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Comportement asymptotique de quelques problèmes dépendants d'un paramètre

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This work is dedicated to my father

Si El Laifa,

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Abstract

In this thesis we study the asymptotic behaviour for the solution of a linear and nonlinear wave equation in a non-cylindrical domain, becoming unbounded in some direction as the time goes to infinity. If the limit of the source term is independent of this direction and t, the wave converges to the solution of an elliptic problem defined on a lower dimensional domain. The rate of convergence depends on the limit behaviour of the source term and on the coefficient of the nonlinear term.

Key words and phrases

Wave equation, Asymptotic behaviour in time, non-cylindrical domains.

Résumé

Dans cette thése, nous étudions le comportement asymptotique de la solution d'une équation d'onde (linéaire et non linéaire) dans un domaine non-cylindrique devenant non bornée, dans une certaine direction lorsque le temps t tend vers l'infini. Si la limite du terme source est indépendante de cette direction et t, la solution de l'équation des ondes converge vers la solution d'un problème elliptique défini sur un domaine de dimension inférieure. Le taux de convergence dépend de la limite du terme source et du coefficient du terme non linéaire.

Mots clés

Equation d'onde, Comportement asymptotique dans le temps, Domaines non cylindrique

Contents

Introduction

Generally the determination of approximate solutions for partial differential equations uses the resolution of simpler models, either by a general technique such as in the case of the finite element method, or by taking advantage of some particularities of our problem to consider another problem simpler.

Our work is motivated by some recent works on the asymptotic behaviour of the solutions of boundary value problems of different types in a cylindrical domain Ω_ℓ, when the size of Ω_ℓ becomes unbounded, in some directions, as the parameter $\ell \to \infty$ (independently of the time), see for instance Chipot [6], Chipot and Mardare [10], Guesmia [19] for elliptic and parabolic and Stokes problems and [5], [18] for hyperbolic problems.

In recent years, there is much interest in evolution problems set in time-dependent domains. These problems arise in many real world applications when the spatial domain of the considered phenomena depends strongly on time, see for instance the survey paper [24] and the references cited therein.

In the present work, we study the asymptotic behaviour for the solution of linear and nonlinear wave equation in a noncylindrical domain, becoming unbounded, in some direction as the time t goes to infinity. If the limit of the source term is independent of this directions and the time t, the wave converges to the solution of an elliptic problem defined on a lower dimensional domain. The rate of convergence depends on the limit behaviour of the source term and on the coefficient of the nonlinear term. We consider a time-dependent family of bounded subset in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ defined as

$$\Omega_t := (-\ell_0 - \ell t; \ell_0 + \ell t)^{n_1} \times \omega, \quad t \geq 0,$$

where n_1, n_2 are positive integers, ω is a bounded open subset of $\mathbb{R}^{n_2}$ with sufficiently smooth boundary, $\ell_0 > 0$ and the speed of expansion ℓ is constant and at hand, we give to ℓt the same role of the parameter ℓ in the previous works. Let us denote the points in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ as

$$x = (X_1, X_2) = \left(x_1, ..., x_{n_1}, x_1', ..., x_{n_2}'\right),$$

We also consider the following noncylindrical domain, and its lateral boundary

$$Q_t := \bigcup_{0<s<t} \{s\} \times \Omega_s, \qquad \Sigma_t := \bigcup_{0<s<t} \{s\} \times \partial\Omega_s.$$

More specifically, we are interested in the asymptotic behaviour, as $t \to \infty$, of the solution of the wave equation in two cases:

-The first case we study the following linear wave equation with a dissipation term

$$\begin{cases} u''(t,x) - \triangle u(t,x) + \beta u'(t,x) = f(t,x) & \text{in } Q_t, \\ u(t,x) = 0 & \text{on } \Sigma_t, \\ u(0,x) = u^0(x), \qquad u'(0,x) = u^1(x) & \text{in } \Omega_0, \end{cases} \tag{1}$$

where the primes stand for the time derivative, $\triangle$ is the Laplace operator and $\beta > 0$ is a positive constant.

-The second case, we study the following nonlinear wave equation

$$\begin{cases} u''(t,x) + \beta u'(t,x) - \triangle u(t,x) + \gamma(t) |u|^{\rho} u = f(t,x) & \text{in } Q_t, \\ u(t,x) = 0 & \text{on } \Sigma_t, \\ u(0,x) = u^0(x), \qquad u'(0,x) = u^1(x) & \text{in } \Omega_0, \end{cases} \tag{2}$$

where f, γ are functions which satisfy some appropriate conditions.

The existence of solutions for linear and nonlinear wave problems in noncylindrical domains has been studied by several authors. Lions [26] introduced the so-called penalty method to show the existence of weak solutions in expanding domains, see also Mederios [29]. Under some time likeness conditions on Q_t, Cooper and Bardos [12], Cooper and Medeiros [13] obtained the existence and uniqueness of weak solutions. Clark [11], Ferreira [16], Ferriera and Lar'kin [17] also considered problems with variable coefficients.

Several works dealt with the asymptotic behaviour in time for the solutions of evolution problems for different types of nonlinearity and boundary conditions in noncylindrical domains.

Using the multiplier method, Bardos and Chen [3] proved that the energy of the linear wave equation decays when the domain is timelike and expanding. Nakao and Narazaki [31] and Rabello [32] studied the decay of the energy for weak solutions of nonlinear wave problems in expanding domains. There idea relays on the penalization method, introduced by Lions [26]. Another method consists in considering a suitable change of variables that transforms the noncylindrical domain to a cylindrical one, establish energy estimates for the new problem, then derive the desired energy estimates for the noncylindrical problem, see for instance [22], [25]. The drawback of this method is that the differential operator of the transformed problem is, in general, more complicated.

We have to mention that in this thesis we study the problem directly in the noncylindrical domains, without any change of variables. The idea is based on the use of some special cut-off functions, depending on $(t; X_1)$, to obtain local estimates of the difference between the wave and its limit, this techniques introduced by Guesmia [20] for a parabolic problem in noncylindrical domain, see also Chipot and Yeressian [9], Roughly speaking, if $f(t; x)$ converges to some $f_\infty(X_2)$ and $\gamma(t)$ converges to 0 (in the nonlinear case), faster enough in a sense to be made precise later, we obtain a convergence $u(t) \to u_\infty$ in interior regions of the domain Q_t. Here u_∞ is the solution of an elliptic problem defined on ω, then the rate of convergence $u(t) \to u_\infty$ is analyzed and improved under some assumptions.

The main features of this work can be summarized as follows:

-In [22], [31], and [32] the size of the domain remains bounded as $t \to \infty$, and the limit of the solution of the considered problem is zero. This situation arises when the decay in the energy of the solution, due to the expansion of the domain and damping terms, overtakes the contribution of the source term. In this work, Ω_t becomes unbounded in n_1 directions and the limit of the solution, in interior regions of the domain, is not necessarily zero, as t goes to infinity. To the best of our knowledge, the asymptotic behaviour of such problems has not been considered before.

-In contrast with [20], the source term f in this work depends on all the variables $(t; x) \in \mathbb{R}^+ \times (-\ell_0 - \ell t; \ell_0 + \ell t)^{n_1} \times \omega$ and not only on $X_2 \in \omega$.

The rest of this thesis is divided into three chapters:

In the first chapter, we present some elementary notions of functional analysis freet, used in the sequel, then we state the basic results about the existence and uniqueness of $u(t)$.

The second chapter is titled *Asymptotic Behaviour of Linear Wave Equation,* is reserved to show some local energy estimates for the difference $u\,(t) - u_\infty$ and we give some convergence results, and discuss some particular cases where the rate of convergences is exponential. Afterwards, we define of u_∞, the candidate limit $u(t)$ as $t \to +\infty$, and the cut-off functions.

The last chapter, titled *Asymptotic Behaviour of Non Linear Wave Equation,* in this chapter, we define u_∞, the candidate limit of $u(t)$ as $t \to +\infty$, and the cut-off functions needed in the sequel, then we give an energy estimate for $u(t)$ as well as a local energy estimate for the difference $u(t) - u_\infty$. At the convergence results and discuss some particular cases where the rate of convergence increase exponentially.

for the convenience of the reader, we will agree some standard notations commonly used during this work, see for instance [4], [7]. However, to make this thesis consistent and to avoid ambiguity, a list of notations for estimates and definitions is included the appendix, as well as some elementary inequalities frequently used in through this work.

Notations

General notations

$\mathbb{R}^+$	:	set of all positive real numbers.
$\mathbb{R}^n$	:	n-dimensional real Euclidean space, $\mathbb{R}=\mathbb{R}^1$.
$\mathbb{R}^{n+1}$	:	denote the point of $\mathbb{R}^{n+1}$ as $(x_1,X_2)=(x_1,x_2,...,x_n)$.
Ω	:	open domain in $\mathbb{R}^n$.
$\partial\Omega$	:	is the Γ boundary of Ω.
Ω_ℓ	:	denote the rectangle $(-\ell,\ell)\times\omega,$, where ω is a bounded subset in $\mathbb{R}^n$.
Ω_t	:	$\{(-\ell_0-\ell t,\ell_0+\ell t)\times\omega, \quad t\geq 0\}$
$\Omega_t^{n_1\times n_2}$	:	$\{(-\ell_0-\ell t,\ell_0+\ell t)^{n_1}\times\omega, \quad t\geq 0\}$.
$S\subset\subset\Omega$	:	S strongly included in Ω, i.e the closure of S $\left(\overline{S}\subset\Omega\right)$.
S_m^t	:	$\{(s,x_1);\ t-m<s<t,\ -t_0-\sigma(m-t+s)<x_1<t_0+\sigma(m-t+s)\}$
$:=$	:	equal by definition.
x	:	$(X_1,X_2)\in\mathbb{R}^n$ where $X_1=(x_1,...,x_q)\in\mathbb{R}^q, \quad X_2=(x_{q+1},...,x_n)\in\mathbb{R}^{n-q}$.

Functions and differential operators

For a function $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$, we note

$u(x)$	:	$u(x_1,x_2,...,x_n), \quad x\in\Omega$.
$u(t,x)$	:	$u(t,x_1,x_2,...,x_n), \quad$ for $t\geq 0,\ x\in\Omega$.
∇u	:	$(\partial_{x_1}u,\partial_{x_2}u,...,\partial_{x_n}u)^T$ gradient of u.
$\triangle u$	:	$\partial_{x_1}^2u+\partial_{x_2}^2u+...+\partial_{x_n}^2u$ Laplacian of u.
$u'(t,x)$	:	$\dfrac{\partial u}{\partial t}$, similarly $u''(t,x)=\dfrac{\partial^2u}{\partial t^2}$.
$\nabla_{x,t}u$	:	$\left(\dfrac{\partial u}{\partial t},\partial_{x_1}u,\partial_{x_2}u,...,\partial_{x_n}u\right)^T$, for $t\geq 0,\ x\in\Omega$.
$f(t,x)$	:	the source term.
$\varrho(t,x)$	:	special cut-off functions.
χ_{ij}	:	the characteristic function denoted by $\chi_{ij}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$

Function spaces

For a function $u : \Omega \to \mathbb{R}$, we note

$\mathcal{C}^k(\Omega)$	:	u is k times continuously differentiable, $k \in \mathbb{N}$.
$\mathcal{D}(\Omega)$	:	u is infinitely differentiable with compact support in Ω.
$\mathcal{D}'(\Omega)$	:	space of distributions on Ω.
$L^p(\Omega)$	:	u is measurable and $\int_\Omega \lvert u(x)\rvert^p \, dx < \infty, \quad 1 \leq p < \infty$.
$L^\infty(\Omega)$	:	u is measurable and $\lvert u(x)\rvert \leq C$ a.e. in Ω for some constant C.
$H^k(\Omega)$	:	$u \in L^2(\Omega) \setminus D^\alpha u \in L^2(\Omega), \quad \forall \alpha \in \mathbb{N}, \ \lvert\alpha\rvert \leq k$ where $K \in \mathbb{N}$.
$H_0^k(\Omega)$	:	the closure of $\mathcal{D}(\Omega)$ in $H^k(\Omega)$.
$W^{k,p}(\Omega)$	:	$u \in L^p(\Omega) \setminus D^\alpha u \in L^p(\Omega), \quad \forall \alpha \in \mathbb{N}^n, \ \lvert\alpha\rvert \leq k$ where $K \in \mathbb{N}$, $1 \leq p \leq \infty$.
$\mathcal{C}^k([a,b]\,;X)$	:	space of k times continuously differentiable functions from an interval $[a,b]$ to the Banach space X.
$L^p([a,b]\,;X)$	:	space of measurable functions u on $[a,b]$ with values in X and such that $\lvert u\rvert^p$ is integrable, $1 \leq p < \infty$.
$L^\infty([a,b]\,;X)$	:	space of measurable functions u on $[a,b]$ such that there exists a constant C such that $\lvert u\rvert_X^p \leq C$ for almost every $x \in [a,b]$.

Chapter 1

Preliminaries

We expose in this chapter some functional analysis tools that we will need in the sequel of the thesis. In particular, we will recall the main results that deal with existence and uniqueness of the solution of the linear and nonlinear wave equations.
Let us denote by X a Banach space (i.e., X is a complete normed vector space equipped with the norms are given by $\|.\|_X$) which will specified later.

1.1 Basic notions

The space $L^p(a,b;X)$

Definition 1 *For $a,\ b\in\mathbb{R}$ we denote by $L^p(a,b;X), 1\leq p<+\infty$ the space of (class of functions) $f:(a,b)\to X$ that are measurable and such that*

$$\int_a^b \|f(t)\|_X^p\,dt<+\infty.$$

Definition 2 ([6]) *We denote by $L^\infty(a,b;X)$ the space of functions essentially bounded on (a,b), i.e., such that there exists $M>0$ such that*

$$\|f(t)\|_X\leqslant M \qquad a.e. \qquad t\in(a,b).$$

Then it is easy to show

Theorem 3 ([6]) *Equipped with the norms*

$$\begin{aligned}\|f\|_{L^p(a,b;X)} &= \left(\int_a^b \|f(t)\|_X^p\,dt\right)^{\frac{1}{p}}, \qquad 1\leq p<+\infty,\\ \|f\|_{L^\infty(a,b;X)} &= \inf\left\{M\in\mathbb{R}_+^*\setminus\ \|f(t)\|_X\leqslant M \qquad a.e. \qquad t\in(a,b)\right\},\end{aligned}$$

the spaces $L^p(a,b;X), 1\leq p<+\infty$, are Banach spaces.

Proof. Let (f_n) be a Cauchy sequence in $L^p(a,b;X)$, it suffices to show that an extracted a converges subsequence in $L^p(a,b;X)$. We extract a subsequence (f_{n_k}) such that

$$\left\|f_{n_{k+1}}-f_{n_k}\right\|_{L^p(a,b;X)}\leq\frac{1}{2^k}\qquad \forall k\geq 1.$$

We will show that (f_{n_k}) converges in $L^p(a,b;X)$. We set

$$g_n(t)=\sum_{k=1}^n\left|f_{n_{k+1}}(t)-f_{n_k}(t)\right|,$$

he comes

$$\|g_n\|_{L^p(a,b;X)}\leq 1.$$

We deduce from the monotonic convergence theorem that

$$g_n(t) \text{ converge to } g(t) \quad a.e. \qquad \text{with } g(t) \in L^p.$$

On the other hand, we have for $m \geq n \geq 2$

$$|f_m(t) - f_n(t)| \leq |f_m(t) - f_{m-1}(t)| + ... + |f_{n+1}(t) - f_n(t)| \leq g(t) - g_{n-1}(t),$$

it follows that $(f_n)(t)$ is from Cauchy and converges towards a limit noted by $f(t)$. We have

$$|f(t) - f_n(t)| \leq g(t) \quad \text{for } n \geq 2, \quad a.e.$$

we deduce that $f(t) \in L^p$. Finally $\|f_n - f\|_{L^p(a,b;X)} \to 0$ ■

The space $\mathcal{D}'(a, b; X)$

We denote by $\mathcal{D}(a, b; X)$ the space of infinitely differentiable functions on (a, b) with compact support contained in X.

Definition 4 ([6]) *A vector valued distribution on (a, b) with values in X is a linear mapping*

$$\begin{aligned} T : \mathcal{D}(a, b) &\to X \\ \varphi &\mapsto \langle T, \varphi \rangle, \end{aligned}$$

which is continuous on $\mathcal{D}(a, b)$ in the sense that

$$\lim_{i \to +\infty} \langle T, \varphi_i \rangle = \langle T, \varphi \rangle \qquad \text{in } X,$$

for any sequence $\varphi_i \to \varphi$ in $\mathcal{D}(a, b)$.

We can now give the definition of the derivative of a distribution.

