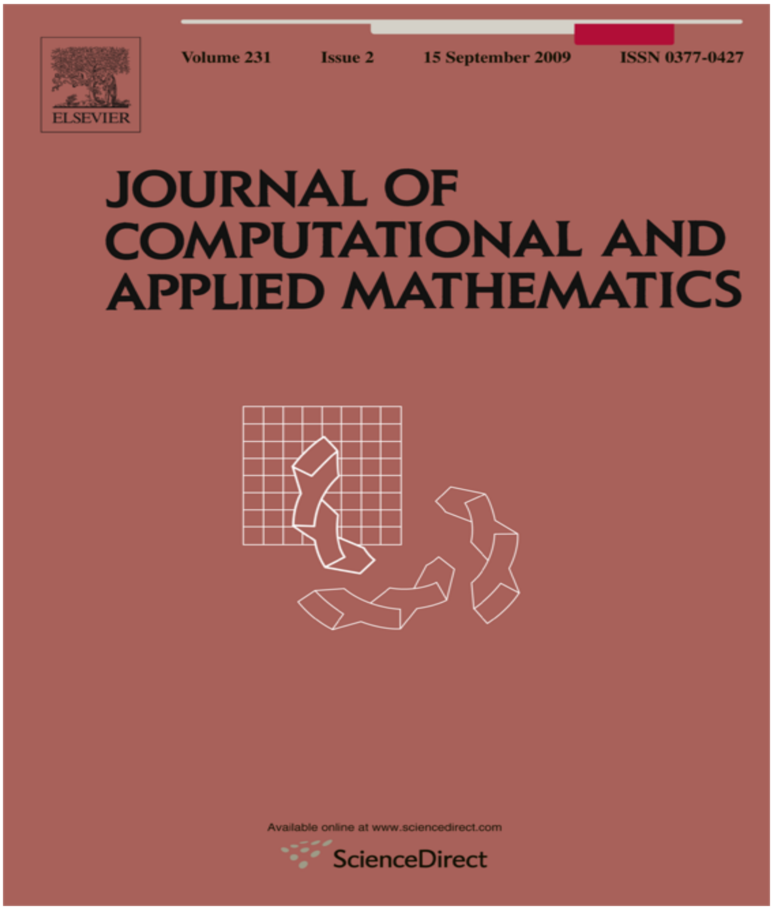
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A numerical approach for a semilinear parabolic equation with a nonlocal boundary condition

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ABSTRACT

A semilinear reaction-diffusion problem with a nonlocal boundary condition is studied. This paper presents a new and very easy implementable numerical algorithm for computations. This is based on a suitable linearization in time and on the principle of linear superposition. Any method for the space discretization (FEM was taken in this analysis) can be chosen. The derived algorithm is implicit and it does not need any iteration scheme to get a solution with the nonlocal boundary condition. Stability analysis has been performed and the optimal error estimates have been derived. Numerical results have been compared with other known techniques.

1. Introduction

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ (with a Lipschitz boundary Γ) and a finite time interval $[0, T]$. First, we denote by $(w, z)_M$ the standard L_2-scalar product of the functions w and z on a set M, i.e.,

$$(w, z)_M = \int_M wz$$

and the corresponding norm

$$\|w\|_M^2 = (w, w)_M .$$

The subscript will be suppressed if $M = \Omega$. Let $\boldsymbol{\nu}$ be the outer unit normal vector to Γ.

This paper is concerned with a numerical solution to the following semilinear evolution equation with a nonlocal Robin boundary condition (BC)

$$\begin{aligned} &\partial_t u - \Delta u = f(t, u) && \text{in } (0, T) \times \Omega \\ &-\nabla u \cdot \boldsymbol{\nu} = \alpha u + \beta + (K, u) && \text{on } \Gamma \\ &u(0) = u_0 && \text{in } \Omega, \end{aligned} \tag{1}$$

where α, β, K and f are given functions.

Nonlocal Dirichlet BC in 1D arises in the theory of thermoelasticity, c.f. [1,2]. The well-posedness of this problem has been studied by various authors - see [3–5] under the additional condition

$$\int_\Omega |K(x)| \, dx < 1.$$

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Numerical studies for the Dirichlet nonlocal BC have been performed by a number of authors — c.f. [6–9] under some stronger conditions on K. The stability of various numerical methods has been studied in [10].

Solvability subject to a nonlocal Robin BC has been studied in [11,12]. Numerical study based on monotonicity methods has been performed in [13].

The main numerical problem for nonlocal (Dirichlet or Robin) BC is the special character of the algebraic matrix obtained by the full discretization. Independently of the fact, which space discretization method has been used (finite differences, finite elements), the algebraic matrix will have one full line. This needs a special solver to get a result. In this paper we design a very easy numerical algorithm, based on linearization and on the superposition principle. This algorithm is directly in contrast to the monotone iterative method from [13]. Moreover, the algebraic matrices obtained after a full discretization are standard (having only a few non-zero elements per row/column).

The paper is organized as follows. First we discretize the problem (backward Euler in time and finite elements in space).[2] Then we introduce some auxiliary problems with standard BCs. A suitable linear combination of solutions of these temporary problems gives the solution of a discreet problem with the nonlocal BC on a fixed time step. Further we derive suitable stability estimates for an approximate solution and finally we get the error estimates for the whole method. The last section supports the developed algorithm by some numerical experiments.

Finally, as is usual in papers of this sort, C, ε and C_ε will denote generic positive constants depending only on a priori known quantities, where ε is small and C_ε is large.

