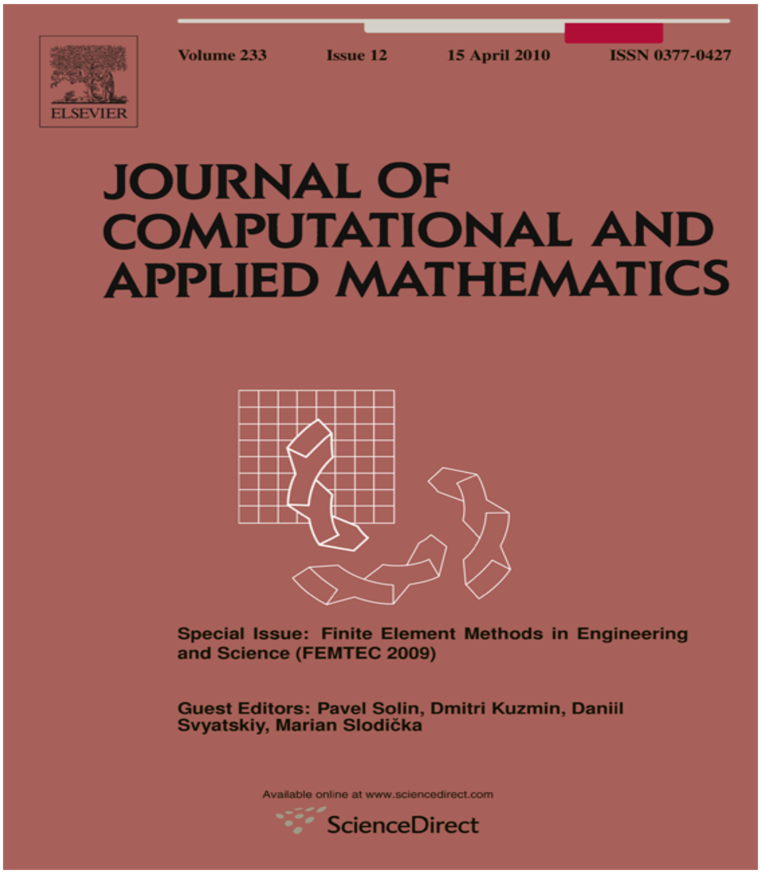
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A nonlinear parabolic equation with a nonlocal boundary term

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ABSTRACT

A nonlinear parabolic problem with a nonlocal boundary condition is studied. We prove the existence of a solution for a monotonically increasing and Lipschitz continuous nonlinearity. The approximation method is based on Rothe's method. The solution on each time step is obtained by iterations, convergence of which is shown using a fixed-point argument. The space discretization relies on FEM. Theoretical results are supported by numerical experiments.

1. Introduction

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ (with a Lipschitz boundary Γ) and a finite time interval $[0, T]$. Let $\boldsymbol{\nu}$ be the outer unit normal vector to Γ.

This paper deals with the solvability of the following nonlinear evolution equation with a nonlocal Robin-type boundary condition (BC)

$$\begin{aligned} &\partial_t g(u) - \Delta u = f(u) && \text{in } (0,T)\times\Omega \\ &-\nabla u \cdot \boldsymbol{\nu} = \alpha u + \beta + \int_\Omega Ku && \text{on } \Gamma \\ &u(0) = u^0 && \text{in } \Omega, \end{aligned} \tag{1}$$

where α, β, g, K and f are given functions. We adopt the following conditions on the data:

$$\begin{aligned} &g(0) = 0, \quad 0 < \gamma \leq g' \leq L \\ &0 \leq \alpha, \qquad |\alpha'| \leq C \\ &|f(x) - f(y)| \leq C|x-y|, \quad \forall x, y \\ &\beta' \in L_2\left((0,T), L_2(\Gamma)\right), \qquad K \in L_2(\Omega), \qquad u^0 \in H^1(\Omega). \end{aligned} \tag{2}$$

Linear parabolic problems with a nonlocal BC have been studied in the last decades. Dirichlet BC in 1D arises in the theory of linear thermoelasticity; cf. [1,2]. The well-posedness of this linear problem has been discussed in [3–5] under the additional condition

$$\int_\Omega |K(x)|\,\mathrm{d}x < 1.$$

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Stronger conditions on K were needed in numerical studies — cf. [6–9]. The stability of various numerical algorithms has been addressed in [10].