Definition 5 ([6]) *We will denote by $\mathcal{D}'(a, b; X)$, the space of all distributions on (a, b) with values in X.*

Definition 6 ([6]) *Let $T \in \mathcal{D}'(a,b;X)$, then for every integer k, we define a distribution $T^{(k)}$ by the formula*

$$\left\langle T^{(k)},\varphi\right\rangle = (-1)^k \left\langle T, \frac{d^k\varphi}{dt^k}\right\rangle.$$

$T^{(k)}$ is called the k^{th} derivative of T on (a,b) and will also be denoted by $\dfrac{d^k\varphi}{dt^k} = T^{(k)}$.

Remark 7 *If X, Y are two Banach spaces such that $X \hookrightarrow Y$ is continuously embedding. Then clearly*

$$\mathcal{D}'(a,b;X) \hookrightarrow \mathcal{D}'(a,b;Y)$$

and

$$L^p(a,b;X) \hookrightarrow L^p(a,b;Y) \quad \forall 1 \le p \le +\infty.$$

The space $H^1(a,b;V,V')$

Let H, V are two separable Hilbert spaces, we suppose that V is dense in H ($V \hookrightarrow H$) and we identify H with his dual H'. So we have

$$V \hookrightarrow H \hookrightarrow V',$$

that is, each space being dense in the next with continuous embedding.
We will denote by

$(\cdot,\cdot)$	the scalar product in H or the product of duality V', V,
$\lVert\cdot\rVert_H$	the norm in H,
$((\cdot,\cdot))$, $\lVert\cdot\rVert_V$	respectively the scalar product and the norm in V,
$\lVert\cdot\rVert_{V'}$	the norm in V'.

A suitable choice could be

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega),$$

then we can define

Definition 8 ([6]) *We define the space $H^1(a,b;V,V')$ by*

$$H^1(a,b;V,V') = \left\{u \in L^2(a,b;V),\ u' \in L^2(a,b;V')\right\},$$

where u' is calculated in the sense of the distribution $\mathcal{D}'(a,b;X)$. And we set

$$\|u\|_{H^1}^2 = \|u\|_V^2 + \|u'\|_{V'}^2 . \tag{1.1}$$

$H^1(a,b;V,V')$ is the Hilbert space.

Now we can state

Theorem 9 *Let $u \in H^1(a,b;V,V')$, then u can be identified with a continuous function on $[a,b]$ with values in H. moreover that*

$$H^1(a,b;V,V') \hookrightarrow \mathcal{C}^0([a,b]\,;H),$$

where $\mathcal{C}^0([a,b]\,;H)$ denote the space of continuous functions on $[a,b]$ with values in H equipped with the topology of the uniform convergence on $[a,b]$.

Density

Theorem 10 ([6]) *$\mathcal{D}(a,b;V)$ is dense in $H^1(a,b;V,V')$.*

Extension

Theorem 11 ([6]) *There exists a continuous linear operator of extension that we denote by P from $H^1(a,b;V,V')$ into $H^1(-\infty,+\infty;V,V')$ such that*

$$P(u) = u \qquad a.e \ on \ (a,b) \qquad \forall u \in H^1(a,b;V,V').$$

Proof. [6] We assume that b is finite. If $a = -\infty$ then one sets

$$Pu(t) = \begin{cases} u(t) & t \leq b \\ u(2b-t) & t \geq b. \end{cases}$$

It is easy to show that this solves the problem. If a is finite, then setting

$$u = \theta_1 u + \theta_2 u,$$

where θ_1, θ_2 are two smooth functions such that

$$\theta_1 = 0 \quad \text{on } (-\infty, a], \quad \theta_2 = 0 \quad \text{on } [b, +\infty), \quad \theta_1 + \theta_2 = 1 \quad \text{for } a \leq t \leq b.$$

We have recall (1.1)

$$\begin{aligned}\|\theta_1 u\|_{H^1} &\leq C\|u\|_{H^1}, \\ \|\theta_2 u\|_{H^1} &\leq C\|u\|_{H^1},\end{aligned}$$

and we conclude by using the previous step ■

Integration by parts

Theorem 12 ([6]) *Let $u, v \in H^1(a,b;V,V')$ with is a, b finite. Then*

$$\int_a^b \langle u'(t), v(t)\rangle\, dt + \int_a^b \langle u(t), v'(t)\rangle\, dt = ((u(b), v(b))) - ((u(a), v(a))).$$

Proof. For $u, v \in \mathcal{D}(a,b;V)$, we have

$$\begin{aligned}\int_a^b \langle u'(t), v(t)\rangle\, dt + \int_a^b \langle u(t), v'(t)\rangle\, dt &= u(t)v(t)|_a^b - \int_a^b \langle u(t), v'(t)\rangle\, dt + \int_a^b \langle u(t), v'(t)\rangle\, dt \\ &= u(b)v(b) - u(a)v(a) \\ &= ((u(b), v(b))) - ((u(a), v(a))).\end{aligned}$$

Then the result follows by density ■

Proposition 13 ([6]) *Let $u \in H^1(a,b;V,V')$, and let $v \in V$. We have*

$$\langle u'(.), v\rangle = \frac{d}{dt}((u(.), v)) \qquad in\ \mathcal{D}'(]a,b[).$$

Proof. Let $\varphi \in \mathcal{D}(]a,b[)$. From the above proposition, we derive

$$\int_a^b \langle u'(t), v\varphi(t)\rangle + \langle v\varphi'(t), u(t)\rangle\, dt = ((u(b), v\varphi(b))) - ((u(a), v\varphi(a))) = 0 \qquad \forall v \in V.$$

This can also be written

$$\int_a^b \langle u'(t), v\rangle\, \varphi(t)\, dt = -\int_a^b \langle v, u(t)\rangle\, \varphi'(t)\, dt = -\int_a^b ((u(t), v))\, \varphi'(t)\, dt.$$

Hence the result ■

- For more details on $L^p, \mathcal{D}', H^1$ at vector values see, [6], [27] ...

1.2 Functions spaces on the noncylindrical domain

We precise some functions spaces and their norms are in order on the noncylindrical domain. Let Ω be an open and bounded set of $\mathbb{R}^n$ with smooth boundary Γ, $Q = \Omega \times (0,T)$ an open cylindrical domain. We will use the following notations

$$\Omega_s = Q \cap \{t = s\} \quad \text{for } s > 0, \qquad \Omega_0 = \overline{Q} \cap \{t = 0\},$$

such that

$$\Omega_s \neq \varnothing \text{ for all } s \geq 0, \qquad \Gamma_s = \partial\Omega_s, \qquad \Sigma = \underset{0<s<T}{\cup} \Gamma_s \qquad \partial Q = \Omega_0 \cup \Sigma,$$

where ∂Q the boundary of Q. Our assumptions on Q are

$$\begin{cases} i)\ \Omega_t \text{ is monotone increasing, that is, } \Omega_t^* \subset \Omega_s^* \text{ if } t < s, \\ \text{where } \Omega_s^* \text{ is the projection of } \Omega_t \text{ on the hyperplane } t = 0. \\ ii) \text{ we assume the following regularity property:} \\ \text{For each } t \in \left]0,T\right[, \qquad \text{if } u \in H_0^1(\Omega) \text{ and } u = 0 \text{ in } \Omega \backslash \Omega_t \\ \text{then } u \in H_0^1(\Omega_t). \end{cases} \tag{1.2}$$

In order to simplify the notation we will identify Ω_t^* with Ω_t and we define

The space $L^q(0,T;L^p(\Omega_t))$

Definition 14 ([26]) *The spaces $L^p(\Omega_t)$ (resp. $H_0^1(\Omega_t)$) identify, for all t in $[0,T]$ to closed subspaces of $L^p(\Omega)$ (resp$H_0^1(\Omega)$).*

Definition 15 ([16]) *We define the spaces $L^q(0,T;L^p(\Omega_t))$, $L^q(0,T;H_0^1(\Omega_t))$ as*

$$\begin{aligned} L^q(0,T;L^p(\Omega_t)) &= \{v : \quad v \in L^q(0,T;L^p(\Omega)),\ \textit{such that } v(t) \in L^p(\Omega_t) \ \textit{a.e.} \quad (1 \leq q \leq \infty)\}, \\ L^q\left(0,T;H_0^1(\Omega_t)\right) &= \left\{v : \quad v \in L^q\left(0,T;H_0^1(\Omega)\right),\ \textit{such that } v(t) \in H_0^1(\Omega_t) \ \textit{a.e.} \quad (1 \leq q \leq \infty)\right\}. \end{aligned}$$

Definition 16 ([26]) *i) If $1 \leq q < \infty$, we consider the norms*

$$\|v\|_{L^q(0,T;L^p(\Omega_t))} = \left[\int_0^T \|v(t)\|_{L^p(\Omega_t)}^q \, dt\right]^{\frac{1}{q}}, \qquad \|v\|_{L^q\left(0,T;H_0^1(\Omega_t)\right)} = \left[\int_0^T \|v(t)\|_{H_0^1(\Omega_t)}^q \, dt\right]^{\frac{1}{q}}.$$

ii) If $q = \infty$ we consider the norms

$$\|v\|_{L^\infty(0,T;L^p(\Omega_t))} = \underset{0<t<T}{ess\ sup} \|v(t)\|_{L^p(\Omega_t)}, \qquad \|v\|_{L^\infty\left(0,T;H_0^1(\Omega_t)\right)} = \underset{0<t<T}{ess\ sup} \|v(t)\|_{H_0^1(\Omega_t)}.$$

Definition 17 ([26]) *The spaces* $L^q(0,T;L^p(\Omega_t))$ $(resp.\ L^q(0,T;H_0^1(\Omega_t)))$ *identify, identify, for all* t *in* $[0,T]$ *to the closed subspaces of* $L^q(0,T;L^p(\Omega))$ $(resp. L^q(0,T;H_0^1(\Omega)))$.

Timelikness condition

Let $\nu = (\nu_1, \nu_2, ..., \nu_{n+1}, \nu_t) = (\nu_x, \nu_t)$ be the unit outward normal at (t,x) on Σ_t, Throughout we assume that

$$\text{(timelikness condition of } \Sigma_t) \qquad |\nu_t| < |\nu_x| \quad \text{on } \Sigma_t \text{ for any } t > 0, \tag{1.3}$$

where $|.|$ denotes the usual Euclidean norm. Since the regular solution u satisfy $u = 0$ on Σ_t, then all the tangential derivatives of u are also vanishing on Σ_t, so

$$\nabla_{x,t} u = \frac{\partial u}{\partial \nu}.\nu \quad \text{on } \Sigma_t,$$

which implies that

$$u' = \frac{\partial u}{\partial \nu}\nu_t, \qquad \nabla u = \frac{\partial u}{\partial \nu}\nu_x \qquad \text{on } \Sigma_t.$$

Chapter 2

Asymptotic behaviour of linear wave equations

In this chapter, we study the asymptotic behaviour in time for the solution of a damped wave equation in a noncylindrical domain becoming unbounded, in some direction as $t \to \infty$. If the limit of the source term is independent of this direction and t, the wave converges to the solution of an elliptic problem defined on a lower dimensional domain. The rate of convergence depends on the limit behaviour of the source term.

2.1 Problem setting

We consider, for an evolution problem, a bounded subset in $\mathbb{R}^{n+1}$ defined as:

$$\Omega_t := (-\ell_0 - \ell t, \ell_0 + \ell t) \times \omega, \quad t \geq 0, \tag{2.1}$$

where n is a positive integer, ω is a bounded open subset of $\mathbb{R}^n$ with sufficiently smooth boundary, $\ell_0 > 0$ and ℓ are positive constants. We also consider the following noncylindrical domain, and its lateral boundary

$$Q_t := \bigcup_{0<s<t} \{s\} \times \Omega_s, \quad \Sigma_t := \bigcup_{0<s<t} \{s\} \times \partial\Omega_s.$$

We denote the points in Q_t as $(t,x) = (t, x_1, X_2)$, and we set

$$X_2 = (x_1', ..., x_n') \in \mathbb{R}^n, \quad \nabla u = \begin{pmatrix} \partial_{x_1} u \\ \nabla_{X_2} u \end{pmatrix}, \quad \nabla_{x,t} u = \begin{pmatrix} u' \\ \nabla u \end{pmatrix}.$$

We are going to consider the asymptotic behavior, as $t \to +\infty$, of the solution of the following wave equation, with a dissipation term

$$u'' - \Delta u + \beta u' = f \quad \text{in } Q_t, \tag{2.2}$$

with a homogeneous boundary condition

$$u(t,x) = 0 \quad \text{on } \Sigma_t, \tag{2.3}$$

and two initial ones

$$u(0,x) = u^0(x), \quad u'(0,x) = u^1(x). \tag{2.4}$$

where the primes stand for the time derivatives, Δ is the Laplacian operator and $\beta > 0$ is a positive constant.

2.2 Existence and uniqueness of a weak solution

We consider the following problem defined on the noncylindrical domain Q_t

$$\begin{cases} u''(t,x) + \beta u'(t,x) - \triangle u(t,x) = f(t,x) & \text{in } Q_t \\ u(t,x) = 0 & \text{on } \Sigma_t \\ u(0,.) = u^0(x), \qquad u'(0,.) = u^1(x) & \text{in } \Omega_0, \end{cases} \tag{2.5}$$

where the initial data and the source term satisfy

$$u^0 \in H_0^1(\Omega_0), \quad u^1 \in L^2(\Omega_0), \quad f \in L^2\left(0,T; L^2\left(\Omega_s\right)\right),$$

Now we are in a position to state our existence and uniqueness result.

Existence

The idea is to transform the noncylinder problem in the cylinder problem, through the penalization function $M \in L^\infty\left(\Omega \times \left]0,T\right[\right)$ that was introduced by Lions [26].

Transformation of the problem

Let the domain $\mathcal{S} = \Omega \times (0,T)$ as those introduced in [[26] Chap 3 page 414] where

$$Q_t \subset \mathcal{S} = \Omega \times (0,T),$$

where Ω_t, Ω are verifying the assumptions (1.2). In $\mathcal{S}$ we consider the penalized problem

$$\begin{cases} u_\varepsilon'' + \beta u_\varepsilon' - \triangle u_\varepsilon + \frac{1}{\varepsilon} M u_\varepsilon' = \widetilde{f} & \varepsilon > 0, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times \left]0,T\right[, \\ u_\varepsilon(0,x) = \widetilde{u}_0, \qquad u_\varepsilon'(0,x) = \widetilde{u}_1 & x \in \Omega, \end{cases} \tag{2.6}$$

where $M \in L^\infty\left(\Omega \times \left]0,T\right[\right)$, the penalty function defined by

$$M(t,x) = \begin{cases} 1 & \text{in } \Omega \times \left]0,T\right[- Q_t, \\ 0 & \text{in } Q_t, \end{cases}$$

and $\widetilde{f}$, $\widetilde{u}_0$, $\widetilde{u}_1$ be the extensions by zero outside of Q, Ω_0 of f, u_0 and u_1 respectively. i.e.,

$$\widetilde{f} = \begin{cases} f & \text{in } Q_t, \\ 0 & \text{in } \mathcal{S} - Q_t \end{cases}, \quad \widetilde{u}_0 = \begin{cases} u_0 & \text{in } \Omega_0, \\ 0 & \text{in } \Omega - \Omega_0 \end{cases}, \quad \widetilde{u}_1 = \begin{cases} u_1 & \text{in } \Omega_0, \\ 0 & \text{in } \Omega - \Omega_0 \end{cases}$$

Accordingly, we first prove the following lemma

Lemma 18 *For each $\varepsilon > 0$ there exists a function u_ε defined in the cylinder $\mathcal{S} = \Omega \times \left]0,T\right[$ such that*

$$u_\varepsilon \in L^\infty\left(0,T; H_0^1\left(\Omega\right)\right), \tag{2.7}$$

$$u_\varepsilon' \in L^\infty\left(0,T;L^2(\Omega)\right), \tag{2.8}$$

$$u_\varepsilon'' \in L^2\left(0,T;H^{-1}(\Omega)\right), \tag{2.9}$$

$$u_\varepsilon'' + \beta u_\varepsilon' - \triangle u_\varepsilon + \frac{1}{\varepsilon} M u_\varepsilon' = \widetilde{f}, \qquad in\ L^2\left(0,T;H^{-1}(\Omega)\right), \tag{2.10}$$

$$u_\varepsilon(0,.) = \widetilde{u}_0 \qquad in\ \Omega, \tag{2.11}$$

$$u_\varepsilon'(0,.) = \widetilde{u}_1 \qquad in\ \Omega. \tag{2.12}$$

Proof of lemma. Let $(w_v)_{v\in\mathbb{N}}$ be a basis of $H_0^1(\Omega)$ and $V_m = [w_1, ..., w_m]$ the subspace generated by the m first vectors of the basis $(w_v)_{v\in\mathbb{N}}$. For $m \in \mathbb{N}$, consider the function $u_{\varepsilon\ m} : [0, t_{\varepsilon\ m}) \to V_m$ as solutions of the following system

$$\begin{cases} i)\ \left(u_{\varepsilon\ m}'', w\right) + \left(\beta u_{\varepsilon\ m}', w\right) - \left(\triangle u_{\varepsilon\ m}, w\right) + \frac{1}{\varepsilon}\left(M u_{\varepsilon\ m}', w\right) = \left(\widetilde{f}, w\right) & \forall w \in V_m, \\ ii)\ u_{\varepsilon\ m}(0) = u_{0\ m} = \sum_{i=1}^m \alpha_{im} w_i \to \widetilde{u}_0 \qquad \text{in } H_0^1(\Omega) & u_{0\ m} \in V_m, \\ iii)\ u_{\varepsilon\ m}'(0) = u_{1\ m} = \sum_{i=1}^m \beta_{im} w_i \to \widetilde{u}_1 \qquad \text{in } L^2(\Omega) & u_{1\ m} \in V_m. \end{cases} \tag{2.13}$$

The system (2.13) has solution in $[0, t_{\varepsilon\ m})$, $0 < t_{\varepsilon\ m} < T$.