2. Algorithm

Rothe's method represents a constructive method suitable for solving evolution problems. Using a simple discretization in time, a time-dependent problem is approximated by a sequence of elliptic problems which have to be solved successively with increasing time steps. This standard procedure is in our case complicated by the nonstandard problem setting, i.e., by the nonlocal BC on Γ. But there exists a simple way how to avoid this complication. We explain it briefly.

First, we divide the time interval $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals (t_{i-1}, t_i) for $t_i = i\tau$, where $\tau = \frac{T}{n}$. We introduce the following notation

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}$$

for any function z. We are left with a recurrent system of elliptic BVPs at each successive time point t_i for $i = 1, \dots, n$

$$\begin{aligned} &\delta u_i - \Delta u_i = f(t_i, u_{i-1}) && \text{in } \Omega \\ &-\nabla u_i \cdot \boldsymbol{\nu} = \alpha_i u_i + \beta_i + (K, u_i) && \text{on } \Gamma \\ &u_0 = u(0) && \text{in } \Omega. \end{aligned} \tag{2}$$

For any given i we introduce the following two auxiliary problems

$$\begin{aligned} &v_i - \tau\Delta v_i = u_{i-1} + \tau f(t_i, u_{i-1}) && \text{in } \Omega \\ &-\nabla v_i \cdot \boldsymbol{\nu} = \alpha_i v_i + \beta_i && \text{on } \Gamma \end{aligned} \tag{3}$$

and

$$\begin{aligned} &z_i - \tau\Delta z_i = 0 && \text{in } \Omega \\ &-\nabla z_i \cdot \boldsymbol{\nu} = \alpha_i z_i + 1 && \text{on } \Gamma. \end{aligned} \tag{4}$$

Let us note that both temporary problems are standard problems. These are well-posed under appropriate conditions on the data functions f, α, β and u_0. The principle of linear superposition gives that $w_i := v_i + \omega_i z_i$ solves

$$\begin{aligned} &w_i - \tau\Delta w_i = u_{i-1} + \tau f(t_i, u_{i-1}) && \text{in } \Omega \\ &-\nabla w_i \cdot \boldsymbol{\nu} = \alpha_i w_i + \beta_i + \omega_i && \text{on } \Gamma. \end{aligned} \tag{5}$$

We are looking for an ω_i such that

$$\omega_i = (K, u_i) = (K, v_i + \omega_i z_i),$$

from which we get

$$\omega_i = \frac{(K, v_i)}{1 - (K, z_i)}. \tag{6}$$

Here, we have to check if the nominator cannot vanish.

3. Stability

In this section we derive the existence of a weak solution to (1), which is given as

$$(\partial_t u, \varphi) + (\nabla u, \nabla\varphi) + (\alpha u, \varphi)_\Gamma + (K, u)\,(1, \varphi)_\Gamma = (f(t, u), \varphi) - (\beta, \varphi)_\Gamma \tag{7}$$

for any $\varphi \in H^1(\Omega)$ and a.e. in $(0, T)$.

[2] One can also use finite differences for the space discretization.

The variational formulation of (2) reads as

$$(\delta u_i, \varphi) + (\nabla u_i, \nabla\varphi) + (\alpha_i u_i, \varphi)_\Gamma + (K, u_i)\,(1, \varphi)_\Gamma = (f(t_i, u_{i-1}), \varphi) - (\beta_i, \varphi)_\Gamma \tag{8}$$

for any $\varphi \in H^1(\Omega)$. The variational formulations of temporary problems are

$$(v_i, \varphi) + \tau\,(\nabla v_i, \nabla\varphi) + \tau\,(\alpha_i v_i, \varphi)_\Gamma = (u_{i-1} + \tau f(t_i, u_{i-1}), \varphi) - \tau\,(\beta_i, \varphi)_\Gamma \tag{9}$$

and

$$(z_i, \varphi) + \tau\,(\nabla z_i, \nabla\varphi) + \tau\,(\alpha_i z_i, \varphi)_\Gamma = -\tau\,(1, \varphi)_\Gamma \tag{10}$$

for any $\varphi \in H^1(\Omega)$.

Lemma 3.1. *Let f be a global Lipschitz continuous function in both variables. Moreover assume that $u_0 \in L_2(\Omega)$, $\beta_i \in L_2(\Gamma)$ and $0 \le \alpha_i \le C$ for $i = 1, \ldots, n$. Then the problems* (9) *and* (10) *are well-posed if $\tau < \tau_0$.*

The assertion of this lemma directly follows from the theory of linear elliptic equations as a consequence of the Lax–Milgram lemma, c.f. [14,15].

The following inequality holds true — see [16]

$$\|z\|_\Gamma^2 \le \varepsilon\,\|\nabla z\|^2 + C_\varepsilon\,\|z\|^2, \quad \forall z \in H^1(\Omega), 0 < \varepsilon < \varepsilon_0. \tag{11}$$

Lemma 3.2. *Let the assumptions of Lemma* 3.1 *be satisfied. Moreover assume that $K \in H^1(\Omega)$. Then there exists $\tau_0 > 0$ such that*

$$|\,(K, z_i)\,| < 1, \quad i = 1, \ldots, n, 0 < \tau < \tau_0.$$

Proof. First, we set $\varphi = z_i \in H^1(\Omega)$ into (10) and we get

$$\|z_i\|^2 + \tau\,\|\nabla z_i\|^2 + \tau\left(\alpha_i, z_i^2\right)_\Gamma = -\tau\,(1, z_i)_\Gamma\,.$$