Solvability subject to a nonlocal Robin BC has been studied in [11–13]. Numerical study based on monotonicity methods has been performed in [14].

The paper is organized as follows. First we perform the time discretization based on Rothe's method. The nonlinear problem on each time point is solved using a relaxation technique. In this way a solution to the nonlinear problem is approached by a sequence of linear BVPs with a nonlocal BC. The well-posedness of these linear problems is based on the superposition principle - cf. [13]. We show the convergence of the relaxation iterations. Further we perform the stability analysis and finally we prove the existence of a solution to the original nonlinear parabolic problem. The last section is devoted to some numerical experiments.

Finally, as it is usual in papers of this sort, C, ε and C_ε will denote generic positive constants depending only on a priori known quantities, where ε is small and C_ε is large.

2. Time discretization

First, we denote by $(w, z)_M$ the standard L_2-scalar product of the functions w and z on a set M, i.e.,

$$(w, z)_M = \int_M wz$$

and the corresponding norm

$$\|w\|_M^2 = (w, w)_M .$$

The subscript will be suppressed if $M = \Omega$.

We divide the time interval $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals (t_{i-1}, t_i) for $t_i = i\tau$, where $\tau = \frac{T}{n}$. We introduce the following notation

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}$$

for any function z.

The Rothe method is an efficient tool for solving evolution problems. Replacing the time derivative by the backward difference, a time-dependent problem is approximated by a sequence of nonlinear elliptic problems with a nonlocal BC, which have to be solved successively with increasing time step t_i for $i = 1, \dots, n$. More exactly,

$$\begin{aligned} &\delta g(u_i) - \Delta u_i = f(u_{i-1}) && \text{in } \Omega \\ &-\nabla u_i \cdot \boldsymbol{\nu} = \alpha_i u_i + \beta_i + (K, u_i) && \text{on } \Gamma \\ &u_0 = u^0 && \text{in } \Omega . \end{aligned} \tag{3}$$

The solution u_i on any time step t_i will be approached in the following relaxation process with running index k

$$\begin{aligned} &Lu_{i,k} - \tau \Delta u_{i,k} = \tau f(u_{i-1}) + g(u_{i-1}) + Lu_{i,k-1} - g(u_{i,k-1}) && \text{in } \Omega \\ &-\nabla u_{i,k} \cdot \boldsymbol{\nu} = \alpha_i u_{i,k} + \beta_i + \left(K, u_{i,k}\right) && \text{on } \Gamma \\ &u_{i,0} = u_{i-1} && \text{in } \Omega . \end{aligned} \tag{4}$$

This is a linear BVP with a nonlocal BC for any fixed i and k. Its solvability is addressed in the following lemma.

Lemma 2.1. *Assume (2). Then there exists $0 < \tau_0$ such that the problem (4) is well-posed for all i and k and $0 < \tau < \tau_0$ and there exists a unique weak solution $u_{i,k} \in H^1(\Omega)$ to (4).*

Proof. First we introduce a function h as

$$h(s) = Ls - g(s).$$

Clearly

$$0 \le h'(s) = L - g'(s) \le L - \gamma .$$

The variational formulation of (4) reads as

$$\begin{aligned} &L\left(u_{i,k}, \varphi\right) + \tau\left(\nabla u_{i,k}, \nabla\varphi\right) + \tau\alpha_i\left(u_{i,k}, \varphi\right)_\Gamma + \tau\left(K, u_{i,k}\right)(1, \varphi)_\Gamma \\ &\quad = \left(\tau f(u_{i-1}) + g(u_{i-1}) + h(u_{i,k-1}), \varphi\right) - \tau\left(\beta_i, \varphi\right)_\Gamma \end{aligned} \tag{5}$$

for any $\varphi \in H^1(\Omega)$. The left-hand side

$$a(u_{i,k}, \varphi) := L\left(u_{i,k}, \varphi\right) + \tau\left(\nabla u_{i,k}, \nabla\varphi\right) + \tau\alpha_i\left(u_{i,k}, \varphi\right)_\Gamma + \tau\left(K, u_{i,k}\right)(1, \varphi)_\Gamma$$

represents a continuous bilinear form in $H^1(\Omega)$.