A priori estimates: Setting $w = u_{\varepsilon\ m}'(t)$ in (2.13), we obtain

$$\left(u_{\varepsilon\ m}'', u_{\varepsilon\ m}'\right) + \left(\beta u_{\varepsilon\ m}', u_{\varepsilon\ m}'\right) + \left(\nabla u_{\varepsilon\ m}, \nabla u_{\varepsilon\ m}'\right) + \frac{1}{\varepsilon}\left(M u_{\varepsilon\ m}', u_{\varepsilon\ m}'\right) = \left(\widetilde{f}, u_{\varepsilon\ m}'\right),$$

or else

$$\frac{1}{2}\frac{d}{dt}\left|u_{\varepsilon\ m}'\right|^2 + \frac{1}{2}\frac{d}{dt}\left|\nabla u_{\varepsilon\ m}\right|^2 + \beta\left|u_{\varepsilon\ m}'\right|^2 + \frac{1}{\varepsilon}\int_\Omega \left|M\left(u_{\varepsilon\ m}'\right)\right|^2 dx \le \left(\widetilde{f}, u_{\varepsilon\ m}'\right).$$

Integrating form 0 to $t < t_{\varepsilon\ m}$, we have

$$\begin{aligned} &\frac{1}{2}\left|u_{\varepsilon\ m}'\right|^2 + \frac{1}{2}\left|\nabla u_{\varepsilon\ m}\right|^2 + \beta\int_0^t \left|u_{\varepsilon\ m}'\right|^2 ds + \frac{1}{\varepsilon}\int_0^t\int_\Omega \left|M\left(u_{\varepsilon\ m}'\right)\right|^2 dxds \\ \le\ &\int_0^t \left(\widetilde{f}, u_{\varepsilon\ m}'\right) ds + \frac{1}{2}\left[\left|u_{1\ m}\right|^2 + \left|\nabla u_{0\ m}\right|^2\right]. \end{aligned} \tag{2.14}$$

Applying Cauchy-Schwartz and Young inequalities, we deduce that

$$\int_0^t \left(\widetilde{f}, u_{\varepsilon\ m}'\right) ds \le \frac{2}{\beta}\int_0^t \left|\widetilde{f}\right|^2 dxds + \frac{\beta}{2}\int_0^t \left|u_{\varepsilon\ m}'\right|^2 ds,$$

since $\widetilde{f} \in L^2(\mathcal{S})$, Then

$$\int_0^t \left(\widetilde{f}, u_{\varepsilon\ m}'\right) ds \le C + \frac{\beta}{2}\int_0^t \left|u_{\varepsilon\ m}'\right|^2 ds.$$

Then (2.14) becomes

$$\begin{aligned}
&\frac{1}{2}\left[\left|u'_{\varepsilon\ m}\right|^2+\left|\nabla u_{\varepsilon\ m}\right|^2\right]+\frac{\beta}{2}\int_0^t\left|u'_{\varepsilon\ m}\right|^2 ds+\frac{1}{\varepsilon}\int_0^t\int_\Omega\left|M\left(u'_{\varepsilon\ m}\right)\right|^2 dxds\\
\leq\ & C+\frac{1}{2}\left[\left|u_{1\ m}\right|^2+\left|\nabla u_{0\ m}\right|^2\right],
\end{aligned}$$

or else

$$\frac{1}{2}\left[\left|u'_{\varepsilon\ m}\right|^2+\left|\nabla u_{\varepsilon\ m}\right|^2\right]+\frac{\beta}{2}\int_0^t\left|u'_{\varepsilon\ m}\right|^2 ds+\frac{1}{\varepsilon}\int_0^t\int_\Omega\left|M\left(u'_{\varepsilon\ m}\right)\right|^2 dxds\leq C_1. \tag{2.15}$$

where C_1 is a positive constant independent of m and ε. *Pass to the limit:* Observe that the estimates (2.15) implies that there exists a subsequence of $(u_{\varepsilon\ m})$, which we still denote by $(u_{\varepsilon\ m})$ and a function u_ε, such that

$$\begin{cases}
1)\ u_{\varepsilon\ m}\overset{*}{\rightharpoonup}u_\varepsilon & \text{in } L^\infty\left(0,T;H_0^1(\Omega)\right),\\
2)\ u'_{\varepsilon\ m}\overset{*}{\rightharpoonup}u'_\varepsilon & \text{in } L^\infty\left(0,T;L^2(\Omega)\right),\\
3)\ \frac{1}{\sqrt{\varepsilon}}Mu'_{\varepsilon\ m}\rightharpoonup\frac{1}{\sqrt{\varepsilon}}Mu'_\varepsilon & \text{in } L^2\left(0,T;L^2(\Omega)\right).
\end{cases} \tag{2.16}$$

It follows from (2.16), that we can pass to the limit in the approximate system (2.13), we obtain

$$u''_\varepsilon+\beta u'_\varepsilon-\triangle u_\varepsilon+\frac{1}{\varepsilon}Mu'_\varepsilon=\widetilde{f},\qquad \text{in } L^2\left(0,T;H^{-1}(\Omega)\right),$$

There is no difficulty to verify that the initial conditions: So from $(2.16)_1$, $((2.16)_2$ and lemma 1.2 [chap 1 in [26]] we have

$$u_{\varepsilon\ m}(0)=u_{0\ m}\to\widetilde{u}_0\qquad \text{in } H_0^1(\Omega),$$

or

$$u_{\varepsilon\ m}(0)=u_{0\ m}\rightharpoonup\widetilde{u}_0\qquad \text{weakly in } L^2(\Omega)$$

then (2.11) it holds. Furthermore, according to lemma 1.3 [chap 1 in [26]], we get

$$\left(u''_{\varepsilon\ m},w_j\right)\overset{*}{\rightharpoonup}\left(u''_\varepsilon,w_j\right)\quad \text{in } L^\infty(0,T)$$

then according to lemma 1.2 [chap 1 in [26]]

$$\left(u'_{\varepsilon\ m}(0),w_j\right)\to\left(u'_\varepsilon(0),w_j\right)$$

and according to $(2.13)_{iii}$, we have

$$\left(u'_\varepsilon(0),w_j\right)=\left(\widetilde{u}_1,w_j\right),$$

so (2.12) is satisfied ■

Theorem of existence

Theorem 19 *Assume that (1.2) holds, then given* $u^0 \in H_0^1(\Omega_0)$, $u^1 \in L^2(\Omega_0)$ *and* $f \in L^2(Q_t)$ *there exists a function* u *defined in* Q_t *such that*

$$u \in L^\infty\left(0,T;H_0^1(\Omega_t)\right), \tag{2.17}$$

$$u' \in L^\infty\left(0,T;L^2(\Omega_t)\right), \tag{2.18}$$

$$u'' \in L^2\left(0,T;H^{-1}(\Omega_t)\right), \tag{2.19}$$

$$u'' + \beta u' - \triangle u = f, \qquad \text{in } L^2\left(0,T;H^{-1}(\Omega_t)\right), \tag{2.20}$$

$$u(0,.) = u_0, \qquad u'(0,.) = u_1, \qquad \text{in } \Omega_0.$$

Remark 20 *We observe that initial conditions make sense since by (2.17), (2.18), (2.19).*

Proof of theorem. Observe that the estimates obtained are independent of ε too. Therefore, by the same argument used to obtain u_ε from $u_{\varepsilon\, m}$, which is the solution of (2.13) we can pass to the limit when $\varepsilon \to 0$ in u_ε, or a subsequence, obtaining a function u independent of ε and m, such that

$$\begin{cases} u_\varepsilon \overset{*}{\rightharpoonup} v & \text{in } L^\infty(0,T;H_0^1(\Omega)), \\ u'_\varepsilon \overset{*}{\rightharpoonup} v' & \text{in } L^\infty(0,T;L^2(\Omega)), \\ u_\varepsilon \to v & \text{in } L^2(\Omega\times]0,T[) \end{cases} \tag{2.21}$$

From estimate (2.15), we have

$$\frac{1}{2}\left[|u'_\varepsilon|^2 + |\nabla u_\varepsilon|^2\right] + \frac{\beta}{2}\int_0^t |u'_\varepsilon|^2\, ds + \int_0^t \int_\Omega \left|\frac{1}{\sqrt{\varepsilon}} M(u'_\varepsilon)\right|^2 dxds \le C_2, \tag{2.22}$$

where C_2 is a positive constant independent of ε. From (2.22) we obtain that

$$\int_0^T \int_\Omega |M(u'_\varepsilon)|^2\, dxdt \le C_2\varepsilon,$$

we conclude that

$$Mv' = 0 \qquad \text{in } \Omega\times]0,T[\,,$$

but $M = 1$ in $\Omega\times]0,T[-Q_t$, therefore $v\prime = 0$ in $\Omega\times]0,T[-Q_t$, but $v(x,0) = \widetilde{u}_0$, $v(x,0) = 0$ in $\Omega - \Omega_0$ and when applying $(1.2)_i$, we obtain that

$$v = 0 \qquad \text{in } \Omega\times]0,T[-Q_t.$$

Then by $(1.2)_{ii}$ and if u be a restrecte of v in Q, then

$$u \in L^{\infty}\left(0, T; H_0^1(\Omega_t)\right).$$

Now, we consider the equation of the lemma i. e

$$(u_\varepsilon'', w) + (\beta u_\varepsilon', w) + (\nabla u_\varepsilon, \nabla w) + \frac{1}{\varepsilon}(M u_\varepsilon', w) = (f, w), \qquad \text{for all } w \in H_0^1(\Omega).$$

Therefore, restricting the above equation on Q_t, and with the functions M satisfy $Mu_\varepsilon' = 0$ we obtain in $\mathcal{D}'(0,T)$

$$(u_\varepsilon'', w) + (\beta u_\varepsilon', w) + (\nabla u_\varepsilon, \nabla w) = (f, w), \qquad \text{for all } w \in H_0^1(\Omega_t). \tag{2.23}$$

It follows from (2.21), that we can pass to the limit, $\varepsilon \to 0$, in the equation (2.23) thus obtaining (2.20). In order to check up that $u(0) = u_0$ it is sufficient to use $(2.21)_1$ followed by an integration by parts. By the same argument, using $(2.21)_2$, it can be shown that $u'(0) = u_1$. We observe that

$$u'' = f + \triangle u - \beta u', \tag{2.24}$$

since $\triangle \in \mathcal{L}\left(H_0^1(\Omega_t); H^{-1}(\Omega_t)\right)$, then

$$\triangle u \in L^{\infty}\left(0, T; H^{-1}(\Omega_t)\right)$$

and (2.18), so that (2.24) leads to

$$u'' \in L^2\left(0, T; L^2(\Omega_t)\right) + L^{\infty}\left(0, T; H^{-1}(\Omega_t) + L^2(\Omega_t)\right),$$

hence, in particular $u'' \in L^{\infty}\left(0, T; H^{-1}(\Omega_t) + L^2(\Omega_t)\right)$. Thanks to Lemma 1.2 (ch 1 in [26]), that u' is continuous $[0,T] \to H^{-1}(\Omega_t) + L^2(\Omega_t)$, such that $u'(0,x) = u_1$, makes sense. This completes the proof of the theorem ■

Existence of regular solutions

Theorem 21 *Let*

$$u^0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0), \quad u^1 \in H_0^1(\Omega_0) \ \textit{and}\ f \in H^1\left(0, T; L^2(\Omega_t)\right) \tag{2.25}$$

Then there exists a regular solution u satisfying

$$u \in L^{\infty}\left(0, T; H_0^1(\Omega_t) \cap H^2(\Omega_t)\right), \quad u' \in L^{\infty}\left(0, T; H^1(\Omega_t)\right), \quad u'' \in L^{\infty}\left(0, T; L^2(\Omega_t)\right).$$

Remark 22 *The following identity holds*

$$\int_{\Omega_t} (u'' + \beta u' - \triangle u)\, u'(t)dx = \int_{\Omega_t} f(t)u'(t)dx, \qquad \textit{for a.e} \quad t \in [0,T]\,. \tag{2.26}$$

So we can take u' as a test function.

Uniqueness

The uniqueness of solution for problem (2.5) is based to the timelikness condition of Q_t.

Theorem 23 *For Ω_t, defined by (2.1), the solution of Problem (2.5) is unique under the following assumption for the timelikness condition of Σ_t*

$$0 \le \ell \le 1. \tag{2.27}$$

Proof. Lets u, v be two solutions in the sense of the theorem 20, we put $J = u - v$, then we have

$$J'' - \triangle J + \beta J' = 0 \tag{2.28}$$

$$J(0) = 0, \qquad J'(0) = 0 \tag{2.29}$$

$$J \in L^{\infty}\left(0,T; H_0^1(\Omega_t) \cap H^2(\Omega_t)\right), \quad J' \in L^{\infty}\left(0,T; H^1(\Omega_t)\right) \tag{2.30}$$

Formally first, multiply both members of (2.28) by J', we deduce that

$$\frac{1}{2}\frac{\partial}{\partial t}\left[(J')^2 + |\nabla J|^2\right] + \beta\,(J')^2 - div\,[J'\nabla J] = 0$$

Integrating over Q_t, $0 \le t \le T$, we obtain

$$\frac{1}{2}\int_{\Omega_t}\left[|J'|^2 + |\nabla J|^2\right]dx + \beta\int_{Q_t}|J'|^2\,dxds = \frac{1}{2}\int_{\Omega_0}\left[|J'|^2 + |\nabla J|^2\right]dx$$
$$-\frac{1}{2}\int_{\Sigma_t}\left((J')^2 + |\nabla J|^2\right)\nu_t d\sigma + \int_{\Sigma_t}(J'\ \nabla J)\,\nu_x d\sigma.$$

Or else

$$\frac{1}{2}\int_{\Omega_t}\left[(J')^2 + |\nabla J|^2\right]dx + \beta\int_{Q_t}(J')^2\,dxds = \frac{1}{2}\int_{\Omega_0}\left[|J'|^2 + |\nabla J|^2\right]dx + \frac{1}{2}\int_{\Sigma_t}\left(\left(\frac{\partial J}{\partial \nu}\right)^2 \nu_t(|\nu_x|^2 - \nu_t^2\right)d\sigma$$

the second term is always non-positive because $\nu_t \le 0$ for expending domains, then $J = 0$. This completes the proof of the theorem ■

2.3 The limit problem

The candidate limit of u, as $t \to +\infty$, is the solution to the following elliptic problem defined on ω

$$\begin{cases} -\triangle_{X_2} u_\infty = f_\infty & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega, \end{cases} \tag{2.31}$$

where $\triangle_{X_2}$ is the Laplace operator in X_2 defined by $\triangle_{X_2} = \partial^2_{x'_1} + ... + \partial^2_{x'_n}$, and

$$f_\infty \in L^2(\omega), \tag{2.32}$$

then assuming that $f \to f_\infty$ as $t \to +\infty$, in a sense to be made precise below, one expects that the limit problem becomes independent of t and x_1.