Omitting the third non-negative term on the left-hand side, using the Cauchy inequality and (11) we easily obtain

$$\|z_i\|^2 + \tau\,\|\nabla z_i\|^2 \le C\tau + C\tau\,\|z_i\|_\Gamma^2 \le C\tau + \varepsilon\tau\,\|\nabla z_i\|^2 + C_\varepsilon\tau\,\|z_i\|^2\,,$$

which yields that

$$(1 - C_\varepsilon\tau)\,\|z_i\|^2 + \tau(1 - \varepsilon)\,\|\nabla z_i\|^2 \le C_\varepsilon\tau.$$

Choosing a sufficiently small positive ε and τ_0, we deduce that

$$\|z_i\|^2 + \|\nabla z_i\|^2 \le C$$

for sufficiently small time step τ. Now, we set $\varphi = K \in H^1(\Omega)$ into (10) and we obtain

$$(z_i, K) + \tau\,(\nabla z_i, \nabla K) + \tau\,(\alpha_i z_i, K)_\Gamma = -\tau\,(1, K)_\Gamma\,.$$

Applying the Cauchy inequality and the trace theorem we readily arrive at

$$|(z_i, K)| \le C\tau\,(\|\nabla z_i\|\,\|\nabla K\| + \|z_i\|_\Gamma\,\|K\|_\Gamma + \|K\|_\Gamma) \le C\tau,$$

which concludes the proof. □

Following the considerations from Section 2, according to (6) and Lemmas 3.1 and 3.2 we may say that

Lemma 3.3. *Let the assumptions of Lemma* 3.2 *be satisfied. Then the problem* (8) *is well-posed for $i = 1, \ldots, n$ and $0 < \tau < \tau_0$ and there exists a unique weak solution $u_i \in H^1(\Omega)$.*

The stability estimates for u_i can be obtained readily using the standard technique for Rothe's method. We describe very briefly the main steps.

Lemma 3.4. *Let the assumptions of Lemma* 3.2 *be satisfied. Then there exists a positive constant C such that*

$$\left\|u_j\right\|^2 + \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j}\|\nabla u_i\|^2\,\tau + \sum_{i=1}^{j}\int_\Gamma \alpha_i {u_i}^2\tau \le C$$

for $j = 1, \ldots, n$.

Proof. Put $\varphi = u_i\tau$ in (8) and sum it for $i = 1, \ldots, j$. Using Abel's summation

$$2\sum_{i=1}^{j} a_i(a_i - a_{i-1}) = a_j^2 - a_0^2 + \sum_{i=1}^{j}(a_i - a_{i-1})^2$$

we get

$$\frac{1}{2}\left[\left\|u_j\right\|^2 - \|u_0\|^2 + \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2\right] + \sum_{i=1}^{j}\|\nabla u_i\|^2\tau + \sum_{i=1}^{j}\int_\Gamma \alpha_i {u_i}^2\tau$$
$$= \sum_{i=1}^{j}(f(t_i, u_{i-1}), u_i)\,\tau - \sum_{i=1}^{j}(\beta_i, u_i)_\Gamma\,\tau - \sum_{i=1}^{j}(K, u_i)\,(1, u_i)_\Gamma\,\tau.$$

Applying the Cauchy and Young inequalities together with (11) to the right-hand side we easily arrive at

$$\left\|u_j\right\|^2 + \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j}\|\nabla u_i\|^2\tau + \sum_{i=1}^{j}\int_\Gamma \alpha_i {u_i}^2\tau \le \varepsilon\sum_{i=1}^{j}\|\nabla u_i\|^2\tau + C_\varepsilon + C_\varepsilon\sum_{i=1}^{j}\|u_i\|^2\tau.$$

Fixing a sufficiently small ε and using the Gronwall lemma we obtain the desired result. $\square$

Let $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ be any sequences of real numbers such that all b_i are nonnegative. We start with an obvious identity

$$a_i(a_i - a_{i-1}) = \frac{1}{2}\left[a_i^2 - a_{i-1}^2 + (a_i - a_{i-1})^2\right],$$

which after summation gives

$$\begin{aligned}
\sum_{i=1}^{j} b_i a_i(a_i - a_{i-1}) &= \frac{1}{2}\sum_{i=1}^{j} b_i\left[a_i^2 - a_{i-1}^2 + (a_i - a_{i-1})^2\right]\\
&= \frac{1}{2}\sum_{i=1}^{j} b_i(a_i - a_{i-1})^2 + \frac{1}{2}\sum_{i=1}^{j} b_i(a_i^2 - a_{i-1}^2)\\
&= \frac{1}{2}\sum_{i=1}^{j} b_i(a_i - a_{i-1})^2 + \frac{1}{2}\left[b_j a_j^2 - b_0 a_0^2 - \sum_{i=1}^{j}\delta b_i\, a_{i-1}^2\tau\right]\\
&\ge \frac{1}{2}\left[b_j a_j^2 - b_0 a_0^2 - \sum_{i=1}^{j}\delta b_i\, a_{i-1}^2\tau\right].
\end{aligned} \tag{12}$$

We will use this inequality in the following lemma.

Lemma 3.5. *Let the assumptions of Lemma 3.2 be satisfied. Moreover we suppose that*

$$|\alpha'| \le C, \quad \beta' \in L_2\left((0, T), L_2(\Gamma)\right), u_0 \in H^1(\Omega).$$

Then there exists a positive constant C such that

$$\sum_{i=1}^{j}\|\delta u_i\|^2\tau + \sum_{i=1}^{j}\|\nabla u_i - \nabla u_{i-1}\|^2 + \left\|\nabla u_j\right\|^2 + \int_\Gamma \alpha_j {u_j}^2 \le C$$

for $j = 1, \ldots, n$.