Using the Cauchy inequality, trace theorem and the Young inequality we can write that

$$\begin{aligned}\left(K, u_{i,k}\right)\left(1, u_{i,k}\right)_\Gamma &\le \|K\| \left\|u_{i,k}\right\| \sqrt{|\Gamma|} \left\|u_{i,k}\right\|_\Gamma \\ &\le C \left\|u_{i,k}\right\| \left(\left\|u_{i,k}\right\| + \left\|\nabla u_{i,k}\right\|\right) \\ &\le C \left\|u_{i,k}\right\|^2 + \frac{1}{2}\left\|\nabla u_{i,k}\right\|^2 .\end{aligned}$$

Therefore

$$a(u_{i,k}, u_{i,k}) \ge (L - C\tau)\left\|u_{i,k}\right\|^2 + \frac{\tau}{2}\left\|\nabla u_{i,k}\right\|^2 ,$$

which implies the coercivity of a in $H^1(\Omega)$ if $\tau < \tau_0$ (τ_0 is a sufficiently small positive number). The right-hand side of (5) is a liner bounded functional on $H^1(\Omega)$. The rest of the proof is a consequence of the Lax–Milgram lemma. □

Convergence of the relaxation process is discussed in the following lemma.

Lemma 2.2. *Suppose (2). Then for sufficiently small τ_0 and $\tau < \tau_0$ we have $\lim_{k\to\infty} u_{i,k} \to u_i$ in $H^1(\Omega)$, where u_i solves (3).*

Proof. We replace k by $k-1$ in (5) and subtract this relation from (5). Then we set $\varphi = u_{i,k} - u_{i,k-1}$ and we get

$$\begin{aligned}&L\left\|u_{i,k} - u_{i,k-1}\right\|^2 + \tau\left\|\nabla[u_{i,k} - u_{i,k-1}]\right\|^2 + \tau\alpha_i\left\|u_{i,k} - u_{i,k-1}\right\|_\Gamma^2 \\ &\quad = \left(h(u_{i,k-1}) - h(u_{i,k-2}), u_{i,k} - u_{i,k-1}\right) - \tau\left(K, u_{i,k} - u_{i,k-1}\right)\left(1, u_{i,k} - u_{i,k-1}\right)_\Gamma .\end{aligned}$$

The first term on the RHS can be estimated as

$$\left(h(u_{i,k-1}) - h(u_{i,k-2}), u_{i,k} - u_{i,k-1}\right) \le (L-\gamma)\left\|u_{i,k} - u_{i,k-1}\right\|\left\|u_{i,k-1} - u_{i,k-2}\right\| .$$

For the second term on the RHS we involve the following inequality — see [15]

$$\|z\|_\Gamma^2 \le \varepsilon\|\nabla z\|^2 + C_\varepsilon\|z\|^2 , \quad \forall z \in H^1(\Omega), 0 < \varepsilon < \varepsilon_0. \tag{6}$$

We can write

$$\begin{aligned}\left(K, u_{i,k} - u_{i,k-1}\right)\left(1, u_{i,k} - u_{i,k-1}\right)_\Gamma &\le C\left\|u_{i,k} - u_{i,k-1}\right\|\left\|u_{i,k} - u_{i,k-1}\right\|_\Gamma \\ &\le C\left\|u_{i,k} - u_{i,k-1}\right\|^2 + C\left\|u_{i,k} - u_{i,k-1}\right\|_\Gamma^2 \\ &\le \varepsilon\left\|\nabla[u_{i,k} - u_{i,k-1}]\right\|^2 + C_\varepsilon\left\|u_{i,k} - u_{i,k-1}\right\|^2 .\end{aligned}$$

Collecting the estimates we arrive at

$$\begin{aligned}&(L - C_\varepsilon\tau)\left\|u_{i,k} - u_{i,k-1}\right\|^2 + \tau(1-\varepsilon)\left\|\nabla[u_{i,k} - u_{i,k-1}]\right\|^2 + \tau\alpha_i\left\|u_{i,k} - u_{i,k-1}\right\|_\Gamma^2 \\ &\quad \le (L-\gamma)\left\|u_{i,k} - u_{i,k-1}\right\|\left\|u_{i,k-1} - u_{i,k-2}\right\| .\end{aligned} \tag{7}$$

Fixing a sufficiently small ε we get for $\tau \le \tau_0$ that

$$\left\|u_{i,k} - u_{i,k-1}\right\| \le \frac{L-\gamma}{L - C_\varepsilon\tau}\left\|u_{i,k-1} - u_{i,k-2}\right\| .$$