We put

$$w := u - u_\infty \quad \text{and} \quad F := f - f_\infty,$$

since u_∞ depends only on X_2, then $u' = w'$, $\Delta u - \Delta_{X_2} u_\infty = \Delta w$ and w satisfies

$$w'' - \Delta w + \beta w' = F \qquad \text{in } Q_t, \tag{2.33}$$

with the following initial conditions

$$w(0, x) = u^0(x) - u_\infty(X_2), \quad w'(0, x) = u^1(x).$$

Remark 24 *Note that if $u_\infty \neq 0$ on Σ_t, then $w \neq 0$ on Σ_t. As a consequence, w is not a valid test function for equation (2.33).*

2.4 Special cut-off functions

For a fixed $t > 1$, let m be a positive integer with $m \leq t - 1$. Consider the family of sets

$$S^t_m := \{(s, x_1);\ t - m < s < t,\ \ -\ell_0 - \ell(m - t + s) < x_1 < \ell_0 + \ell(m - t + s)\},$$

which is increasing in m (see Figure 1), i.e. $S^t_m \subset S^t_{m+1}$, and satisfies

$$S^t_m \times \omega \subset \bigcup_{t-m<s<t} \{s\} \times \Omega_s \subset]t - m, t[\times \mathbb{R} \times \omega.$$

The associated family of smooth cut-off functions is defined as

$$\varrho_m = \varrho_m(s, x_1) : [0,t] \times \mathbb{R} \to \mathbb{R},$$

such that

$$0 \le \varrho_m \le 1, \quad \varrho_m = 1 \text{ in } S_m^t, \qquad \varrho_m = 0 \text{ in } [0,t] \times \mathbb{R} \setminus S_{m+1}^t,$$

$$|\partial_{x_1} \varrho_m|, |\varrho_m'| \le \theta,$$

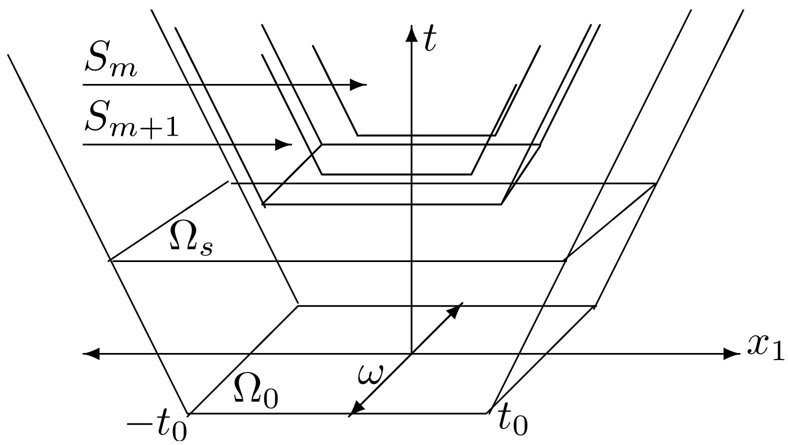

Figure 2.1: The family of sets S_m^t

where θ is a constant independent of t and m. We have in particular $\varrho_m(0, x_1) = 0$ and $\varrho_m = 0$ near the lateral boundary Σ_t. The supports of $\partial_{x_1} \varrho_m$ and ϱ_m' are included in $S_{m+1}^t \backslash S_m^t$.

2.5 Energy Estimates

2.5.1 A priori estimate

We can take the following estimation

Lemma 25 *Under the assumptions (2.25) and (2.27), the unique solution of Problem (2.5) satisfies, for a.e.* $t \in [0,T]$

$$\int_{\Omega_t} |u'(t)|^2 + |\nabla u(t)|^2 \, dx + \beta \int_{Q_t} |u'|^2 \, dx ds \le \int_{\Omega_0} \left(u^1\right)^2 + \left|\nabla u^0\right|^2 dx + \frac{1}{\beta} \int_{Q_t} f^2 dx ds.$$

Proof. We multiply the differential equation of problem (2.5) by u', we get

$$\frac{1}{2}\frac{\partial}{\partial s}\left[(u')^2+|\nabla u|^2\right]+\beta(u')^2-div(u')\nabla u=u'\ f.$$

Integrating over Q_t, $0\leq t\leq T$, we obtain

$$\frac{1}{2}\int_{\Omega_t}\left[|u'|^2+|\nabla u|^2\right]dx+\beta\int_{Q_t}|u'|^2\,dxds\leq\frac{1}{2}\int_{\Omega_0}\left[|u'|^2+|\nabla u|^2\right]dx+\int_{Q_t}(u').f\ dxds$$
$$-\frac{1}{2}\int_{\Sigma_t}\left((u')^2+|\nabla u|^2\right)\nu_t d\sigma+\frac{1}{2}\int_{\Sigma_t}(u'\ \nabla u)\,\nu_x d\sigma.$$

Or else

$$\frac{1}{2}\int_{\Omega_t}\left[|u'|^2+|\nabla u|^2\right]dx+\beta\int_{Q_t}|u'|^2\,dxds\leq\frac{1}{2}\int_{\Omega_0}\left[|u'|^2+|\nabla u|^2\right]dx+\int_{Q_t}(u').f\ dxds$$
$$-\frac{1}{2}\int_{\Sigma_t}\left(\left(\frac{\partial u}{\partial\nu}\right)^2\nu_t+\left(|\nu_x|^2-\nu_t^2\right)\right)d\sigma.$$

The third term in the left part is always non-positive because ν_t is non-positive and (1.3). Therefore we have

$$\frac{1}{2}\int_{\Omega_t}\left[|u'|^2+|\nabla u|^2\right]dx+\beta\int_{Q_t}|u'|^2\,dxds\leq\frac{1}{2}\int_{\Omega_0}\left[|u'|^2+|\nabla u|^2\right]dx+\int_{Q_t}(u').f\ dxds.$$

Applying Cauchy-Schwartz and Young inequalities for the term $\int_{Q_t}(u').f\ dxds$, we get

$$\int_{Q_t}(u').f\ dxds\leq\frac{\beta}{2}\int_{Q_t}(u')^2\,dxds+\frac{1}{2\beta}\int_{Q_t}f^2\ dxds.$$

we deduce the result ■

In order to derive local energy estimates we will make use of ϱ_m and its proprieties. So we have the following lemma.

Lemma 26 *Under the assumptions (2.25) and (2.32), the solutions of Problem (2.5) and Problem (2.31) satisfy*

$$\int_{\Omega_t}D(t)\,\varrho_m^2dx+\int_{S_m^t\times\omega}D\ dxds\leq c_4\int_{\left(S_{m+1}^t\backslash S_m^t\right)\times\omega}D\ dxds+c_4\int_{S_{m+1}^t\times\omega}F^2dxds,$$

for a.e. $t\in[0,T]$. *where*

$$D(t,x):=|w'(t,x)|^2+|\nabla w(t,x)|^2.$$

Proof. • *A local energy identity*

Let us start by taking $2w\varrho_m^2$ as a test function for (2.33), we get

$$\begin{aligned}
\left(2\varrho_m^2 ww''\right) &= \frac{\partial}{\partial s}\left[2\varrho_m^2 ww'\right] - 2\varrho_m^2\left|w'\right|^2 - 4\varrho_m'.\varrho_m ww', \\
\left(\beta\varrho_m^2 ww'\right) &= \frac{\partial}{\partial s}\left[\beta.\varrho_m^2 w^2\right] - 2\beta\varrho_m'\varrho_m w^2, \\
\left(2\varrho_m^2 w\right)\triangle w &= 2\nabla\cdot\left(\varrho_m^2 w\nabla w\right) - 2\nabla\left(\varrho_m^2 w\right)\nabla w \\
&= 2\nabla\cdot\left(\varrho_m^2 w\nabla w\right) - 4\varrho_m w\left(\nabla\varrho_m\cdot\nabla w\right) - 2\alpha.\varrho_m^2\left|\nabla w\right|^2,
\end{aligned}$$

so, (2.33) it comes that

$$\begin{aligned}
&\frac{\partial}{\partial s}\left[\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww'\right] - 2\beta\varrho_m'\varrho_m w^2 - 2\varrho_m^2\left|w'\right|^2 - 4\varrho_m'\varrho_m ww' \\
&+2\varrho_m^2\left|\nabla w\right|^2 - 2\nabla\cdot\left(\varrho_m^2 w\nabla w\right) + 4\varrho_m w\left(\nabla\varrho_m\cdot\nabla w\right) = 2w\varrho_m^2 F.
\end{aligned} \tag{2.34}$$

For some constant $\alpha > 0$, multiplying (2.33) by $2\alpha w'\varrho_m^2$, yields

$$\begin{aligned}
2\alpha\varrho_m^2 w'w'' &= \frac{\partial}{\partial s}\left(\alpha\varrho_m^2\left|w'\right|^2\right) - 2\alpha\varrho_m'\varrho_m\left|w'\right|^2 \\
2\alpha\varrho_m^2 w'\triangle w &= 2\alpha\nabla\cdot\left(\varrho_m^2 w'\nabla w\right) - 2\alpha\nabla\cdot\left(\varrho_m^2 w'\nabla w\right) - \frac{\partial}{\partial s}\left(\varrho_m^2\left|\nabla w\right|^2\right) + 2\alpha\varrho_m'\varrho_m\left|\nabla w\right|^2 \\
2\alpha\beta\varrho_m^2 w'w' &= 2\alpha\beta\varrho_m^2\left|w'\right|^2,
\end{aligned}$$

so, (2.33) it comes that

$$\begin{aligned}
&\frac{\partial}{\partial s}\left[\alpha\varrho_m^2\left|w'\right|^2 + \alpha\varrho_m^2\left|\nabla w\right|^2\right] - 2\alpha\varrho_m'\varrho_m\left|w'\right|^2 + 2\alpha\beta\varrho_m^2\left|w'\right|^2 \\
&-2\alpha\varrho_m'\varrho_m\left|\nabla w\right|^2 - 2\alpha\nabla\cdot\left(\varrho_m^2 w'\nabla w\right) + 4\alpha\varrho_m w'\left(\nabla\varrho_m\cdot\nabla w\right) = 2\alpha w'\varrho_m^2 F.
\end{aligned} \tag{2.35}$$

Summing up the above identities (2.34) and (2.35), we get

$$\begin{aligned}
&\frac{\partial}{\partial s}\left[\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww' + \alpha\varrho_m^2\left|w'\right|^2 + \alpha\varrho_m^2\left|\nabla w\right|^2\right] - 2\beta\varrho_m'\varrho_m w^2 - 2\varrho_m^2\left|w'\right|^2 - 4\varrho_m'\varrho_m ww' \\
&-2\alpha\varrho_m'\varrho_m\left|w'\right|^2 + 2\alpha\beta\varrho_m^2\left|w'\right|^2 + 2\varrho_m^2\left|\nabla w\right|^2 - 2\nabla\cdot\left(\varrho_m^2 w\nabla w\right) + 4\varrho_m w\left(\nabla\varrho_m\cdot\nabla w\right) \\
&-2\alpha\varrho_m'\varrho_m\left|\nabla w\right|^2 - 2\alpha\nabla\cdot\left(\varrho_m^2 w'\nabla w\right) + 4\alpha\varrho_m w'\left(\nabla\varrho_m\cdot\nabla w\right) = 2w\varrho_m^2 F + 2\alpha w'\varrho_m^2 F.
\end{aligned}$$

Then collecting the terms with derivatives of ϱ_m in the right-hand side of the identity, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial s}\left[\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww' + \alpha\varrho_m^2\left|w'\right|^2 + \alpha\varrho_m^2\left|\nabla w\right|^2\right] + 2\left(\alpha\beta - 1\right)\varrho_m^2\left|w'\right|^2 + 2\varrho_m^2\left|\nabla w\right|^2 \\
&= 2\beta\varrho_m'\varrho_m w^2 + 4\varrho_m'\varrho_m ww' + 2\alpha\varrho_m'\varrho_m\left|w'\right|^2 + 2\alpha\varrho_m'\varrho_m\left|\nabla w\right|^2 \\
&-4\varrho_m w\left(\nabla\varrho_m\cdot\nabla w\right) - 4\alpha\varrho_m w'\left(\nabla\varrho_m\cdot\nabla w\right) + 2\alpha\nabla\cdot\left(\varrho_m^2 w'\nabla w\right) \\
&+2\nabla\cdot\left(\varrho_m^2 w\nabla w\right) + 2w\varrho_m^2 F + 2\alpha w'\varrho_m^2 F.
\end{aligned}$$

Integrating on Q_t, and taking into account the fact that $\varrho_m = 0$ for $t = 0$ and on Σ_t, it comes

$$\int_{\Omega_t} \left[\beta \varrho_m^2 w^2(t) + 2\varrho_m^2 w w'(t) + \alpha \varrho_m^2 |w'(t)|^2 + \alpha \varrho_m^2 |\nabla w(t)|^2\right] dx$$
$$+ \int_{Q_t} \left[2(\alpha\beta - 1)\varrho_m^2 |w'|^2 + 2\varrho_m^2 |\nabla w|^2\right] dxds$$
$$= \int_{Q_t} \left[2\beta \varrho_m' \varrho_m w^2 + 4\varrho_m' \varrho_m w w' + 2\alpha \varrho_m' \varrho_m |w'|^2 + 2\alpha \varrho_m' \varrho_m |\nabla w|^2\right] dxds$$
$$- \int_{Q_t} \left[4\varrho_m w (\nabla \varrho_m \cdot \nabla w) - 4\alpha \varrho_m w' (\nabla \varrho_m \cdot \nabla w)\right] dxds + \int_{Q_t} \left[2w\varrho_m^2 F + 2\alpha w' \varrho_m^2 F\right] dxds.$$

To make the left-hand side as positive definite quadratic form, we use the fact that

$$2ww' \geq -\left(\beta w^2 + \frac{1}{\beta}|w'|^2\right),$$

then choose $\alpha > \dfrac{1}{\beta}$, we obtain

$$\beta \varrho_m^2 w^2 + 2\varrho_m^2 w w' + \alpha \varrho_m^2 |w'|^2 + \alpha \varrho_m^2 |\nabla w|^2 \geq \delta_0 \varrho_m^2 |w'|^2 + \alpha \varrho_m^2 |\nabla w|^2,$$

where $\delta_0 = \left(\alpha - \dfrac{1}{\beta}\right) > 0$. Integrating on Q_t we obtain

$$\int_{\Omega_t} \left(\delta_0 |w'(t)|^2 + \alpha |\nabla w(t)|^2\right) \varrho_m^2 dx + \int_{Q_t} \left(\beta\delta_0 |w'|^2 + 2|\nabla w|^2\right) \varrho_m^2 dxds$$
$$\leq \int_{Q_t} 2\beta \varrho_m' \varrho_m w^2 + 4\varrho_m' \varrho_m w w' + 2\alpha \varrho_m' \varrho_m |w'|^2 - 4\varrho_m w (\nabla \varrho_m \cdot \nabla w)\, dxds$$
$$+ \int_{Q_t} 2\alpha \varrho_m' \varrho_m |\nabla w|^2 - 4\alpha \varrho_m w' (\nabla \varrho_m \cdot \nabla w)\, dxds + \int_{Q_t} 2w\varrho_m^2 F + 2\alpha w' \varrho_m^2 F dxds.$$

Using Young's inequality in the two last integrals, we get

$$\int_{\Omega_t} \left(\delta_0 |w'(t)|^2 + \alpha |\nabla w(t)|^2\right) \varrho_m^2 dx + \int_{Q_t} \left(\beta\delta_0 |w'|^2 + 2|\nabla w|^2\right) \varrho_m^2 dxds$$
$$\leq \int_{Q_t} 2\beta \varrho_m' \varrho_m w^2 + 4\varrho_m' \varrho_m w w' + 2\alpha \varrho_m' \varrho_m |w'|^2 - 4\varrho_m w (\nabla \varrho_m \cdot \nabla w)\, dxds$$
$$+ \int_{Q_t} 2\alpha \varrho_m' \varrho_m |\nabla w|^2 - 4\alpha \varrho_m w' (\nabla \varrho_m \cdot \nabla w)\, dxds$$
$$+ \varepsilon \int_{Q_t} \left(w^2 + |w'|^2\right) \varrho_m^2 dxds + c_0 \int_{Q_t} F^2 \varrho_m^2 dxds$$

for some constant $\varepsilon > 0$. Here and in the sequel, c_i denotes positive constants depending (at most) on θ, α and ω, but not on t.

• *End of proof* Again applying Young's inequality $\left(AB \leq \delta A^2 + \frac{1}{4\delta}B^2\right)$ several times for different choices of $\delta > 0$, and noting that the supports of ϱ_m' and $|\nabla \varrho_m|$ are included in $S_{m+1}^t \backslash S_m^t$, we end up with

$$\begin{aligned}
&\int_{\Omega_t} \left(|w'(t)|^2 + |\nabla w(t)|^2\right) \varrho_m^2 dx + \int_{Q_t} \left(|w'|^2 + |\nabla w|^2\right) \varrho_m^2 dxds \\
&\leq c_1 \int_{\left(S_{m+1}^t \backslash S_m^t\right)\times\omega} |w'|^2 + |w|^2 + |\nabla w|^2 \, dxds \\
&+ c_2\varepsilon \int_{Q_t} \left(|w'|^2 + |w|^2\right) \varrho_m^2 dxds + c_3 \int_{Q_t} F^2 \varrho_m^2 dxds
\end{aligned}$$

Applying the Poincaré inequality in the X_2-direction yields

$$\int_{\Omega_t} |w(t)|^2 \varrho_m^2 dx \leq c_\omega^2 \int_{\Omega_t} |\nabla_{X_2} w|^2 \varrho_m^2 dx \leq c_\omega^2 \int_{\Omega_t} |\nabla w|^2 \varrho_m^2 dx,$$

where c_ω is the Poincaré constant. Choosing ε such that $c_2\varepsilon < 1$ and setting

$$D(t,x) := |w'(t,x)|^2 + |\nabla w(t,x)|^2 .$$

This completes the proof ■

2.6 Main results

2.6.1 Convergence theorems

To obtain the convergences, we use an iteration technique for we read the following result.