Proof. $\varphi = u_i - u_{i-1}$ in (8) and sum it up for $i = 1, \ldots, j$. Using Abel's summation we get

$$\sum_{i=1}^{j}\|\delta u_i\|^2\tau + \frac{1}{2}\left[\left\|\nabla u_j\right\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^{j}\|\nabla u_i - \nabla u_{i-1}\|^2\right] + \sum_{i=1}^{j}\int_\Gamma \alpha_i u_i \delta u_i\tau$$
$$= \sum_{i=1}^{j}(f(t_i, u_{i-1}), \delta u_i)\,\tau - \sum_{i=1}^{j}(\beta_i, \delta u_i)_\Gamma\,\tau - \sum_{i=1}^{j}(K, u_i)\,(1, \delta u_i)_\Gamma\,\tau.$$

The last term on the left can be estimated using (12) and Lemma 3.4 as follows

$$\sum_{i=1}^{j}\int_\Gamma \alpha_i u_i \delta u_i\tau \ge \frac{1}{2}\left[\int_\Gamma \alpha_j {u_j}^2 - \int_\Gamma \alpha_0 {u_0}^2 - C\sum_{i=0}^{j}\|u_i\|_\Gamma^2\,\tau\right] \ge \frac{1}{2}\int_\Gamma \alpha_j {u_j}^2 - C.$$

For the terms on the RHS we successively deduce that

$$\sum_{i=1}^{j} (f(t_i, u_{i-1}), \delta u_i)\, \tau \le \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2\, \tau + C_\varepsilon$$

and

$$\left| \sum_{i=1}^{j} (\beta_i, \delta u_i)_\Gamma\, \tau \right| = \left| \left(\beta_j, u_j\right)_\Gamma - (\beta_0, u_0)_\Gamma - \sum_{i=1}^{j} (\delta \beta_i, u_{i-1})_\Gamma\, \tau \right| \le \varepsilon \left\|\nabla u_j\right\|^2 + C_\varepsilon$$

and

$$\begin{aligned}\sum_{i=1}^{j} (K, u_i)\,(1, \delta u_i)_\Gamma\, \tau &= \left(K, u_j\right)\left(1, u_j\right)_\Gamma - (K, u_0)\,(1, u_0)_\Gamma - \sum_{i=1}^{j} (K, \delta u_i)\,(1, u_{i-1})_\Gamma\, \tau \\ &\le \varepsilon \left\|\nabla u_j\right\|^2 + C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2\, \tau .\end{aligned}$$

Putting things together and fixing a sufficiently small ε we conclude the proof. □

Now, let us introduce the following piecewise linear in time function

$$u_n(0) = u_0$$
$$u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i \quad \text{for } t \in (t_{i-1}, t_i],$$

and the step function $\overline{u}_n$

$$\overline{u}_n(0) = u_0, \qquad \overline{u}_n(t) = u_i, \quad \text{for } t \in (t_{i-1}, t_i].$$

Similarly we define $\overline{\alpha}_n$, $\overline{\beta}_n$. The variational formulation (8) can be rewritten as

$$(\partial_t u_n, \varphi) + (\nabla \overline{u}_n, \nabla \varphi) + (\overline{\alpha}_n \overline{u}_n, \varphi)_\Gamma + (K, \overline{u}_n)\,(1, \varphi)_\Gamma = \left(\overline{f}(t, \overline{u}_n(t-\tau)), \varphi\right) - \left(\overline{\beta}_n, \varphi\right)_\Gamma \tag{13}$$

for any $\varphi \in H^1(\Omega)$.

Now, we are in a position to show the well-posedness of the problem (8).

Theorem 3.1. *Let the assumptions of Lemma 3.5 be fulfilled. Then there exists a unique solution to (7).*

Proof. A priori estimates together with [17, Lemma 1.3.13] imply the existence of a function $u \in C([0,T], L_2(\Omega)) \cap L_\infty\left((0,T), H^1(\Omega)\right)$ obeying $\partial_t u \in L_2((0,T), L_2(\Omega))$ and the existence a subsequence of $\{u_n\}$ (denoted by the same symbol again), for which

$$\begin{aligned} u_n &\to u && \text{in } C([0,T], L_2(\Omega)) \\ \partial_t u_n &\rightharpoonup \partial_t u && \text{in } L_2((0,T), L_2(\Omega)) \\ \overline{u}_n(t) &\rightharpoonup u(t) && \text{in } H^1(\Omega) \text{ for all } t \in [0,T]. \end{aligned} \tag{14}$$

Using (11) and (14) we deduce

$$u_n, \overline{u}_n \to u \quad L_2((0,T), L_2(\Gamma)). \tag{15}$$

Integrating (13) over $(0,t)$, passing to the limit for $n \to \infty$ and differentiating the result with respect to the time variable, we get the existence of a weak solution to (8).