If τ_0 is sufficiently small then $\frac{L-\gamma}{L-C_\varepsilon\tau} < 1$ and due to the Banach fixed-point theorem we get

$$\lim_{k\to\infty} u_{i,k} = u_i, \quad \text{in } L_2(\Omega).$$

Using this fact in (5) we can analogously see that the sequence of gradients is a Cauchy sequence, which implies that

$$\lim_{k\to\infty} u_{i,k} = u_i, \quad \text{in } H^1(\Omega)$$

and

$$\lim_{k\to\infty} u_{i,k} = u_i \quad \text{a.e. in } \Omega.$$

Now, we may pass to the limit for $k \to \infty$ in (5) and we see that u_i is a weak solution to (3). □

3. Solvability

In this section we derive the existence of a weak solution to (1), which is given as

$$(\partial_t g(u), \varphi) + (\nabla u, \nabla\varphi) + \alpha\,(u, \varphi)_\Gamma + (K, u)\,(1, \varphi)_\Gamma = (f(u), \varphi) - (\beta, \varphi)_\Gamma \tag{8}$$

for any $\varphi \in H^1(\Omega)$ and a.e. in $(0, T)$.

The variational formulation of (3) reads as

$$(\delta g(u_i), \varphi) + (\nabla u_i, \nabla\varphi) + \alpha_i (u_i, \varphi)_\Gamma + (K, u_i)(1, \varphi)_\Gamma = (f(u_{i-1}), \varphi) - (\beta_i, \varphi)_\Gamma \tag{9}$$

for any $\varphi \in H^1(\Omega)$. The stability estimates for u_i can be obtained using the standard technique for Rothe's method.

Lemma 3.1. *Assume (2). Then there exists a positive constant C such that*

$$\|u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau + \sum_{i=1}^{j} \alpha_i \|u_i\|_\Gamma^2 \tau \le C$$

for $j = 1, \ldots, n$.

Proof. We introduce the following notation

$$\Phi_\beta(z) = \int_0^z \beta(s)\,\mathrm{d}s \qquad \forall z \in \mathbb{R}$$

for any function β. If β is a monotonically increasing function with the Lipschitz coefficient L_β and $\beta(0) = 0$, then Φ_β is convex and

$$\frac{\beta^2(z)}{2L_\beta} \le \Phi_\beta(z) \le \frac{L_\beta z^2}{2},$$

which follows from the following simple considerations.
The continuous function $\xi(z) := \Phi_\beta(z) - \frac{\beta^2(z)}{2L_\beta}$ fulfills $\xi(0) = 0$ and $\xi'(z) = \beta(z)\left(1 - \frac{\beta'(z)}{L_\beta}\right)$. Therefore $\xi(z) \ge 0$.
Assume $0 \le s \le z$. Applying the mean value theorem we get $\beta(s) = \beta'(\theta_s)s \le L_\beta s$. An integration over $(0, z)$ implies $\Phi_\beta(z) \le \frac{L_\beta z^2}{2}$.
Assume $z \le s \le 0$. Then $\beta(s) = \beta'(\theta_s)s \ge L_\beta s$. An integration over $(z, 0)$ implies $\Phi_\beta(z) \le \frac{L_\beta z^2}{2}$.

Moreover one can easily verify that

$$\beta(z_1)(z_2 - z_1) \le \Phi_\beta(z_2) - \Phi_\beta(z_1) \le \beta(z_2)(z_2 - z_1)$$

for any $z_1, z_2 \in \mathbb{R}$, which follows from the monotone character of β.

Put $\varphi = u_i\tau$ in (9) and sum it up for $i = 1, \ldots, j$. We get

$$\begin{aligned}\sum_{i=1}^{j} (g(u_i) - g(u_{i-1}), u_i) &+ \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau + \sum_{i=1}^{j} \alpha_i \|u_i\|_\Gamma^2 \tau \\ &= \sum_{i=1}^{j} (f(u_{i-1}), u_i)\tau - \sum_{i=1}^{j} (\beta_i, u_i)_\Gamma \tau - \sum_{i=1}^{j} (K, u_i)(1, u_i)_\Gamma \tau.\end{aligned} \tag{10}$$

The first term on the LHS can be estimated as follows

$$\begin{aligned}\sum_{i=1}^{j} (g(u_i) - g(u_{i-1}), u_i) &\ge \sum_{i=1}^{j} \int_\Omega \left[\Phi_{g^{-1}}(g(u_i)) - \Phi_{g^{-1}}(g(u_{i-1}))\right] \\ &= \int_\Omega \left[\Phi_{g^{-1}}(g(u_j)) - \Phi_{g^{-1}}(g(u_0))\right] \\ &\ge \frac{\gamma \|u_j\|^2}{2} - C\|u_0\|^2.\end{aligned}$$