Theorem 27 *Under the assumptions (2.25), (2.27), (2.32), (2.39) and (2.38), we have*

$$u' \to 0, \quad \partial_{x_1} u \to 0, \quad \nabla_{X_2} u \to \nabla_{X_2} u_\infty \quad \text{in } L^2\left(S_1^t \times \omega\right), \quad \text{as } t \to +\infty.$$

Moreover, if $f = f_\infty$, the above convergence is exponential.

Proof. First we note that

$$\int_{\left(S_{m+1}^t \backslash S_m^t\right)\times\omega} D \; dxds = \int_{S_{m+1}^t \times\omega} D \; dxds - \int_{S_m^t \times\omega} D \; dxds,$$

then, dropping the first integral term in the inequality of Lemma 26, it comes that

$$(1+c_4)\int_{S_m^t\times\omega} D\ dxds \le c_4\int_{S_{m+1}^t\times\omega} D\ dxds + c_4\int_{S_{m+1}^t\times\omega} F^2 dxds,$$

where $c_4>0$. We have then

$$\int_{S_m^t\times\omega} D\ dxds \le k\int_{S_{m+1}^t\times\omega} D\ dxds + k\int_{S_{m+1}^t\times\omega} F^2 dxds,$$

where $k=\dfrac{c_4}{1+c_4}<1$. Iterating the above inequality for $m=1,\cdots,[t]-1$, ($[\cdot]$ denotes the integer part), we have

$$\begin{aligned}\int_{S_1^t\times\omega} D\ dxds &\le k\int_{S_2^t\times\omega} D\ dxds + k\int_{S_2^t\times\omega} F^2 dxds\\ &\le k^2\int_{S_3^t\times\omega} D\ dxds + k^2\int_{S_3^t\times\omega} F^2 dxds + k\int_{S_2^t\times\omega} F^2 dxds\\ &\ \vdots\\ &\le k^{[t]-1}\int_{S_{[t]}^t\times\omega} D\ dxds + \sum_{j=1}^{[t]-1}\left(k^j\int_{S_{1+j}^t\times\omega} F^2 dxds\right).\end{aligned}$$

Noting that $t-1<[t]\le t$, we get

$$\int_{S_1^t\times\omega} D\ dxds \le c_5 e^{-\gamma t}\int_{S_{[t]}^t\times\omega} D\ dxds + \sum_{j=1}^{[t]-1}\left(k^j\int_{S_{1+j}^t\times\omega} F^2 dxds\right). \tag{2.36}$$

where $\gamma=-\ln k$ and $c_5>0$. In order to estimate the right-hand side, we write

$$\begin{aligned}\int_{S_{[t]}^t\times\omega} D\ dxds &\le \int_{Q_t} D\ dxds\\ &\le 2\int_{Q_t} |u'|^2+|\nabla u|^2+|\nabla_{X_2}u_\infty|^2\ dxds\\ &\le 2\int_{Q_t} |u'|^2+|\nabla u|^2\ dxds + 2\,|\nabla_{X_2}u_\infty|^2_{L^2(\omega)}\int_0^t\left(\int_{-\ell_0-\ell s}^{\ell_0+\ell s} dx_1\right) ds.\end{aligned}$$

One can easily show that $|\nabla u_\infty|_{L^2(\omega)}\le |f_\infty|_{L^2(\omega)}$, and taking into account Lemma 25, it follows that

$$\begin{aligned}\int_{S_{[t]}^t\times\omega} D\ dxds &\le 2t\left(\int_{\Omega_0}\left(u^1\right)^2+\left|\nabla u^0\right|^2 dx + \frac{1}{\beta}\int_{Q_t} f^2 dxds\right) + 4\,|f_\infty|^2_{L^2(\omega)}\left(\ell t^2+2t_0 t\right)\\ &\le c_6\left(t^2+t\int_{Q_t} f^2 dxds\right)\end{aligned}$$

for large t. Substituting this in (2.36), we obtain

$$\int_{S_1^t\times\omega} D\ dxds \le c_7\left(t^2 + t\int_{Q_t} f^2 dxds\right)e^{-\gamma t} + g\left(t\right) \tag{2.37}$$

where g is a positive real function defined by

$$g\left(t\right) = \sum_{j=1}^{[t]-1}\left(k^j\int_{S_{1+j}^t\times\omega} F^2 dxds\right).$$

The estimate (2.37) may ensure the convergence $u\to u_\infty$ if we assume that

$$g\left(t\right)\to 0,\quad \text{as } t\to+\infty, \tag{2.38}$$

$$t\int_{Q_t} f^2 dxds = o\left(e^{\gamma t}\right). \tag{2.39}$$

This completes the proof of the theorem ■

Remark 28 *The assumption (2.39) holds for example if $\int_{Q_t} f^2 dxds$ grows polynomially in time. In particular, this holds if*

$$\int_{\Omega_t}\left|f-f_\infty\right|^2 dx\to 0,\ \textit{as } t\to+\infty.$$

Using Lemma 26 we have in particular, for $m=1$,

$$\int_{\Omega_1}\left|w'(t)\right|^2+\left|\nabla w(t)\right|^2 dx\le c_4\int_{S_2^t\times\omega}\left|w'\right|^2+\left|\nabla w\right|^2 dxds + c_4\int_{S_2^t\times\omega}F^2 dxds,$$

then applying the precedent iteration for $m=2,\cdots,[t]-1$ and arguing as above, we end up with the following corollary.

Corollary 29 *Under the assumptions (2.25), (2.27), (2.32), (2.39) and (2.38), we have*

$$u'\left(t\right)\to 0,\quad \partial_{x_1}u\left(t\right)\to 0,\quad \nabla_{X_2}u\left(t\right)\to\nabla_{X_2}u_\infty\quad \textit{in } L^2\left(\Omega_1\right),\quad \textit{as } t\to+\infty.$$

Moreover, if $f=f_\infty$, the above convergence is exponential.

Finally, iterating m between $\left[\frac{t}{2}\right]$ and $[t]-1$ as above, and assuming that

$$\sum_{j=\left[\frac{t}{2}\right]}^{[t]-1}\left(k^j\int_{S_{1+j}^t\times\omega} F^2 dxds\right)\to 0,\quad \text{as } t\to+\infty, \tag{2.40}$$

leads to the following corollary.

Corollary 30 *Let O be a bounded subset of $\mathbb{R}\times\omega$, then we can choose t large enough such that $O\subset\subset\Omega_{\frac{t}{2}}$, and under the assumptions (2.25), (2.27), (2.32), (2.38) and (2.40), we have*

$$u'(t)\to 0,\quad \nabla_{X_1}u(t)\to 0,\quad \nabla_{X_2}u(t)\to\nabla_{X_2}u_\infty\quad \text{in } L^2(O),\quad \text{as } t\to+\infty.$$

Moreover, if $f=f_\infty$, the above convergence is exponential.

This means that in any interior bounded region of $\mathbb{R}\times\omega$, the local energy decays to 0.

2.6.2 Exponential convergences

The rate of convergence, in the precedent theorems, depends essentially on the convergence of f towards f_∞. We now give some examples of source terms, other than the trivial case $f=f_\infty$, where we get an exponential rate of convergence in Theorem 27.

• Assume that

$$\int_{\Omega_t}|f(t)-f_\infty|^2\,dx\leq C_1e^{-\gamma_1 t},\quad \text{for } \forall t>0,$$

for some positive constants C_1,γ_1 such that $\gamma_1>0$. In one hand the necessary assumption (2.39) holds since

$$\int_{Q_t}f^2dxds\leq\int_{Q_t}|f_\infty|^2\,dxds+C_1\int_0^t e^{-\gamma_1 s}ds\leq c_9t^2 \tag{2.41}$$

for large t. On the other hand we have

$$\begin{aligned}\int_{S_{1+j}^t\times\omega}F^2dxds &\leq \int_{t-(1+j)}^t\int_{\Omega_s}F^2dxds\\ &\leq C_1\int_{t-(1+j)}^t e^{-\gamma_1 s}ds\leq\frac{C_1}{\gamma_1}(1+j)e^{-\gamma_1 t}\times e^{\gamma_1(1+j)}.\end{aligned}$$

Then, since $k^j=e^{-\gamma j}$,

$$k^j\int_{S_{1+j}^t\times\omega}F^2dxds\leq c_{10}te^{-\gamma_1 t}\times e^{(\gamma_1-\gamma)j},\qquad 1\leq j\leq[t]-1,$$

i.e.

$$g(t)\leq c_{10}te^{-\gamma_1 t}\left(\sum_{j=1}^{[t]-1}e^{(\gamma_1-\gamma)j}\right).$$

If $\gamma_1 < \gamma$ then $\sum_{j=1}^{[t]-1} e^{(\gamma_1-\gamma)j}$ is bounded, and if $\gamma_1 > \gamma$ then $\sum_{j=1}^{[t]-1} e^{(\gamma_1-\gamma)j} \leq c_{11} e^{(\gamma_1-\gamma)t}$. Thus, due to (2.37) and (2.41), we end up with

$$\int_{S_1^t\times\omega} |u'|^2 + |\nabla (u-u_\infty)|^2\, dxds \leq M_1 t^3\ e^{-\min\{\gamma,\gamma_1\}t},$$

for some positive constant M_1.

• Another assumption, analogous to that made in [31], is to assume that

$$\int_{t-1}^{t}\int_{\Omega_s} |f-f_\infty|^2\, dxds \leq C_2 e^{-\gamma_2 t}, \quad \text{for } \forall t>0,$$

for some positive constants C_2, γ_2 such that $\gamma_2 > 0$. Then assumption (2.39) holds and again we have $\int_{Q_t} |f(t)|^2\, dxds \leq c_{12} t^2$. To estimate g, we note that

$$\int_{S_{1+j}^t\times\omega} F^2 dxds \leq \int_{t-(1+j)}^{t}\int_{\Omega_s} F^2 dxds = \sum_{i=0}^{j}\left(\int_{t-(j-i)-1}^{t-(j-i)}\int_{\Omega_s} F^2 dxds\right)$$

and it follows that

$$\int_{S_{1+j}^t\times\omega} F^2 dxds \leq C_2 (j+1)\ e^{-\gamma_2 t} \leq C_2 t e^{-\gamma_2 t}, \qquad 1\leq j\leq [t]-1.$$

Thus

$$g(t) \leq C_2 t e^{-\gamma_2 t}\left(\sum_{j=1}^{[t]-1} e^{-\gamma j}\right) \leq c_{13} t\ e^{-\gamma_2 t},$$

and due to (2.37), it follows that

$$\int_{S_1^t\times\omega} |u'|^2 + |\nabla (u-u_\infty)|^2\, dxds \leq M_2 t^3\ e^{-\min\{\gamma,\gamma_2\}t},$$

for some positive constants M_2.

Chapter 3

Asymptotic behaviour of nonlinear wave equations

We study the asymptotic behaviour for the solution of nonlinear wave equations in a noncylindrical domain, becoming unbounded in some directions, as the time t goes to infinity. If the limit of the source term is independent of these directions and t, the wave converges to the solution of an elliptic problem defined on a lower dimensional domain. The rate of convergence depends on the limit behaviour of the source term and on the coefficient of the nonlinear term.

3.1 Problem setting

Let as denote the points in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ as

$$x = (X_1, X_2) = \left(x_1, ..., x_{n_1}, x'_1, ..., x'_{n_2}\right),$$

where n_1, n_2 are positive integers and we set

$$\begin{aligned} \nabla_{X_1} u &= (\partial x_1 u, ..., \partial x_{n_1} u)^T, \qquad \nabla_{X_2} u = (\partial x'_1 u, ..., \partial x'_{n_2} u)^T, \\ \nabla u &= \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}, \quad \nabla_{x,t} u = \begin{pmatrix} u' \\ \nabla u \end{pmatrix}. \end{aligned}$$

Then we consider a time-dependent family of bounded subsets in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ defined as

$$\Omega_t := (-\ell_0 - \ell t, \ell_0 + \ell t)^{n_1} \times \omega, \quad t \geq 0,$$

where ω is a bounded open subset of $\mathbb{R}^{n_2}$. In this chapter we study the asymptotic bihaviour of the solution of the following nonlinear wave equation when $t \to +\infty$

$$\begin{cases} u''(t,x) + \beta u'(t,x) - \triangle u(t,x) + \gamma(t) \left|u\right|^{\rho} u = f(t,x) & \text{in } Q_t, \\ u(t,x) = 0 & \text{on } \Sigma_t, \\ u(0,.) = u^0(x), \qquad u'(0,.) = u^1(x) & \text{in } \Omega_0, \end{cases} \tag{3.1}$$

where Q_t is a noncylindrical domain of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with lateral boundary Σ_t such that

$$Q_t :=_{0<s<t} \{s\} \times \Omega_s, \qquad \Sigma_t :=_{0<s<t} \{s\} \times \partial\Omega_s.$$

The prime stands for the time derivative, $\triangle$ is the Laplace operator, β is a positive constant and $\gamma(t)$ is a nonnegative function.

3.2 Existence and uniqueness of regular solutions

Transformation problem in the cylindrical domain

Let us consider the subsets Ω_s of $\mathbb{R}^{n_1} \times \omega$, given by

$$\Omega_s := \{(X_1, X_2) \in \mathbb{R}^{n_1} \times \omega \setminus X_1 = K(s)Y_1, \qquad Y_1 = (-\ell_0, \ell_0)^{n_1}\}, \quad s \in (0,t),$$

where $K(s) = 1 + \dfrac{\ell}{\ell_0} s$.

At first we will study our problem in a cylinder $Q = (0,s) \times \Omega, \quad \Omega = (-\ell_0, \ell_0)^{n_1} \times \omega$, where domains Q and Q_t are related by the diffeomorphism h defined by

$$\begin{aligned} &h : Q_t \to Q \\ &(s,x) \to \left(s, \frac{X_1}{K(s)}, X_2\right), \end{aligned}$$

so h^{-1} defined by

$$\begin{aligned} &h^{-1} : Q \to Q_t \\ &(s,y) \to (s, K(s)Y_1, X_2). \end{aligned} \tag{3.2}$$

For each $u \in L^2(Q_t)$; $v(s, Y_1) = u(s, K(s)Y_1, X_2)$. By change of variables $X_1 = K(s)Y_1$, we obtain $v \in L^2(Q)$. Our object consists of changing of variables $v(s, Y_1) = u(s, K(s)Y_1, X_2)$. Under this transformation problem (3.1) in Q_t is formulated in the cylindrical domain Q as follows

$$\begin{cases} v''(t,y) + \beta v'(t,y) - \sum\limits_{i,j=1}^{n_1+n_2} \dfrac{\partial}{\partial y_i}\left(a_{ij}(t,y)\dfrac{\partial v}{\partial y_j}\right) + \sum\limits_{i=1}^{n_1+n_2} b_i(t,y)\dfrac{\partial v'}{\partial y_i} \\ \qquad + \sum\limits_{i=1}^{n_1+n_2} c_i(t,y)\dfrac{\partial v}{\partial y_i} + \gamma(t)\left|v(t,y)\right|^{\rho} v(t,y) = \Phi(t,y) & \text{in } Q, \\ v = 0 & \text{on } \Sigma = \partial\Omega \times (0,t), \\ v(0) = v^0(y) = u^0(K(0)Y_1) = u^0(Y_1) & Y_1 \in \Omega, \\ v'(0) = v^1(y) = u^1(K(0)y) + \dfrac{K'(0)}{K(0)} \sum\limits_{i=1}^{n_1+n_2} \dfrac{\partial v^0}{\partial y_i} y_i & Y_1 \in \Omega. \end{cases}$$

where

$$\begin{aligned}
f(t,x) &= f(s,K(s)Y_1,X_2)=\Phi(t,y),\\
a_{ij}(t,y) &= \left[\chi_{ij}-K'^2 y_i y_j\right]K^{-2}\\
&= \left[\chi_{ij}-\left(\frac{\ell}{\ell_0}\right)^2\right]\left(1+\frac{\ell}{\ell_0}t\right)^{-2} y_i y_j,\quad \chi_{ij}\text{ the characteristic function,}\\
b_i(t,y) &= -2K'K^{-1}y_i=-2\left(\frac{\ell_0}{\ell}+t\right)^{-1} y_i,\\
c_i(t,Y_1) &= \left[1-(n_1+n_2)K'^2-K''^2K-\beta K'K\right]K^{-2}y_i\\
&= \left[1-(n_1+n_2)-\beta\left(\frac{\ell_0}{\ell}+t\right)\right]\left(\frac{\ell_0}{\ell}+t\right)^{-2} y_i.
\end{aligned}$$

Operators

Let us consider the following family of operators in $\mathcal{L}\left(H_0^1(\Omega),H^{-1}(\Omega)\right)$ (i.e., $\mathcal{L}\left(H_0^1(\Omega),H^{-1}(\Omega)\right)=$ the continuous linear application space of $H_0^1(\Omega)$ in $H^{-1}(\Omega)$).

$$\mathcal{A}(s)=-\sum_{1\leqslant i,j\leqslant n_1+n_2}\frac{\partial}{\partial_{y_i}}\left(a_{ij}(t,y)\frac{\partial}{\partial y}\right),t\geq 0.$$

where

$$a_{ij}=a_{ji}\quad\text{and } a_{ij}\in W^{3,\infty}\left(0,T;C^0\left(\overline{\Omega}\right)\right)\text{ for all } i,j=1,...,n_1+n_2. \tag{3.3}$$

We suppose that

$$\sum_{1\leqslant i,j\leqslant n_1+n_2} a_{ij}(s,y)\zeta_i\zeta_j\geqslant\lambda\left\|\zeta\right\|^2$$

where λ is a positive constant.