The uniqueness of a solution follows by standard arguments using Gronwall's lemma. □

4. Error estimates for full discretization

To get the optimal rate of convergence we shall need the following compatibility condition between the initial datum and the BC:
There exists $v \in L_2(\Omega)$ such that

$$(v, \varphi) + (\nabla u_0, \nabla \varphi) + (\alpha_0 u_0, \varphi)_\Gamma + (K, u_0)\,(1, \varphi)_\Gamma = (f(0, u_0), \varphi) - (\beta_0, \varphi)_\Gamma \tag{16}$$

for any $\varphi \in H^1(\Omega)$. This relation gives us the possibility to differentiate (7) with respect to the time variable and to get better stability for a solution when putting $\varphi = \partial_t u = u_t$ and integrating over the time variable. More exactly we get

$$\begin{aligned}&\int_0^t (u_{tt}, u_t) + \int_0^t \|\nabla u_t\|^2 + \int_0^t (\alpha u_t, u_t)_\Gamma \\ &= \int_0^t (\partial_t f(t,u), u_t) - \int_0^t \left(\beta', u_t\right)_\Gamma - \int_0^t \left(\alpha' u, u_t\right)_\Gamma - \int_0^t (K, u_t)\,(1, u_t)_\Gamma .\end{aligned}$$

Taking into account the fact that $u_t(0) = v \in L_2(\Omega)$ and using the standard estimates (Cauchy, Young inequalities, (11) and the Gronwall lemma) we easily arrive at

$$\max_{t\in[0,T]} \|u_t(t)\|^2 + \int_0^T \|\nabla u_t\|^2 + \int_0^T (\alpha u_t, u_t)_\Gamma \le C. \tag{17}$$

Let V_h be a system of finite dimensional subspaces of $H^1(\Omega)$ for $h > 0$. The corresponding Ritz projector is denoted by $P_h : H^1(\Omega) \to V_h$. We assume that

$$\begin{aligned} &\forall u \in H^1(\Omega): && \lim_{h\to 0} \|u - P_h u\|_{H^1(\Omega)} = 0 \\ &\exists C > 0\, \forall u \in H^1(\Omega)\, \forall h: && \|u - P_h u\|_{H^1(\Omega)} \le C \|u\|_{H^1(\Omega)}. \end{aligned} \tag{18}$$

The full discrete variational formulation of (2) reads as

$$\left(\delta u_{i,h}, \varphi\right) + \left(\nabla u_{i,h}, \nabla\varphi\right) + \left(\alpha_i u_{i,h}, \varphi\right)_\Gamma + \left(K, u_{i,h}\right)(1,\varphi)_\Gamma = \left(f(t_i, u_{i-1,h}), \varphi\right) - (\beta_i, \varphi)_\Gamma \tag{19}$$

for any $\varphi \in V_h$ and $u_{0,h} = P_h u_0$. The well-posedness of (19) follows immediately from the Lax–Milgram lemma using the coercivity and continuity of the bilinear form

$$a(u,\varphi) = \left(\frac{u}{\tau}, \varphi\right) + (\nabla u, \nabla\varphi) + (\alpha_i u, \varphi)_\Gamma + (K, u)(1,\varphi)_\Gamma$$

for sufficiently small τ, as well as the continuity of the linear functional

$$\langle F, \varphi\rangle = \left(f(t_i, u_{i-1,h}), \varphi\right) - (\beta_i, \varphi)_\Gamma .$$

For practical computations we will of course use the discrete auxiliary problems (3) and (4) to avoid the problems with special solvers for algebraic systems. The equivalent of Lemmas 3.4 and 3.5 reads as

Lemma 4.1. *Let the assumptions of Lemma 3.5 be fulfilled. Then there exists $C > 0$ such that*

(i)

$$\left\|u_{j,h}\right\|^2 + \sum_{i=1}^{j} \left\|u_{i,h} - u_{i-1,h}\right\|^2 + \sum_{i=1}^{j} \left\|\nabla u_{i,h}\right\|^2 \tau + \sum_{i=1}^{j} \int_\Gamma \alpha_i {u_{i,h}}^2 \tau \le C$$

(ii)

$$\sum_{i=1}^{j} \left\|\delta u_{i,h}\right\|^2 \tau + \sum_{i=1}^{j} \left\|\nabla u_{i,h} - \nabla u_{i-1,h}\right\|^2 + \left\|\nabla u_{j,h}\right\|^2 + \int_\Gamma \alpha_j {u_{j,h}}^2 \le C$$

holds uniformly for all $j = 1, \ldots, n$ and $h > 0$.

The proof follows exactly the same line as in Lemmas 3.4 and 3.5 and therefore we omit it.
Using the compatibility condition one can prove better stability results, namely:

Lemma 4.2. *Let the assumptions of Lemma 3.5 be fulfilled. Moreover assume (16). Then there exists $C > 0$ such that*

$$\left\|\delta u_{j,h}\right\|^2 + \sum_{i=1}^{j} \left\|\delta u_{i,h} - \delta u_{i-1,h}\right\|^2 + \sum_{i=1}^{j} \left\|\nabla\delta u_{i,h}\right\|^2 \tau + \sum_{i=1}^{j} \int_\Gamma \alpha_i \delta {u_{i,h}}^2 \tau \le C$$

holds uniformly for all $j = 1, \ldots, n$ and $h > 0$.

Proof. The proof follows the standard strategy. We point out the main idea, only. Subtract (19) for $i = i - 1$ from (19) for $i = i$. Set $\varphi = \delta u_{i,h}\tau$ and sum the result for $i = 1, \ldots, j$. The compatibility condition is used instead of (19) for $i = 0$. The rest is a usual combination of the energy estimates technique already presented for the time-discretization. $\square$

The full discrete Rothe functions are defined as follows for $\sigma = (\tau, h)$

$$\begin{aligned} &u_\sigma(0) = P_h u_0 \\ &u_\sigma(t) = u_{i-1,h} + (t - t_{i-1})\delta u_{i,h} \quad \text{for } t \in (t_{i-1}, t_i], \end{aligned}$$

and the step function $\overline{u}_\sigma$

$$\overline{u}_\sigma(0) = P_h u_0, \qquad \overline{u}_\sigma(t) = u_{i,h}, \quad \text{for } t \in (t_{i-1}, t_i].$$

We rewrite (19) into the following form

$$(\partial_t u_\sigma, \varphi) + (\nabla\overline{u}_\sigma, \nabla\varphi) + (\overline{\alpha}_n \overline{u}_\sigma, \varphi)_\Gamma + (K, \overline{u}_\sigma)(1,\varphi)_\Gamma = \left(\overline{f}_n(t, \overline{u}_\sigma(t-\tau)), \varphi\right) - \left(\overline{\beta}_n, \varphi\right)_\Gamma \tag{20}$$

for any $\varphi \in V_h$.