Applying the Cauchy and Young inequalities together with (6) to the RHS of (10) we easily arrive at

$$\|u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau + \sum_{i=1}^{j} \alpha_i \|u_i\|_\Gamma^2 \tau \le \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau + C_\varepsilon + C_\varepsilon \sum_{i=1}^{j} \|u_i\|^2 \tau.$$

Fixing a sufficiently small ε and using the Gronwall lemma we conclude the proof. □

Let $\{a_i\}_{i=0}^\infty$ and $\{b_i\}_{i=0}^\infty$ be any sequences of real numbers such that all b_i are nonnegative. We start with an obvious identity

$$a_i(a_i - a_{i-1}) = \frac{1}{2}\left[a_i^2 - a_{i-1}^2 + (a_i - a_{i-1})^2\right],$$

which after summation gives

$$\begin{aligned}\sum_{i=1}^{j} b_i a_i (a_i - a_{i-1}) &= \frac{1}{2}\sum_{i=1}^{j} b_i \left[a_i^2 - a_{i-1}^2 + (a_i - a_{i-1})^2\right] \\ &= \frac{1}{2}\sum_{i=1}^{j} b_i (a_i - a_{i-1})^2 + \frac{1}{2}\sum_{i=1}^{j} b_i (a_i^2 - a_{i-1}^2) \\ &= \frac{1}{2}\sum_{i=1}^{j} b_i (a_i - a_{i-1})^2 + \frac{1}{2}\left[b_j a_j^2 - b_0 a_0^2 - \sum_{i=1}^{j} \delta b_i \, a_{i-1}^2 \tau\right] \\ &\geq \frac{1}{2}\left[b_j a_j^2 - b_0 a_0^2 - \sum_{i=1}^{j} \delta b_i \, a_{i-1}^2 \tau\right]. \end{aligned} \tag{11}$$

We will use this relation in the following lemma.

Lemma 3.2. *Suppose (2). Then there exists a positive constant C such that*

$$\sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 + \left\|\nabla u_j\right\|^2 + \alpha_j \left\|u_j\right\|_\Gamma^2 \leq C$$

for $j = 1, \ldots, n$.

Proof. $\varphi = u_i - u_{i-1}$ in (9) and sum it up for $i = 1, \ldots, j$. Using Abel's summation

$$2\sum_{i=1}^{j} a_i (a_i - a_{i-1}) = a_j^2 - a_0^2 + \sum_{i=1}^{j} (a_i - a_{i-1})^2$$

we get

$$\begin{aligned}&\sum_{i=1}^{j} (\delta g(u_i), \delta u_i)\, \tau + \frac{1}{2}\left[\left\|\nabla u_j\right\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2\right] + \sum_{i=1}^{j}\int_\Gamma \alpha_i u_i \delta u_i \tau \\ &\quad = \sum_{i=1}^{j} (f(u_{i-1}), \delta u_i)\, \tau - \sum_{i=1}^{j} (\beta_i, \delta u_i)_\Gamma\, \tau - \sum_{i=1}^{j} (K, u_i)\, (1, \delta u_i)_\Gamma\, \tau. \end{aligned} \tag{12}$$

The last term on the left can be estimated using (11) and Lemma 3.1 as follows

$$\sum_{i=1}^{j}\int_\Gamma \alpha_i u_i \delta u_i \tau \geq \frac{1}{2}\left[\alpha_j \left\|u_j\right\|_\Gamma^2 - \alpha_0 \|u_0\|_\Gamma^2 - C\sum_{i=0}^{j} \|u_i\|_\Gamma^2\, \tau\right] \geq \frac{1}{2}\alpha_j \left\|u_j\right\|_\Gamma^2 - C.$$