If we denote by $a(s,.,.)$ the family of bilinear forms associated with $\mathcal{A}(s)$, we have

$$a(s,u,v)=\sum_{1\leqslant i,j\leqslant n}\int_\Omega a_{ij}(t,y)\frac{\partial u}{\partial_{y_i}}\frac{\partial v}{\partial_{y_j}}dy,\qquad u,v\in H_0^1(\Omega).$$

From the hypothesis on a_{ij}, we obtain that $a(t,u,v)$ is symmetric and

$$a(t,u,u)\geq\lambda\left\|u\right\|^2_{H_0^1(\Omega)}\qquad\text{for all } u\in H_0^1(\Omega)\qquad s\in(0,t). \tag{3.4}$$

Suppose that functions $\rho, \gamma, f, u^0, u^1, K$ satisfy the following conditions

- Concerning the speed of expansion, in the n_1 first directions, it satisfies

$$0 \leq \ell \leq 1. \tag{3.5}$$

This ensures that Σ_t satisfies the so-called timelikness condition

$$|\nu_t| \leq |\nu_x| \quad \text{on } \Sigma_t, \text{ for } t > 0,$$

- The nonlinear term in Problem (3.1) is subject to the following assumptions (Recall that $x \in \mathbb{R}^{n_1+n_2}$)

$$\begin{cases} 0 < \rho \leq \frac{2}{(n_1+n_2)-2} & \text{if } n_1 + n_2 > 2, \\ 0 < \rho \leq \infty & \text{if } n_1 = n_2 = 1. \end{cases} \tag{3.6}$$

$$\gamma(t) \geq 0, \quad \gamma'(t) \leq 0, \quad \gamma(t), \gamma'(t) \in L^\infty(0,t) \tag{3.7}$$

- The initial data and the source term satisfy

$$u^0 \in H_0^2(\Omega_0), \quad u^1 \in H_0^1(\Omega_0), \quad f \in H^1\left(0, t; L^2(\Omega_s)\right), \tag{3.8}$$

- $H.1$:

$$\begin{aligned} & K \in C^4(0,t), \\ & \min_{0 \leq s < t} K(t) = \alpha_0 > 0, \qquad \max_{0 \leq s < t} K(t) = \alpha_1 > 0, \\ & \sup_{0 \leq s < t} K'(t) = \epsilon < \tfrac{1}{M}, \qquad M = \sup_{\mathbb{R}^{n_1+n_2}} \{|y|, \quad y \in \Omega\}, \\ & K'(s) \geq 0, \quad |K''(s)|, \quad |K'''(s)|, \quad \left|K^{(4)}(s)\right| \leq C, \quad \forall s \in (0,t), \\ & m_1 = \int_0^t K'(s) ds < \infty, \qquad m_2 = \int_0^t |K''(s)| \, ds < \infty, \\ & m_3 = \int_0^t |K'''(s)| \, ds < \infty, \qquad m_4 = \int_0^t \left|K^{(4)}(s)\right| ds < \infty, \\ & m_5 = \int_0^t (K'(s))^2 \, ds < \infty, \qquad m_6 = \int_0^t (K''(s))^2 \, ds < \infty, \\ & m_7 = \int_0^t (K'(s))^3 \, ds < \infty, \qquad m_8 = \int_0^t |K''(s)|^3 \, ds < \infty. \end{aligned} \tag{3.9}$$

Now we are in a position to state our existence and uniqueness result.

Existence of regular solutions

Theorem 31 *Let $t>0$. Under the assumptions eqreftlike (3.8), Then there exists a unique solution for Problem (3.1), in the sense that*

$$u\in L^{\infty}\left(0,t;H_0^1\left(\Omega_s\right)\cap H^2\left(\Omega_s\right)\right),\quad u'\in L^{\infty}\left(0,t;H^1\left(\Omega_s\right)\right),\quad u''\in L^2\left(0,t;L^2\left(\Omega_s\right)\right).$$

We can take u' as a test function, i.e. the following identity holds

$$\int_{\Omega_s}\left(u''+\beta u'-\triangle u+\gamma(s)\left|u\right|^{\rho}u\right)u'(s)dx=\int_{\Omega_s}f\left(s\right)u'(s)dx\quad \text{for a.e. } s\in(0,t). \tag{3.10}$$

Remark 32 *Here and in the sequel we use notations of [17].*

At first we will study our problem in a cylinder Q. Taking into account $H.1$, we can verify that

- $H.2$:

$$a_{ij}=a_{ji}\qquad \text{and } a_{ij}\in W^{3,\infty}\left(0,t;C^0\left(\overline{\Omega}\right)\right),$$
$$a(t,v,v)\geqslant\lambda\left\|v\right\|^2\qquad \text{in } Q\qquad (\lambda>0).$$

Let f, u^0, u^1 satisfy (3.8), by (3.2), we obtain

$$v^0\in H_0^2(\Omega),\quad v^1\in H_0^1(\Omega). \tag{3.11}$$

Theorem 33 *Under conditions of Theorem 31, for any $f\in H^1\left(0,t;L^2\left(\Omega\right)\right)$, there exists a unique function $v(t,y)$ satisfying initial data (3.11)*

$$v\in L^{\infty}\left(0,t;H_0^1\left(\Omega\right)\cap H^2\left(\Omega\right)\right),\quad v'\in L^{\infty}\left(0,t;H_0^1\left(\Omega\right)\right),\quad v''\in L^2\left(0,t;L^2\left(\Omega\right)\right), \tag{3.12}$$

for $t>0$, the identity holds

$$\left(\left\{\begin{array}{c} v''+\beta v'-\sum\limits_{i,j=1}^{n_1+n_2}\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial v}{\partial y_j}\right)+\sum\limits_{i=1}^{n_1+n_2}b_i\frac{\partial v'}{\partial y_i} \\ +\sum\limits_{i=1}^{n_1+n_2}c_i\frac{\partial v}{\partial y_i}+\gamma\left(t\right)\left|v\right|^{\rho}v \end{array}\right\},w\left(t\right)\right)=\left(\Phi,w\right)\left(t\right) \tag{3.13}$$

where w is an arbitrary function from $L^2\left(\Omega\right)$.

Proof. For small $\varepsilon > 0$ we consider in a cylinder Q the following nonlinear problem

$$\begin{cases} K_\varepsilon v_\varepsilon'' + \beta v_\varepsilon' - \sum\limits_{i,j=1}^{n_1+n_2} \dfrac{\partial}{\partial y_i}\left(a_{ij}\dfrac{\partial v_\varepsilon}{\partial y_j}\right) + \sum\limits_{i=1}^{n_1+n_2} b_i \dfrac{\partial v_\varepsilon'}{\partial y_i} + \sum\limits_{i=1}^{n_1+n_2} c_i \dfrac{\partial v}{\partial y_i} + \gamma(t)\left|v_\varepsilon\right|^\rho v_\varepsilon = \Phi & \text{in } Q, \\ v_\varepsilon = 0 & \text{on } \partial\Omega \times (0,t), \\ v_\varepsilon(0) = v^0(Y_1) = u^0(K(0)Y_1) = u_0(Y_1) & Y_1 \in (-\ell_0, \ell_0)^{n_1}, \\ v_\varepsilon'(0) = v^1(y) = u^1(y) + \dfrac{\ell}{\ell_0}\sum\limits_{i=1}^{n_1+n_2} \dfrac{\partial v^0}{\partial y_i} y_i & Y_1 \in (-\ell_0, \ell_0)^{n_1}. \end{cases} \tag{3.14}$$

where $K_\varepsilon = 1 + \varepsilon$.

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis in $H_0^2(\Omega)$. For each $m \in \mathbb{N}$ we define

$$v_{m,\varepsilon}(t,y) = \sum_{r=1}^{m} \Psi_{rm\varepsilon}(t)\, w_r(y)$$

where unknown functions $\Psi_{rm\varepsilon}(t)$ are solutions to the following Cauchy problem for the system of ordinary differential equations

$$\begin{aligned} &(K_\varepsilon v_{m,\varepsilon}'', w_r) + \beta(v_{m,\varepsilon}', w_r) + a(t, v_{m,\varepsilon}, w_r) - 2\frac{K'}{K}\sum_{i=1}^{n_1+n_2}\left(y_i \frac{\partial v_{m,\varepsilon}'}{\partial y_i}, w_r\right) \\ &+ \sum_{i=1}^{n_1+n_2}\left(c_i(t)\frac{\partial v_{m,\varepsilon}}{\partial y_i}, w_r\right) + \left(\gamma(t)\left|v_{m,\varepsilon}\right|^\rho v_{m,\varepsilon}, w_r\right) \\ &= (\Phi, w_r) \qquad\qquad 1 \le r \le m, \end{aligned} \tag{3.15}$$

$$\Psi_{rm\varepsilon}(0) = \left(v^0, w_r\right), \qquad \Psi_{rm\varepsilon}'(0) = \left(v^1, w_r\right).$$

This problem has solutions $\Psi_{rm\varepsilon} \in \mathcal{C}^2((0, T_{m\varepsilon}))$, $0 < T_{m\varepsilon} < T$.

A priori estimate 1. In our calculations we will omit indices m, ε. Multiplying (3.15) by $2\Psi_r'$, summing over r, using the hypothesis $H.2$, (3.6) and(3.7) we get

$$\begin{aligned} &\frac{d}{dt}\left[\left|\sqrt{K_\varepsilon} v'(t)\right|^2 + a(t, v(t), v(t)) + (\gamma(t), N(u))\right] + 2\beta\left|v'(t)\right|^2 - (\gamma'(t), N(u)) \\ &- a'(t, v(t), v(t)) - 4\frac{K'(t)}{K(t)}\sum_{i=1}^{n_1+n_2}\left(\frac{\partial v'}{\partial y_i}, v' y_i\right) + 2\sum_{i=1}^{n_1+n_2}\left(c_i(s)\frac{\partial v}{\partial y_i}, v'\right) \\ &= 2(\Phi(t), v'), \end{aligned} \tag{3.16}$$

where $N(u) = \int_0^u |s|^\rho s ds \ge 0$. Integrating (3.16) from 0 to t, using the hypothesis $H.1$-$H.2$, (3.6) and (3.7), and observing that $K_\varepsilon (v')^2 \ge (v')^2 \ge 0$, we obtain

$$\left|v'(t)\right|^2 + \lambda\left\|v\right\|^2 \le C + \int_0^t \gamma_1(s)\left(\left|v_s\right|^2 + \left\|v\right\|^2(s)\right) ds + \int_0^t \left|\Phi\right|^2(s)\, ds, \tag{3.17}$$

where C is a positive constant independent of m and t, $\gamma_1 \in L^1(0,t)$. Hence, by Gronwall's Lemma

$$|v'(t)|^2 + \lambda \|v\|^2 + \beta \int_0^t |v_s|^2 \, ds \le C \tag{3.18}$$

A priori estimate 2. Now we differentiate equation (3.14) with respect to t, multiply the result by $2\Psi_r''$ and sum over r to obtain

$$\begin{aligned}&\frac{d}{dt}\left[\left|\sqrt{K_\varepsilon} v''(t)\right|^2 + a(t, v'(t), v'(t)) + 2a'(t, v(t), v'(t))\right] + 2\beta |v''(t)|^2 - 2a''(t, v(t), v'(t)) \\ &-3a'(t, v'(t), v'(t)) + 2\sum_{i=1}^{n_1+n_2}\left(\left(b_i \frac{\partial v}{\partial y_i}\right)', v''\right) + 2\sum_{i=1}^{n_1+n_2}\left(\left(c_i \frac{\partial v}{\partial y_i}\right)', v''\right) \\ &+2\left(\left(\gamma(t)|v|^\rho v\right)', v''\right) = 2\left(\Phi'(t), v''\right),\end{aligned} \tag{3.19}$$

Integrating (3.19) from 0 to t, using the hypothesis $H.1$-$H.2$, (3.6) and (3.7) and observing that $K_\varepsilon (v')^2 \ge (v')^2 \ge 0$, we have

$$\begin{aligned}&|v''(t)|^2 + \lambda \|v'(t)\|^2 + \beta \textstyle\int_0^t |v_{ss}|^2 \, ds \le C_1 + |(v''(0), v''(0))| \\ &\textstyle\int_0^t \gamma_1(s)\left(|v_{ss}|^2 + \|v_s\|^2(s)\right) ds + \int_0^t |\Phi_s|^2(s) \, ds,\end{aligned}$$

Remark 34 *We need an estimate for $v''(0)$. Putting $t = 0$ in (3.14), we obtain $|v_{tt}(0)| \le C$, where C does not depend of m and t.*

Now, using the above Remark, observing that $\gamma_1 \in L^1(0,t)$, by Gronwall's Lemma we get

$$|v''(t)|^2 + \lambda \|v(t)\|^2 + \frac{\beta}{4}\int_0^t |v_{ss}|^2 \, ds \le C. \tag{3.20}$$

where C is a positive constant independent of m and t.

Let us now study the nonlinear term. Since $\gamma \in L^1(0,t) \cap L^\infty(0,t)$, we have from (3.18) and (3.20)

$$\left|\gamma(t)|v_{m,\varepsilon}|^{\rho+1}\right|_{L^2(0,T;L^2(\Omega))} \le C. \tag{3.21}$$

By compactness arguments

$$\gamma(t)|v_{m,\varepsilon}|^\rho v_{m,\varepsilon} \to \gamma(t)|v^\varepsilon|^\rho v^\varepsilon \qquad a.e \text{ in } Q, \qquad m \to \infty. \tag{3.22}$$

From (3.21), (3.22), we conclude

$$\gamma(t)|v_{m,\varepsilon}|^\rho v_{m,\varepsilon} \to \gamma(t)|v_\varepsilon|^\rho v_\varepsilon \qquad \text{weakly in } L^2(Q). \tag{3.23}$$

From the a priori estimates obtained we can see that there exists a subsequence of $(v_{m,\varepsilon})$, which we still denote by $((v_{m,\varepsilon})_{m\in\mathbb{N}}$, such that

$$\begin{cases} v_{m,\varepsilon} \overset{*}{\rightharpoonup} v_\varepsilon \quad & \text{in } L^\infty\left(0,t;H_0^1\left(\Omega\right)\right), \\ v'_{m,\varepsilon} \overset{*}{\rightharpoonup} v'_\varepsilon \quad & \text{in } L^\infty\left(0,t;H_0^1\left(\Omega\right)\right), \\ v''_{m,\varepsilon} \rightharpoonup v''_\varepsilon \quad \text{in } L^2\left(Q\right), & K_\varepsilon v''_{m,\varepsilon} \overset{*}{\rightharpoonup} K_\varepsilon v''_\varepsilon \quad \text{in } L^\infty\left(0,t;L^2\left(\Omega\right)\right), \\ \gamma\left(t\right)\left|v_{m,\varepsilon}\right|^\rho v_{m,\varepsilon} \rightharpoonup \gamma\left(t\right)\left|v_\varepsilon\right|^\rho v_\varepsilon \quad & \text{in } L^2\left(Q\right). \end{cases}$$

Letting m tend to ∞, we deduce that

$$\begin{aligned} &(K_\varepsilon v''_\varepsilon, w) + \beta(v'_\varepsilon, w) + \left(\sum_{i=1}^{n_1+n_2} \frac{\partial}{\partial y_i}\left(a_{ij}\left(t,y\right)\frac{\partial v_\varepsilon}{\partial y_i}\right), w\right)(t) \\ &+ \left(\sum_{i=1}^{n_1+n_2} b_i \frac{\partial v'_\varepsilon}{\partial y_i}, w\right)(t) + \left(\sum_{i=1}^{n_1+n_2} c_i \frac{\partial v_\varepsilon}{\partial y_i}, w\right)(t) + \left(\gamma\left(t\right)\left|v_\varepsilon\right|^\rho v_\varepsilon, w\right)(t) \\ &= \left(\Phi, w\right)(t) \end{aligned}$$

where w is an arbitrary function from $H_0^1\left(\Omega\right)$.