Theorem 4.1. *Let the assumptions of Lemma 4.2 be satisfied. Then*

$$\max_{t\in[0,T]} \|u(t)-u_\sigma(t)\|^2 + \int_0^T \|\nabla u(t) - \nabla u_\sigma(t)\|^2 \le C\left(\tau^2 + \int_0^T \|\nabla(u-P_hu)\|^2 + \max_{t\in[0,T]}\|u(t)-P_hu(t)\|^2 + \int_0^T \|\partial_t(u-P_hu)\|^2\right).$$

Proof. Subtract (20) from (7), set $\varphi = P_hu - u_\sigma$ and integrate the result in time to get

$$\begin{aligned}
&\int_0^t (\partial_t(u-u_\sigma), u-u_\sigma) + \int_0^t \|\nabla[u-\overline{u}_\sigma]\|^2 + \int_0^t (\alpha(u-u_\sigma), u-u_\sigma)_\Gamma \\
&= -\int_0^t (K, u \pm u_\sigma - \overline{u}_\sigma)(1, P_hu \pm u \pm \overline{u}_\sigma - u_\sigma)_\Gamma \\
&\quad + \int_0^t (\partial_t(u-u_\sigma), u-P_hu) + \int_0^t (\nabla[u-\overline{u}_\sigma], \nabla[u-P_hu+u_\sigma-\overline{u}_\sigma]) \\
&\quad + \int_0^t (\alpha(u-u_\sigma), u-P_hu)_\Gamma + \int_0^t (\alpha(\overline{u}_\sigma - u_\sigma), P_hu \pm u - u_\sigma)_\Gamma + \int_0^t ((\overline{\alpha}_n-\alpha)\overline{u}_\sigma, P_hu \pm u - u_\sigma)_\Gamma \\
&\quad + \int_0^t \left(f(t,u) - \overline{f}_n(t, \overline{u}_\sigma(t-\tau)), P_hu \pm u - u_\sigma\right) - \int_0^t \left(\beta - \overline{\beta}_n, P_hu \pm u \pm \overline{u}_\sigma - u_\sigma\right)_\Gamma .
\end{aligned}$$

The left-hand side equals

$$\frac{1}{2}\left(\|u(t)-u_\sigma(t)\|^2 - \|u_0 - P_hu_0\|^2\right) + \int_0^t \|\nabla[u-\overline{u}_\sigma]\|^2 + \int_0^t (\alpha(u-u_\sigma), u-u_\sigma)_\Gamma .$$

The right-hand side can be estimated in a standard way using the Cauchy, Young and triangle inequalities and the trace theorem. For the second term we also use the integration by parts in time. The following upper bound can be derived

$$\begin{aligned}
&\varepsilon\,\|u(t)-u_\sigma(t)\|^2 + \varepsilon\int_0^t \|\nabla[u-\overline{u}_\sigma]\|^2 \\
&\quad + C_\varepsilon\left(\tau^2 + \int_0^T \|\nabla(u-P_hu)\|^2 + \max_{t\in[0,T]}\|u(t)-P_hu(t)\|^2 + \int_0^T \|\partial_t(u-P_hu)\|^2\right) + C_\varepsilon \int_0^t \|u-u_\sigma\|^2 .
\end{aligned}$$

Hence, choosing a sufficiently small ε we get

$$\begin{aligned}
&\|u(t)-u_\sigma(t)\|^2 + \int_0^t \|\nabla[u-\overline{u}_\sigma]\|^2 + \int_0^t (\alpha(u-u_\sigma), u-u_\sigma)_\Gamma \\
&\le C\int_0^t \|u-u_\sigma\|^2 + C\left(\tau^2 + \int_0^T \|\nabla(u-P_hu)\|^2 + \max_{t\in[0,T]}\|u(t)-P_hu(t)\|^2 + \int_0^T \|\partial_t(u-P_hu)\|^2\right).
\end{aligned}$$

Finally, an application of Gronwall's lemma concludes the proof of the theorem. □

If we assume the following approximation property of the subspaces V_h

$$\exists C > 0\ \forall u \in H^1(\Omega)\ \forall h: \quad \|u - P_hu\| \le Ch\,\|u\|_{H^1(\Omega)} \tag{21}$$

which is fulfilled for standard finite elements, then we can write:

Theorem 4.2. *Let the assumptions of Lemma 4.2 be satisfied. Suppose (21) and $u \in L_2\left((0,T), H^2(\Omega)\right)$. Then*

$$\max_{t\in[0,T]} \|u(t)-u_\sigma(t)\|^2 + \int_0^T \|\nabla u(t) - \nabla u_\sigma(t)\|^2 \le C\left(\tau^2 + h^2\right).$$

For smooth solutions we have

$$\max_{t\in[0,T]} \|u(t)-u_\sigma(t)\|^2 + \int_0^T \|\nabla u(t) - \nabla u_\sigma(t)\|^2 \le C\left(\tau^2 + h^4\right).$$