For the terms on the RHS of (12) we successively deduce that

$$\sum_{i=1}^{j} (f(u_{i-1}), \delta u_i)\, \tau \leq \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau + C_\varepsilon$$

and

$$\left|\sum_{i=1}^{j} (\beta_i, \delta u_i)_\Gamma\, \tau\right| = \left|\left(\beta_j, u_j\right)_\Gamma - (\beta_0, u_0)_\Gamma - \sum_{i=1}^{j} (\delta \beta_i, u_{i-1})_\Gamma\, \tau\right| \leq \varepsilon \left\|\nabla u_j\right\|^2 + C_\varepsilon$$

and

$$\begin{aligned}\left|\sum_{i=1}^{j} (K, u_i)\, (1, \delta u_i)_\Gamma\, \tau\right| &= \left|\left(K, u_j\right)\left(1, u_j\right)_\Gamma - (K, u_0)\, (1, u_0)_\Gamma - \sum_{i=1}^{j} (K, \delta u_i)\, (1, u_{i-1})_\Gamma\, \tau\right| \\ &\leq \varepsilon \left\|\nabla u_j\right\|^2 + C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau. \end{aligned}$$

Further we can write

$$\sum_{i=1}^{j} (\delta g(u_i), \delta u_i)\, \tau \geq \gamma \sum_{i=1}^{j} \|\delta u_i\|^2 \tau.$$

Collecting the estimates above and fixing a sufficiently small ε we obtain the desired result. □

Now, let us introduce the following piecewise linear in time functions

$$\begin{aligned} &G_n(0) = g(u_0)\\ &G_n(t) = g(u_{i-1}) + (t - t_{i-1})\delta g(u_i) \quad \text{for } t \in (t_{i-1}, t_i], \end{aligned}$$

and

$$\begin{aligned} &u_n(0) = u_0\\ &u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i \quad \text{for } t \in (t_{i-1}, t_i]. \end{aligned}$$

We also introduce a step function $\overline{u}_n$

$$\overline{u}_n(0) = u_0, \qquad \overline{u}_n(t) = u_i, \quad \text{for } t \in (t_{i-1}, t_i].$$

Similarly we define $\overline{\alpha}_n, \overline{\beta}_n$. The variational formulation (9) can be rewritten as

$$(\partial_t G_n, \varphi) + (\nabla \overline{u}_n, \nabla\varphi) + (\overline{\alpha}_n \overline{u}_n, \varphi)_\Gamma + (K, \overline{u}_n)\,(1, \varphi)_\Gamma = \left(\overline{f}(\overline{u}_n(t-\tau)), \varphi\right) - \left(\overline{\beta}_n, \varphi\right)_\Gamma \tag{13}$$

for any $\varphi \in H^1(\Omega)$.

Now, we are in a position to show the solvability of (8).

Theorem 3.1. *Suppose (2). Then there exists a solution to (8).*

Proof. A priori estimates fulfill the conditions of [16, Lemma 1.3.13]. This directly gives the existence of a function $u \in C([0,T], L_2(\Omega)) \cap L_\infty\left((0,T), H^1(\Omega)\right)$ obeying $\partial_t u \in L_2((0,T), L_2(\Omega))$ and the existence a subsequence of $\{u_n\}$ (denoted by the same symbol again), for which

$$\begin{aligned} &u_n \to u && \text{in } C([0,T], L_2(\Omega))\\ &\partial_t u_n \rightharpoonup \partial_t u && \text{in } L_2((0,T), L_2(\Omega))\\ &\overline{u}_n(t) \rightharpoonup u(t) && \text{in } H^1(\Omega) \quad \text{for all } t \in [0,T]. \end{aligned} \tag{14}$$

Using the stability results and (6) we deduce (an analogous deduction is valid for $\overline{u}_n$)

$$\begin{aligned} \int_0^T \|u_n - u\|_\Gamma^2 &\le \varepsilon \int_0^T \|u_n - u\|_{H^1(\Omega)}^2 + C_\varepsilon \int_0^T \|u_n - u\|^2\\ &\le C\varepsilon + C_\varepsilon \int_0^T \|u_n - u\|^2 . \end{aligned}$$

Passing to the limit for $n \to \infty$ and using (14) we get

$$\lim_{n\to\infty} \int_0^T \|u_n - u\|_\Gamma^2 \le C\varepsilon,$$

which for $\varepsilon \to 0$ implies

$$u_n, \overline{u}_n \to u \quad \text{in } L_2((0,T), L_2(\Gamma)) . \tag{15}$$

Lipschitz continuity of g together with (14) give

$$g(u_n) \to g(u) \quad \text{in } C([0,T], L_2(\Omega)) .$$

Integrating (13) over $(0,t)$, passing to the limit for $n \to \infty$ and differentiating the result with respect to the time variable, we get the existence of a weak solution to (8). We show only a short hint for the nonlinear term in (13), which is the most complicated one. The reasoning for other terms is standard.