Obviously, initial conditions (3.14) are satisfied. Observe that estimates obtained are also independent of ε. Therefore, by the same argument we can pass to the limit when ε goes to zero in $\{v_\varepsilon\}$. Thus we obtain a function

$$v \in L^\infty\left(0,t;H_0^1\left(\Omega\right)\right), \qquad v' \in L^\infty\left(0,t;H_0^1\left(\Omega\right)\right), \qquad v'' \in L^2\left(Q\right), \qquad v'' \in L^\infty\left(0,t;L^2\left(\Omega\right)\right)$$

satisfying the identity

$$\begin{aligned} &\sum_{i=1}^{n_1+n_2}\left(\frac{\partial}{\partial y_i}\left(a_{ij}\left(t,y\right)\frac{\partial v}{\partial y_i}\right), Z\right)(t) = \\ &\left(\left\{\Phi - v'' - \beta v' - \sum_{i=1}^{n_1+n_2}\left[b_i\frac{\partial v'}{\partial y_i} + c_i\frac{\partial v}{\partial y_i}\right] - \gamma\left(t\right)\left|v\right|^\rho v\right\}, Z\right)(t) \\ &\equiv \left(P\left(t,y\right), Z\right)(t) \end{aligned}$$

where Z is an arbitrary function from $H_0^1\left(\Omega\right)$ and $P \in L^2\left(\Omega\right)$.

It follows from the properties of a function $v(t,y)$ that $P\left(t,y\right) \in L^\infty\left(0,t;L^2\left(\Omega\right)\right)$. The theory of elliptic equations gives us

$$v \in L^\infty\left(0,T;H_0^1\left(\Omega\right)\cap H^2\left(\Omega\right)\right).$$

This completes the existence part of Theorem 33.

Uniqueness

Let v_1, v_2 be two distinct solutions to (3.13). Putting $w = 2(v_1 - v_2)$, we obtain

$$\begin{cases} \dfrac{d}{dt}\left[\left|w'(t)\right|^2 + a(t, w(t), w(t))\right] + 2\beta\left|w'(t)\right|^2 - a'(t, w(t), w(t)) + 2\left(\sum\limits_{i=1}^{n_1+n_2} b_i \dfrac{\partial w'}{\partial y_i}, w'\right) \\ +2\left(\sum\limits_{i=1}^{n_1+n_2} c_i \dfrac{\partial w}{\partial y_i}, w'\right) + 2\left(\gamma(t)\left|v_1\right|^\rho v_1 - \gamma(t)\left|v_2\right|^\rho v_2, w'\right) = 0, \\ w = 0 \qquad \text{on } \Sigma \\ w(0) = 0, \qquad w_t(0) = 0. \end{cases} \tag{3.24}$$

Green's formula gives

$$2\sum_{i=1}^{n_1+n_2}\left(b_i\frac{\partial w'}{\partial y_i}, w'\right) = -\sum_{i=1}^{n_1+n_2}\left(\frac{\partial b_i}{\partial y_i}, (w')^2\right)$$

and

$$-\sum_{i=1}^{n_1+n_2}\left(\frac{\partial b_i}{\partial y_i}, (w')^2\right) = \sum_{i=1}^{n_1+n_2}\left(2K'K^{-1}, (w')^2\right) \tag{3.25}$$

With regard to the nonlinear term, we obtain

$$\begin{aligned} & 2\left|\left(\gamma(t)\left|v_1\right|^\rho v_1 - \gamma(t)\left|v_2\right|^\rho v_2, w'\right)\right| \\ \leq\ & 2\gamma(t)\int_\Omega \left|\left(\left|v_1\right|^\rho v_1 - \left|v_2\right|^\rho v_2, w'\right)\right| dy \\ \leq\ & 2C_\rho\gamma(t)\int_\Omega \left(\left|v_1\right|^\rho v_1 - \left|v_2\right|^\rho v_2\right)|w|\left|w'\right| dy. \end{aligned} \tag{3.26}$$

Since injection $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is continuous, if $\frac{1}{n_1+n_2} + \frac{1}{2} + \frac{1}{q} = 1$ and $\rho(n_1+n_2) \leq q$, then

$$|u|_{L^p}^\rho,\ \ |v|_{L^p}^\rho \in L^{n_1+n_2}(\Omega).$$

From (3.26), we find

$$2\left|\left(\gamma(t)\left|v_1\right|^\rho v_1 - \gamma(t)\left|v_2\right|^\rho v_2, w'\right)\right| \leq 2C_\rho\gamma(t)\|w\|\left|w'\right|. \tag{3.27}$$

Integrating (3.24) from 0 to $t < \infty$, using the hypothesis $H.1 - H.2$, (3.3), (3.4), (3.24), (3.25), (3.27) and the inequality of Schwartz, we have

$$\begin{aligned} & \left|w'(t)\right|^2 + \lambda\int_\Omega\left|\nabla w(t)\right|^2 dy + 2\beta\int_0^t\left|w_s(s)\right|^2 ds \\ & + \int_0^t\int_\Omega 2(n_1+n_2)K'K^{-1}\left|w_s(s)\right|^2 dyds \\ & \leq C_\varepsilon\int_0^t\gamma_1(s)\left|w_s(s)\right|^2 ds + C_\varepsilon\int_0^t\gamma_1(s)\left|\nabla w(s)\right|^2 ds + \varepsilon\int_0^t\left|w_s(s)\right|^2 ds, \end{aligned}$$

From here

$$\left|w'(t)\right|^2+\lambda\int_\Omega\left|\nabla w(t)\right|^2dy\le C\int_0^t\gamma_2(s)\left(\left|\nabla w(s)\right|^2+\left|w_s(s)\right|^2\right)ds,$$

where $\gamma_2(t)=\max\{\gamma(t),\gamma_2(t)\}\qquad\forall t\in(0,\infty)$.

Since $\gamma_2(t)\in L^1(0,T)$, we have by Gronwall's lemma $\nabla w(t)\equiv 0$ a.e. $t\in(0,\infty)$. With $w-\Sigma=0$ we conclude that $w(t)\equiv 0$ in Q, hence $v_1=v_2$. The proof of the uniqueness part of Theorem 33 is completed.

Proof of theorem 32. Let v be the solution from Theorem 33 and u defined by (3.11). Then

$$u\in L^\infty\left(0,t;H_0^1(\Omega_s)\cap\in H^2(\Omega_s)\right),\quad u'\in L^\infty\left(0,t;H^1(\Omega_s)\right),$$
$$u''\in L^2(Q_t),\quad u''\in L^\infty\left(0,t;H_0^1(\Omega_s)\right),\quad u(0)=u^0,\qquad u'(0)=u^1.$$

If $w\in L^2(0,t;H_0^1(\Omega_s))$, let $\phi(t,y)=w(t,K(t)Y_1)$ for $(t,y)\in Q$. We note that (3.13) is valid. Changing the variable

$$X_1=K(t)Y_1,$$

we obtain (3.10) from (3.13). Let u_1, u_2 be two solutions to (3.10), and v_1,v_2 be the functions obtained through the isomorphism h.

Then u_1,u_2 are the solutions to (3.13). By the uniqueness result of Theorem 33, we have $v_1=v_2$, so $u_1=u_2$. Thus the proof of Theorem 31 is completed ■

3.3 Limit problem

We assume that the source term f becomes independent of the variables (t,X_1), i.e.

$$f(t,X_1,X_2)\to f_\infty(X_2)\qquad\text{as } t\to+\infty,$$

for some

$$f_\infty\in L^2(\omega),\tag{3.28}$$

To handle the nonlinear term, in the estimations below, we need to assume that

$$\gamma(t)\to 0\qquad\text{as } t\to+\infty,$$

The sense of these two convergences will be made precise below.

Passing formally to the limit in (3.1), we obtain the candidate limit of $u(t)$, as $t \to +\infty$, is the solution of the elliptic problem defined on ω

$$\begin{cases} -\triangle_{X_2} u_\infty = f_\infty & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega, \end{cases} \tag{3.29}$$

where is the Laplace operator in X_2 defined by $\triangle_{X_2} = \partial^2_{x'_1} + ... + \partial^2_{x'_{n_2}}$, so the limit problem to become independent of $(t; X_1)$, as $t \to +\infty$.

Remark 35 *the problem (3.29) has a unique solution $u \in H^1_0(\omega)$ and one can check easily that*

$$|\nabla_{X_2} u_\infty|_{L^2(\omega)} \leq |f_\infty|_{L^2(\omega)} \tag{3.30}$$

Remark 36 *By the Sobolev embedding theorem (Recall that $\omega \subset \mathbb{R}^{n_2}$), we have*

- *if $n_2 \in \{1, 2\}$, then $H^1(\omega) \subset L^{\rho+2}(\omega)$ for $0 < \rho \leq \infty$.*
- *if $n_2 \geq 3$, then due to (3.6) we have $0 < \rho \leq \dfrac{2}{(n_1+n_2)-2}$ which implies that $0 < \rho \leq \dfrac{2}{n_2-2}$, hence $H^1(\omega) \subset L^{\rho+2}(\omega)$.*

Therefore, under assumption (3.6), it holds that

$$|u_\infty|_{L^{\rho+2}(\omega)} \leq C_s |\nabla_{X_2} u_\infty|_{L^2(\omega)},$$

for $n_2 \geq 1$ and some constant C_s depending only on ω. Combining this inequality with (3.30) we have

$$|u_\infty|_{L^{\rho+2}(\omega)} \leq C_s |f_\infty|_{L^2(\omega)}. \tag{3.31}$$

3.4 Special cut-off functions

To estimate the converge of $u(t)$ towards u_∞, we consider the functions

$$w(t, X_1, X_2) := u(t, X_1, X_2) - u_\infty(X_2) \quad \text{and} \quad F(t, X_1, X_2) := f(t, X_1, X_2) - f_\infty(_{X_2}),$$

for $(X_1, X_2) \in \Omega_t$ and $t \geq 0$. Since u_∞ depends only on X_2, then the function w satisfies the equation

$$w'' + \beta w' - \triangle w + \gamma(t) |u|^\rho u = F \quad \text{in } Q_t, \tag{3.32}$$

with the initial conditions

$$w(0,.) = u^0(x) - u_\infty(X_2), \qquad w'(0,.) = u^1(x).$$

Remark 37 *Observe that if $u_\infty \neq 0$ on Σ_t, then $w \neq 0$ on Σ_t. As a consequence, $w(t) \notin H_0^1(\omega)$, hence it is not a valid test function for equation (3.32).*

This motivates the consideration of the next cut-o functions. For a fixed $t > 1$, let m be a positive integer with $m \leq t - 1$. On one band, we consider the sequence of sets

$$S_m^t := \{(s, X_1);\ t - m < s < t,\ |x_i| < \ell_0 + \ell(m - t + s), \quad \text{for } i = 1, ..., n_1\},$$

which is increasing in m, i.e. $S_m^t \subset S_{m+1}^t$, and satisfies

$$S_m^t \subset \bigcup_{t-m<s<t} \{s\} \times (-\ell_0 - \ell s, \ell_0 + \ell s) \subset (t - m, t) \times \mathbb{R}^{n_1}.$$

On the other hand, we consider a sequence of smooth cut-off functions, depending on $(0, X_1)$

$$\varrho_m = \varrho_m(s, X_1) : (0, t) \times \mathbb{R}^{n_1} \to \mathbb{R},$$

such that

$$\begin{aligned} \varrho_m &= \begin{cases} 1 & \text{in } S_m^t, \\ 0 & \text{in } \{(0,t) \times \mathbb{R}^{n_1}\} \setminus S_{m+1}^t, \end{cases} \\ 0 &\leq \varrho_m \leq 1, \quad |\nabla_{X_1} \varrho_m|, |\varrho_m'| \leq \theta, \end{aligned}$$

where θ is a constant independent of t and m. We have in particular $\varrho_m(0, X_1) = 0$ and $\varrho_m = 0$ near the lateral boundary Σ_t. The supports of $\nabla_{X_1} \varrho_m$ and ϱ_m' are included in $S_{m+1}^t \backslash S_m^t$.

3.5 Energy estimates

3.5.1 A priori estimate

In this section, we establish some useful lemmas needed in the sequel. The first one gives an estimation for u and its derivatives.

Lemma 38 *Under the assumptions (3.5) (3.8), the solution of Problem (3.1) satisfies,*

$$\int_{\Omega_t}\left[\left|u'(t)\right|^2+\left|\nabla u(t)\right|^2+\frac{\gamma(t)}{\rho+2}\left|u(t)\right|^{\rho+2}\right]dx+\int_{Q_t}\left[\beta\left|u'\right|^2+\frac{2\left|\gamma'\right|}{\rho+2}\left|u(t)\right|^{\rho+2}\right]dxds$$
$$\leq\ C_9\left(1+|f|^2_{L^2(Q_t)}\right),\qquad \textit{for } t>0,$$

where C_9 is a positive constant independent of t.

Proof. Since the solutions u satisfies $u=0$ on Σ_t, then all the tangential derivatives of u are also vanishing on Σ_t, so $\nabla_{x,t}u=\frac{\partial u}{\partial\nu}\ \nu$, on Σ_t, which implies that

$$u'=\frac{\partial u}{\partial\nu}\nu_t,\quad \nabla u=\frac{\partial u}{\partial\nu}\nu_x,\quad \text{on } \Sigma_t.$$

Thanks to Theorem 31, we can take u' as a test function and arguing as in [3], we obtain

$$\frac{1}{2}\int_{\Omega_t}|u'(t)|^2+|\nabla u(t)|^2+\frac{\gamma(t)}{\rho+2}|u(t)|^{\rho+2}dx+\int_{Q_t}\beta|u'|^2-\frac{\gamma'}{\rho+2}|u|^{\rho+2}\,dx\,ds$$
$$=\frac{1}{2}\int_{\Omega_0}|u^1|^2+|\nabla u^0|^2+\frac{\gamma(0)}{\rho+2}|u^0|^{\rho+2}dx+\int_{Q_t}fu'\,dx\,ds$$
$$+\frac{1}{2}\int_{\Sigma_t}(\frac{\partial u}{\partial\nu})^2\ \nu_t\ (|\nu_x|^2-\nu_t^2)\ d\sigma,$$

for $t>0$. Using the fact that $|\nu_t|\leq|\nu_x|$ on Σ_t and noting that $\nu_t\leq 0$ for expanding domains, we infer that the boundary integral term in the right-hand side is non-positive. Then applying Young's inequality $fu'\leq\frac{\beta}{2}(u')^2+\frac{1}{2\beta}f^2$, we obtain

$$\int_{\Omega_t}|u'(t)|^2+|\nabla u(t)|^2+\frac{\gamma(t)}{\rho+2}|u(t)|^{\rho+2}dx+\int_{Q_t}\beta|u'|^2+\frac{2|\gamma'|}{\rho+2}|u|^{\rho+2}\,dx\,ds$$
$$\leq\int_{\Omega_0}|u^1|^2+|\nabla u^0|^2+\frac{\gamma(0)}{\rho+2}|u^0|^{\rho+2}dx+\frac{1}{\beta}\int_{Q_t}f^2\,dx\,ds.$$

This completes the proof ■

The second lemma, gives an estimation for the difference $u(t)-u_\infty$ in interior regions of Ω_t and Q_t. For simplicity, we set

$$D(t,x):=\left|w'(t,x)\right|^2+\left|\nabla w(t,x)\right|^2+\gamma(t)\left|u(t,x)\right|^{\rho+2}\quad\text{for } x\in\Omega_t,\quad t\geq 0.\tag{3.33}$$

Then we have the following energy inequality.

Lemma 39 *Under assumptions (3.5), (3.28), the solutions of Problem (3.29) and Problem (3.1) satisfy*

$$\int_{\Omega_t} D(t)\,\varrho_m^2 dx + \int_{S_m^t\times\omega} D\,dxds \le M_1 \int_{\left(S_{m+1}^t\setminus S_m^t\right)\times\omega} D\;dxds + M_1 \int_{S_{m+1}^t\times\omega} \left[F^2 + \gamma\,|u_\infty|^{\rho+2}\right] dxds,$$

for a.e $t > 0$ and M_1 is a positive constant independent of t .

Proof. To derive local energy estimates, we use ϱ_m and its proprieties.