Proof. The proof can be obtained from Theorem 4.1 using (21), (17) and the regularity of solution. □

Table 1
Numerical solution to (22) with $K = \frac{1}{2}$; $h = \tau = 0.05$ (u—computed solution, u^*—analytical solution, $u^{\clubsuit}$—computed solution by [13]).

x_i	0.0	0.2	0.4	0.6	0.8	1.0
$u_{1,i}$	0.0	0.38333319	0.68370396	0.89962562	1.0297583	1.0733843
$u^*_{1,i}$	0.0	0.38377119	0.68521548	0.90223377	1.0331602	1.0769252
$u^{\clubsuit}_{1,i}$	0.0	0.38431	0.68660	0.90449	1.03595	1.07767
$u_{5,i}$	0.0	0.36718538	0.65321493	0.85779239	0.98071492	1.0220181
$u^*_{5,1}$	0.0	0.36832386	0.65583292	0.86179215	0.98561821	1.0269366
$u^{\clubsuit}_{5,i}$	0.0	0.36988	0.66025	0.86951	0.99496	1.02951
$u_{10,i}$	0.0	0.36181833	0.64325065	0.84434079	0.96506368	1.0054288
$u^*_{10,i}$	0.0	0.36224216	0.64426485	0.84587006	0.96690067	1.0072558
$u^{\clubsuit}_{10,i}$	0.0	0.36353	0.64925	0.85545	0.97855	1.01032

Table 2
Numerical solution to (22) with $K = \frac{1}{2}$; $h = 0.001$, $\tau = 0.0001$ (u—computed solution, u^*—analytical solution).

x_i	0.0	0.2	0.4	0.6	0.8	1.0
$u_{1,i}$	0.0	0.39088566	0.69874787	0.92085944	1.0550559	1.0999477
$u^*_{1,i}$	0.0	0.39088549	0.69874770	0.92085927	1.0550558	1.0999475
$u_{2500,i}$	0.0	0.36832133	0.65582754	0.86178414	0.98560848	1.0269269
$u^*_{2500,1}$	0.0	0.36832386	0.65583292	0.86179215	0.98561821	1.0269366
$u_{5000,i}$	0.0	0.36224081	0.64426214	0.84586623	0.96689619	1.0072514
$u^*_{5000,i}$	0.0	0.36224216	0.64426485	0.84587006	0.96690067	1.0072558

5. Numerical experiment

To test the above algorithm we have used the following example from [13] for $x \in (0, 1)$ and $t \in (0, T)$

$$\begin{aligned} &u_t - D u_{xx} + au = u(1-u) + q \\ &u(t,0) = 0 \\ &u_x(t,1) + u(t,1) = \int_0^1 K(x)\, u(t,x)\, \mathrm{d}x + \beta(t) \\ &u(0,x) = u_0(x), \end{aligned} \tag{22}$$

where D and a are positive constants; q, K and β are some functions to be chosen. Let

$$u^*(t,x) = x(2-x) + 0.1\mathrm{e}^{-\alpha t} \sin\left(\frac{\pi x}{2}\right) \quad \text{with } \alpha = 5 + \frac{\pi^2}{40}$$

be the solution to (22). We set $D = 0.1$, $a = 5$ and

$$\begin{aligned} &q(t,x) = 0.2 + 5x(2-x) - u^*(1-u^*) \\ &\beta(t) = u^*_x(1,t) + u^*(1,t) - \int_0^1 K(x)\, u^*(t,x)\, \mathrm{d}x \\ &u_0(x) = x(2-x) + 0.1 \sin\left(\frac{\pi x}{2}\right). \end{aligned}$$

The function K remains the only free parameter to be chosen. We have used either $K = \frac{1}{2}$ or $K(x) = \mathrm{e}^{-x}$ in our computations. The results for $u_{i,j} = u(t_i, x_j)$ with $h = \tau = 0.05$ – using the Rothe-finite element method developed in this article – are shown in Tables 1 and 3 together with the results from [13]. Computations for smaller discretization parameters h, τ are shown in Tables 2 and 4.

We have applied the Simpson rule for the quadrature in calculations. The convergence rates in τ and h for the maximum $L_2(\Omega)$-error

$$err = \max_{t \in [0,T]} \left\| u^*(t) - u(t) \right\|$$

are depicted in Fig. 1.

We see that the rate of convergence is almost $\mathcal{O}(\tau)$ and $\mathcal{O}\left(h^2\right)$, which corresponds to Theorem 4.2.

Conclusions

We have studied a semilinear problem with a nonlocal BC with applications in thermoelasticity. The nonlocal term in the BC was dependent on a weighted average of solution in the whole domain Ω. We have designed a very easy implementable algorithm for computations, which was based on suitable linearization and an on the superposition principle. We have derived the stability results and we have got the optimal convergence rates, which have been confirmed by numerical

Table 3
Numerical solution to (22) with $K(x) = e^{-x}$; $h = \tau = 0.05$ (u—computed solution, u^*—analytical solution, $u^{\clubsuit}$—computed solution by [13]).