Assume that $t \in (t_{j-1}, t_j]$.

$$\begin{aligned} \int_0^t (\partial_t G_n, \varphi) &= \int_0^{t_j} (\partial_t G_n, \varphi) - \int_t^{t_j} (\partial_t G_n, \varphi)\\ &= (g(\overline{u}_n(t)) - g(u_0), \varphi) - \int_t^{t_j} (\partial_t G_n, \varphi)\\ &= (g(\overline{u}_n(t)) - g(u_0), \varphi) - \int_t^{t_j} \left(\frac{g(u_n(t_j)) - g(u_n(t_{j-1}))}{\tau}, \varphi\right). \end{aligned} \tag{16}$$

According to Lemma 3.2 we have for $s < t$

$$\|g(u_n(t)) - g(u_n(s))\| \le C\,\|u_n(t) - u_n(s)\| \le C \int_s^t \|\partial_t u_n\| \le C\sqrt{|t-s|}.$$

Table 1
Numerical solution of (17) with $L = 1 + 2e^{0.5}$.

x_i	u^*	u with $\tau = 0.001$		u with $\tau = 0.0001$	
		$d = 2$ maxnit = 11	$d = 3$ maxnit = 29	$d = 2$ maxnit = 17	$d = 3$ maxnit = 45
0.0	1.6487213	1.6477674	1.6477819	1.6486307	1.6486314
0.62831855	1.3338435	1.3330053	1.3330726	1.3337659	1.3337696
1.2566371	0.50948284	0.50903394	0.50917884	0.50944467	0.50945361
1.8849556	−0.50948297	−0.50903584	−0.5091806	−0.50944493	−0.50945387
2.5132742	−1.3338436	−1.3330282	−1.3330948	−1.3337679	−1.3337716
3.1415927	−1.6487213	−1.6480338	−1.6480411	−1.6486553	−1.6486556

Table 2
Numerical solution of (17) with $L = 100$.

x_i	u^*	u with $\tau = 0.001$		u with $\tau = 0.0001$	
		$d = 2$ maxnit = 135	$d = 3$ maxnit = 535	$d = 2$ maxnit = 245	$d = 3$ maxnit = 913
0.0	1.6487213	1.6382256	1.6477815	1.6482649	1.6486314
0.62831855	1.3338435	1.3205855	1.3330704	1.3337659	1.3337696
1.2566371	0.50948284	0.49888473	0.50917323	0.50841503	0.50945359
1.8849556	−0.50948297	−0.49890955	−0.50917508	−0.50841637	−0.50945384
2.5132742	−1.3338436	−1.3209115	−1.3330927	−1.3330338	−1.3337716
3.1415927	−1.6487213	−1.6415738	−1.6480409	−1.6484228	−1.6486556

Thus, passing to the limit for $n \to \infty$ in (16) we get

$$\lim_{n\to\infty} \int_0^t (\partial_t G_n, \varphi) = (g(u(t)) - g(u_0), \varphi) .$$

On the other hand, $\int_0^T \|\partial_t G_n\|^2 \le C \int_0^T \|\partial_t u_n\|^2 \le C$ and we may write (due to the reflexivity of $L_2((0,T), L_2(\Omega))$

$$\lim_{n\to\infty} \int_0^t (\partial_t G_n, \varphi) = \int_0^t (\zeta, \varphi) .$$

This relation is valid for any t, thus $\zeta = \partial_t g(u(t))$ and one can differentiate the term $(g(u(t)) - g(u_0), \varphi)$ with respect to the time variable. □

4. Numerical experiment

In this section we will present the numerical results from the solution of the model problem (1) for testing the performance of the presented algorithms. We consider a model problem in the form

$$\begin{aligned} &\partial_t g(u) - u_{xx} = f(u) \quad x \in (0,\pi), t \in (0,T) \\ &u_x(0,t) = 0; \\ &-u_x(\pi,t) - u(\pi,t) = \int_0^\pi K(x)\, u(x,t)\, \mathrm{d}x + \beta(t) \\ &u(x,0) = u^0(x), \end{aligned} \tag{17}$$

where g and K are some functions to be chosen.

Let $u^*(x,t) = e^t \cos(x)$ be the solution to (17). We choose $K(x) = \frac{x}{8}$ and $g(s) = s + s|s|$ in our computations. Then $\beta(t) = \frac{5}{4}e^t, f(u) = \partial_t g(u) - u_{xx} = 2u(1+|u|)$ and $u^0(x) = \cos(x)$.