x_i	0.0	0.2	0.4	0.6	0.8	1.0
$u_{1,i}$	0.0	0.38333319	0.68370396	0.89962560	1.0297580	1.0733792
$u^*_{1,i}$	0.0	0.38377119	0.68521548	0.90223377	1.0331602	1.0769252
$u^{\clubsuit}_{1,i}$	0.0	0.38431	0.68660	0.90448	1.03594	1.07772
$u_{5,i}$	0.0	0.36718537	0.65321488	0.85779203	0.98071232	1.0220022
$u^*_{5,1}$	0.0	0.36832386	0.65583292	0.86179215	0.98561821	1.0269366
$u^{\clubsuit}_{5,i}$	0.0	0.36983	0.66020	0.86950	0.99499	1.02969
$u_{10,i}$	0.0	0.36181830	0.64325052	0.84434018	0.96506088	1.0054172
$u^*_{10,i}$	0.0	0.36224216	0.64426485	0.84587006	0.96690067	1.0072558
$u^{\clubsuit}_{10,i}$	0.0	0.36352	0.64925	0.85549	0.97864	1.01051

Table 4
Numerical solution to (22) with $K(x) = e^{-x}$; $h = 0.001$, $\tau = 0.0001$ (u—computed solution, u^*—analytical solution).

x_i	0.0	0.2	0.4	0.6	0.8	1.0
$u_{1,i}$	0.0	0.39088566	0.69874787	0.92085944	1.0550559	1.0999477
$u^*_{1,i}$	0.0	0.39088549	0.69874770	0.92085927	1.0550558	1.0999475
$u_{2500,i}$	0.0	0.36832133	0.65582754	0.86178414	0.98560847	1.0269268
$u^*_{2500,1}$	0.0	0.36832386	0.65583292	0.86179215	0.98561821	1.0269366
$u_{5000,i}$	0.0	0.36224081	0.64426214	0.84586623	0.96689618	1.0072514
$u^*_{5000,i}$	0.0	0.36224216	0.64426485	0.84587006	0.96690067	1.0072558

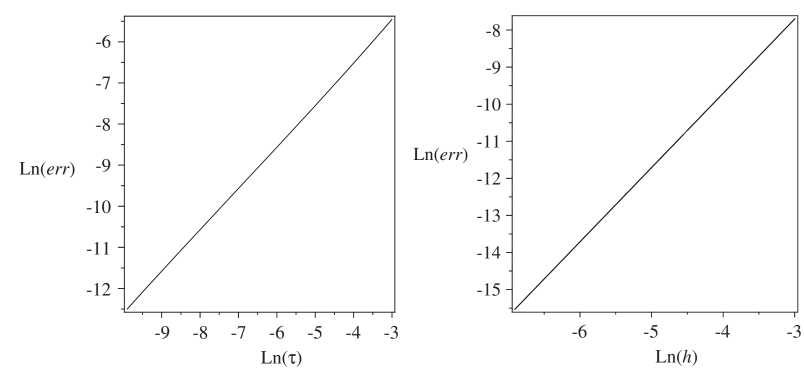

Fig. 1. Relation between $L_2(\Omega)$-error and the discretization parameters τ and h in logarithmic scales.

experiment. Our results in the test example are comparable with those reported by [13]. The advantage of our approach is that we do not need the monotone iterative procedure (which is the core of [13]). Moreover, our approach is valid for $\Omega \subset \mathbb{R}^d$, $d \geq 1$, which was not the case in [13].

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References

[1] W.A. Day, Extensions of a property of the heat equation to linear thermoelasticity and other theories, Quart. Appl. Math. 40 (1982) 319–330.
[2] W.A. Day, A decreasing property of solutions of parabolic equations with applications to thermoelasticity, Quart. Appl. Math. 41 (1983) 468–475.
[3] A. Friedman, Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions, Quart. Appl. Math. 44 (1986) 401–407.
[4] B. Kawohl, Remarks on a paper by w. a. day on a maximum principle under nonlocal boundary conditions, Quart. Appl. Math. 44 (1987) 751–752.
[5] M. Sapagovas, Hypothesis on the solvability of parabolic equations with nonlocal conditions, Nonlinear Anal. Model. Control 7 (1) (2002) 93–104.
[6] G. Ekolin, Finite difference methods for a nonlocal boundary value problem for the heat equation, BIT 31 (2) (1991) 245–261.
[7] Y. Liu, Numerical solution of the heat equation with nonlocal boundary conditions, J. Comput. Appl. Math. 110 (1) (1999) 115–127.
[8] Y. Lin, S. Xu, H-M. Yin, Finite difference approximations for a class of nonlocal parabolic equations, Int. J. Math. Math. Sci. 20 (1) (1997) 147–163.
[9] Z.-Z. Sun, A high-order difference scheme for a nonlocal boundary-value problem for the heat equation, Comput. Methods Appl. Math. 1 (4) (2001) 398–414.
[10] N. Borovykh, Stability in the numerical solution of the heat equation with nonlocal boundary conditions, Appl. Numer. Math. 42 (1–3) (2002) 17–27.
[11] A.A. Amosov, Global solvability of a nonlinear nonstationary problem with a nonlocal boundary condition of radiative heat transfer type, Differ. Equ. 41 (1) (2005) 96–109.
[12] S. Carl, V. Lakshmikantham, Generalized quasilinearization method for reaction-diffusion equations under nonlinear and nonlocal flux conditions, J. Math. Anal. Appl. 271 (1) (2002) 182–205.

[13] C.V. Pao, Numerical solutions of reaction-diffusion equations with nonlocal boundary conditions, J. Comput. Appl. Math. 136 (1–2) (2001) 227–243.
[14] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, in: Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer, 1977.
[15] L.C. Evans, Partial Differential Equations, in: Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 1998.
[16] J. Nečas, Les méthodes Directes en Théorie des Équations Elliptiques, Academia, Prague, 1967.
[17] J. Kačur, Method of Rothe in Evolution Equations, in: Teubner Texte zur Mathematik, vol. 80, Teubner, Leipzig, 1985.