The solution u_i on any time step t_i will be approached by the relaxation process (4). We recall that L is the Lipschitz constant of the function g. The iteration process stops when the following condition is satisfied

$$\|u_{i,k} - u_{i,k-1}\| \le \tau^d,$$

where $d > 1$ is a fixed constant and $\|.\|$ stands for the usual $L_2(\Omega)$-norm.

After stopping the iterations at $k = k_{i,last}$ we denote $u_i := u_{i,k_{i,last}}$. The results for $u(t, x_j)$ with $h = \pi/500$ at the time point $t = 0.5$, using Rothe-finite element method developed in this article and the exact solution u^*, are shown in Table 1 for $L = 1+2e^{0.5}$. Analogous results but for $L = 100$ are shown in Table 2. Here, the symbol "maxnit" denotes the maximum number of iterations. Comparing the number of iterations in both tables we see that the relaxation process converges faster for smaller L.

Fig. 1 shows the "maximum $L_2(\Omega)$-error" $err = \max_{t\in[0,T]} \|u^*(t) - u(t)\|$, which is graphed against τ. We have taken $h = \frac{\pi}{500}$ in our computations. We see that the rate of convergence is almost $\mathcal{O}(\tau)$.

Fig. 2 depicts the "maximum $L_2(\Omega)$-error", which is graphed against h. We have taken $\tau = 0.001$ in our computations.

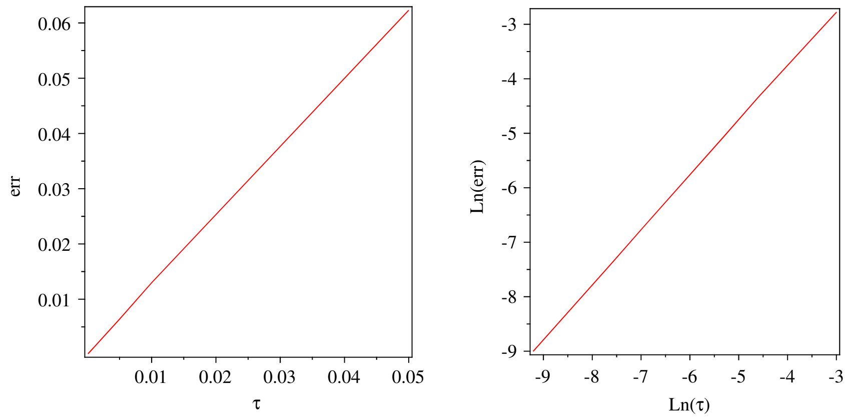

Fig. 1. Relation between the maximum $L_2(\Omega)$-error and the discretization parameter τ.

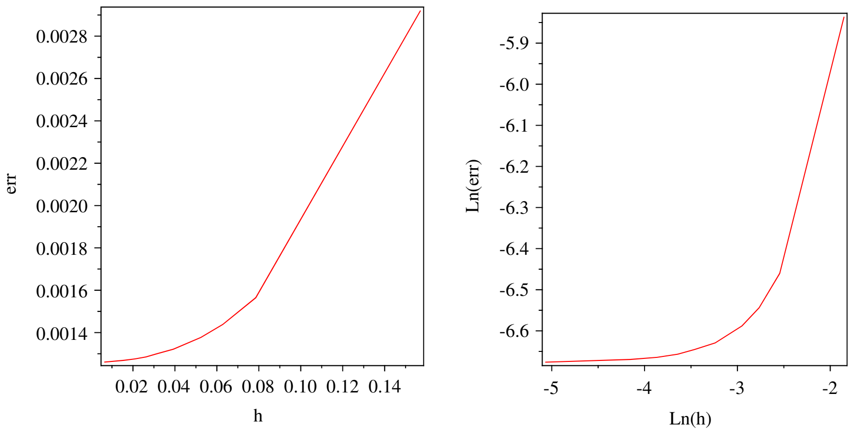

Fig. 2. Relation between the maximum $L_2(\Omega)$-error and the discretization parameter h.

Conclusions

We have studied a nonlinear parabolic problem with a nonlocal BC with applications in thermoelasticity. The nonlocal term in the BC was modelled by a weighted average of a solution over the whole Ω. We have designed a very easy implementable algorithm for computations, based on suitable linearization and an on the superposition principle. We have proved the existence of a weak solution. The uniqueness of a solution remains still an open problem.

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