• *A local energy identity.* Let us multiply (3.32) by $2w\varrho_m^2$, it yields

$$\frac{\partial}{\partial s}(\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww') - 2\beta\varrho_m'\varrho_m w^2 - 2\varrho_m^2|w'|^2 - 4\varrho_m'\varrho_m ww' + 2\gamma|u|^\rho uw\varrho_m^2$$
$$+ 2\varrho_m^2|\nabla w|^2 - 2\nabla\cdot(\varrho_m^2 w\nabla w) + 4\varrho_m w(\nabla\varrho_m\cdot\nabla w) = 2w\varrho_m^2 F.$$

Then, multiplying (3.32) by $2\alpha w'\varrho_m^2$, for some constant $\alpha > 0$, yields

$$\frac{\partial}{\partial s}\left(\alpha\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2 + \frac{2\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m^2\right)$$
$$- 2\alpha\varrho_m'\varrho_m|w'|^2 + 2\alpha\beta\varrho_m^2|w'|^2 - \frac{2\alpha\gamma'}{\rho+2}|u|^{\rho+2}\varrho_m^2 - \frac{4\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m'\varrho_m$$
$$- 2\alpha\varrho_m'\varrho_m|\nabla w|^2 - 2\alpha\nabla\cdot(\varrho_m^2 w'\nabla w) + 4\alpha\varrho_m w'(\nabla\varrho_m\cdot\nabla w) = 2\alpha w'\varrho_m^2 F.$$

Summing the above identities, we obtain

$$\frac{\partial}{\partial s}\left(\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww' + \alpha\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2 + \frac{2\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m^2\right)$$
$$- 2\varrho_m^2|w'|^2 + 2\alpha\beta\varrho_m^2|w'|^2 + 2\varrho_m^2|\nabla w|^2 - 2\alpha\varrho_m'\varrho_m|\nabla w|^2$$
$$+ 2\gamma|u|^{\rho+2}\varrho_m^2 - 2\gamma|u|^\rho uu_\infty\varrho_m^2 - \frac{2\alpha\gamma'}{\rho+2}|u|^{\rho+2}\varrho_m^2 - \frac{4\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m'\varrho_m$$
$$- 2\beta\varrho_m'\varrho_m w^2 - 4\varrho_m'\varrho_m ww' - 2\alpha\varrho_m'\varrho_m|w'|^2 - 2\nabla\cdot(\varrho_m^2 w\nabla w) + 4\varrho_m w(\nabla\varrho_m\cdot\nabla w)$$
$$- 2\alpha\nabla\cdot(\varrho_m^2 w'\nabla w) + 4\alpha\varrho_m w'(\nabla\varrho_m\cdot\nabla w) = 2w\varrho_m^2 F + 2\alpha w'\varrho_m^2 F.$$

Collecting the terms with derivatives of ϱ in the right-hand side of the above identity, we obtain

$$\frac{\partial}{\partial s}\left(\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww' + \alpha\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2 + \frac{2\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m^2\right)$$

$$+2(\alpha\beta-1)\varrho_m^2|w'|^2 + 2\varrho_m^2|\nabla w|^2 + 2\left(\gamma - \frac{\alpha\gamma'}{\rho+2}\right)|u|^{\rho+2}\varrho_m^2$$

$$= 2\beta\varrho_m'\varrho_m w^2 + 4\varrho_m'\varrho_m ww' + 2\alpha\varrho_m'\varrho_m|w'|^2 + 2\alpha\varrho_m'\varrho_m|\nabla w|^2$$

$$-4\varrho_m w(\nabla\varrho_m\cdot\nabla w) - 4\alpha\varrho_m w'(\nabla\varrho_m\cdot\nabla w) + 2\alpha\nabla\cdot(\varrho_m^2 w'\nabla w)$$

$$+2\nabla\cdot(\varrho_m^2 w\nabla w) + \frac{4\alpha}{\rho+2}|u|^{\rho+2}\gamma\varrho_m'\varrho_m + 2\gamma(|u|^\rho u)u_\infty\varrho_m^2$$

$$+2w\varrho_m^2 F + 2\alpha w'\varrho_m^2 F.$$

Integrating on Q_t and taking into account the fact that $\varrho_m = 0$ for $t = 0$ and on Σ_t, we end up with the identity

$$\int_{\Omega_t}\left(\beta w^2 + 2ww' + \alpha|w'|^2 + |\nabla w|^2 + \frac{2\alpha\gamma(t)}{\rho+2}|u|^{\rho+2}\right)\varrho_m^2(t)\,dx \tag{3.34}$$

$$+\int_{Q_t} 2(\alpha\beta-1)\varrho_m^2|w'|^2 + 2\varrho_m^2|\nabla w|^2 + 2\left(\gamma-\frac{\alpha\gamma'}{\rho+2}\right)|u|^{\rho+2}\varrho_m^2 dxds$$

$$=\int_{Q_t} 2\beta\varrho_m'\varrho_m w^2 + 4\varrho_m'\varrho_m ww' + 2\alpha\varrho_m'\varrho_m|w'|^2 + 2\alpha\varrho_m'\varrho_m|\nabla w|^2$$

$$+\frac{4\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m'\varrho_m dxds - \int_{Q_t} 4\varrho_m w(\nabla\varrho_m\cdot\nabla w) - 4\alpha\varrho_m w'(\nabla\varrho_m\cdot\nabla w)dxds$$

$$+\int_{Q_t} 2\gamma(|u|^\rho u)u_\infty\varrho_m^2 dxds + \int_{Q_t} 2w\varrho_m^2 F + 2\alpha w'\varrho_m^2 F dxds.$$

• *Estimate for the left-hand side of (3.34).* Using the inequality

$$2ww' \geq -\left(\beta w^2 + \frac{1}{\beta}|w'|^2\right),$$

then choosing $\alpha > 1/\beta$, we obtain

$$\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww' + \alpha\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2 \geq \delta_0\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2,$$

where $\delta_0 = (\alpha - \frac{1}{\beta}) > 0$. Integrating on Q_t , and taking into account that $\gamma' \leq 0$, we deduce that the left-hand side of (3.34) is bounded below by

$$\int_{\Omega_t}\left(\delta_0|w'(t)|^2 + \alpha|\nabla w(t)|^2 + \frac{2\alpha\gamma(t)}{\rho+2}|u(t)|^{\rho+2}\right)\varrho_m^2(t)dx$$

$$+2\int_{Q_t}\left(\beta\delta_0|w'|^2 + |\nabla w|^2 + \left(\gamma + \frac{\alpha|\gamma'|}{\rho+2}\right)|u|^{\rho+2}\right)\varrho_m^2\,dx\,ds.$$

• *Estimate for the right-hand side of (3.34).* Given that the supports of ϱ_m' and $|\nabla\varrho_m|$ are included in the set $S_{m+1}^t\backslash S_m^t$, the right-hand side of (3.34) can be estimated above by

$$c_0\int_{(S_{m+1}^t\backslash S_m^t)\times\omega}|w'|^2+|w|^2+|\nabla w|^2+\gamma|u|^{\rho+2}\,dx\,ds$$
$$+\int_{Q_t}2\gamma(|u|^\rho u)u_\infty\varrho_m^2\,dx\,ds+\int_{Q_t}2w\varrho_m^2F+2\alpha w'\varrho_m^2F\,dx\,ds.$$

Here and in the sequel, c_i denotes positive constants depending (at most) on θ,α and ω, but not on t. To estimate the second integral, containing $(|u|^\rho u)u_\infty$, we apply Young's inequality $ab\le\frac{\varepsilon a^p}{p}+\frac{1}{\varepsilon^{q/p}}\frac{b^q}{q}$ for $p=\frac{\rho+2}{\rho+1}$, $q=\rho+2$ and $\varepsilon\in(0,1)$. We obtain

$$(|u|^\rho u)u_\infty\le\frac{(\rho+1)\varepsilon}{\rho+2}|u|^{\rho+2}+\frac{1}{(\rho+2)\varepsilon^{(\rho+1)}}|u_\infty|^{\rho+2}.$$

The same inequality, for $p=q=2$, yields

$$2w\varrho_m^2F+2\alpha w'F\le\varepsilon(w^2+|w'|^2)+\frac{1+\alpha^2}{\varepsilon}F^2,$$
$$2ww'\le w^2+|w'|^2,$$
$$2w|\nabla w|\le w^2+|\nabla w|^2.$$

Then, the right-hand side of (3.34) is bounded above by

$$c_0\int_{(S_{m+1}^t\backslash S_m^t)\times\omega}|w'|^2+|w|^2+|\nabla w|^2+\gamma|u|^{\rho+2}\,dx\,ds$$
$$+c_1\varepsilon\int_{Q_t}(|w'|^2+|w|^2+\gamma|u|^{\rho+2})\varrho_m^2\,dx\,ds+\frac{c_1}{\varepsilon^{(\rho+1)}}\int_{Q_t}(F^2+\gamma|u_\infty|^{\rho+2})\varrho_m^2\,dx\,ds.$$

Since ω is bounded, then Poincaré's inequality in the X_2-direction yields

$$\int_{\Omega_t}|w(t)|^2\varrho_m^2(t)dx\le c_\omega^2\int_{\Omega_t}|\nabla_{X_2}w(t)|^2\varrho_m^2(t)dx\le c_\omega^2\int_{\Omega_t}|\nabla w(t)|^2\varrho_m^2(t)dx,$$

where c_ω is the Poincaré constant. Thus the right-hand side of (3.34) is bounded above by

$$c_2\int_{(S_{m+1}^t\backslash S_m^t)\times\omega}|w'|^2+|\nabla w|^2+\gamma|u|^{\rho+2}\,dx\,ds$$
$$+c_2\varepsilon\int_{Q_t}(|w'|^2+|\nabla w|^2+\gamma|u|^{\rho+2})\varrho_m^2\,dx\,ds+\frac{c_2}{\varepsilon^{(\rho+1)}}\int_{Q_t}(F^2+\gamma|u_\infty|^{\rho+2})\varrho_m^2\,dx\,ds.$$

• *End of proof.* The estimations of the two sides of (3.34) yields

$$\int_{\Omega_t}\left(\delta_0|w'(t)|^2+\alpha|\nabla w(t)|^2+\frac{2\alpha\gamma(t)}{\rho+2}|u(t)|^{\rho+2}\right)\varrho_m^2(t)dx$$
$$+2\int_{Q_t}\left(\beta\delta_0|w'|^2+|\nabla w|^2+\left(\gamma+\frac{\alpha|\gamma'|}{\rho+2}\right)|u|^{\rho+2}\right)\varrho_m^2\,dx\,ds$$
$$\leq c_2\int_{(S_{m+1}^t\setminus S_m^t)\times\omega}|w'|^2+|\nabla w|^2+\gamma|u|^{\rho+2}\,dx\,ds$$
$$+c_2\varepsilon\int_{Q_t}\left(|w'|^2+|\nabla w|^2+\gamma|u|^{\rho+2}\right)\varrho_m^2\,dx\,ds+\frac{c_2}{\varepsilon^{(\rho+1)}}\int_{Q_t}\left(F^2+\gamma|u_\infty|^{\rho+2}\right)\varrho_m^2\,dx\,ds.$$

For ε small enough, we end up with

$$\int_{\Omega_t}\left(|w'(t)|^2+|\nabla w(t)|^2+\gamma(t)|u(t)|^{\rho+2}\right)\varrho_m^2(t)dx+\int_{Q_t}(|w'|^2+|\nabla w|^2+\gamma|u|^{\rho+2})\varrho_m^2\,dx\,ds$$
$$\leq c_3\int_{(S_{m+1}^t\setminus S_m^t)\times\omega}|w'|^2+|\nabla w|^2+\gamma|u|^{\rho+2}\,dx\,ds+c_3\int_{Q_t}(F^2+\gamma|u_\infty|^{\rho+2})\varrho_m^2\,dx\,ds.$$

This completes the proof ■

Remark 40 *Thanks to Inequality (3.31), we obtain*

$$\begin{aligned}\int_{S_{m+1}^t\times\omega}\gamma|u_\infty|^{\rho+2}\varrho_m^2\,dx\,ds &= |u_\infty|_{L^{\rho+2}(\omega)}^{\rho+2}\int_{S_{m+1}^t\times\omega}\gamma\varrho_m^2\,dX_1\,ds\\ &\leq C_s^{\rho+2}|f_\infty|_{L^{\rho+2}(\omega)}^{\rho+2}\int_{S_{m+1}^t\times\omega}\gamma\varrho_m^2\,dX_1\,ds\end{aligned}$$

and since $0\leq\varrho_m\leq 1$, *we obtain*

$$\int_{Q_t}\gamma|u_\infty|^{\rho+2}\varrho_m^2\,dx\,ds\leq C_s^{\rho+2}|f_\infty|_{L^{\rho+2}(\omega)}^{\rho+2}2^{n_1}\left(\ell_0+\ell t\right)^{n_1}\int_{t-m-1}^{t}\gamma\left(s\right)ds$$

Thus

$$\int_{S_{m+1}^t\times\omega}\gamma|u_\infty|^{\rho+2}\varrho_m^2\,dx\,ds\leq C_2\left(\ell_0+\ell t\right)^{n_1}\int_{t-m-1}^{t}\gamma\left(s\right)ds$$

where C_2 *is a constant independent of* t *and* m.

3.6 Main results

In this section, we establish the convergence $u(t) \to u_\infty$, in bounded interior region of Ω_t, Q_t, under some assumptions involving the asymptotic behaviour of f and γ as $t \to \infty$.

3.6.1 Convergence theorems

Let us consider the nonnegative real function

$$g_0(t) = \sum_{j=1}^{[t]-1} \int_{S_{j+1}^t \times \omega} \left(k^j \left| f - f_\infty \right|^2 + \gamma \left| u_\infty \right|^{\rho+2} \right) dx ds, \quad t \geq 2 \tag{3.35}$$

where [.] denotes the integer part and $K = \dfrac{M_1}{M_1+1}$, ($M_1 > 0$ is the constant considered in Lemma 39). Then, we have the following convergence on $S_m^t \times \omega$.

Theorem 41 *Assume (3.5), (3.28) and*

$$g_0(t) \to 0 \quad \textit{as } t \to \infty, \tag{3.36}$$

$$t \ \left| f \right|_{L^2(Q_t)}^2 = o\left(e^{\mu_0 t} \right), \qquad \textit{as } t \to \infty \tag{3.37}$$

where $\mu_0 = \ln\left(1 + \dfrac{1}{M_1}\right)$, *then we have*

$$u'(t) \to 0, \quad \nabla_{X_1} u(t) \to 0, \quad \nabla_{X_2} u(t) \to \nabla_{X_2} u_\infty \quad \textit{in } L^2\left(S_1^t \times \omega\right), \tag{3.38}$$

$$\gamma^{\frac{1}{\rho+2}} u \to 0, \qquad \textit{in } L^{\rho+2}\left(S_1^t \times \omega\right), \tag{3.39}$$

as $t \to +\infty$. *Moreover, if* $f = f_\infty$ *and* $\gamma = 0$, *the above convergences are exponential.*

Proof. The main idea is an iteration technique on the increasing sequence of sets $S_m^t \times \omega$. First, we observe that

$$\int_{(S_{m+1}^t \setminus S_m^t) \times \omega} D \, dx \, ds = \int_{S_{m+1}^t \times \omega} D \ dx \, ds - \int_{S_m^t \times \omega} D \ dx \, ds$$

and therefore Lemma 39 yields in particular

$$(1 + C_1) \int_{S_m^t \times \omega} D \ dx \, ds \leq C_1 \int_{S_{m+1}^t \times \omega} D \, dx \, ds + C_1 \int_{S_{m+1}^t \times \omega} F^2 + \gamma |u_\infty|^{\rho+2} \, dx \, ds.$$

Since $k = \dfrac{C_1}{1+C_1}$, then $0 < k < 1$ and we can rewrite the precedent inequality as

$$\int_{S^t_m \times \omega} D \;\; dx\, ds \le k \int_{S^t_{m+1} \times \omega} D \, dx\, ds + k \int_{S^t_{m+1} \times \omega} F^2 + \gamma |u_\infty|^{\rho+2}\, dx\, ds. \qquad (3.40)$$

This is an inequality that we can iterate for $m = 1, \ldots, [t]-1$. It follows that

$$\begin{aligned}
\int_{S^t_1 \times \omega} D \;\; dx\, ds &\le k \int_{S^t_2 \times \omega} D \;\; dx\, ds + k \int_{S^t_2 \times \omega} \left(F^2 + \gamma |u_\infty|^{\rho+2}\right) dx\, ds \\
&\le k^2 \int_{S^t_3 \times \omega} D \;\; dx\, ds + \sum_{j=1}^{2} \left(k^j \int_{S^t_{1+j} \times \omega} F^2 + \gamma |u_\infty|^{\rho+2}\, dx\, ds \right) \\
&\ldots \\
&\le k^{[t]-1} \int_{S^t_{[t]} \times \omega} D \;\; dx\, ds + \sum_{j=1}^{[t]-1} \left(k^j \int_{S^t_{1+j} \times \omega} F^2 + \gamma |u_\infty|^{\rho+2}\, dx\, ds \right).
\end{aligned}$$

Note that $t - 1 < [t] \le t$ and $\mu_0 = -\ln k > 0$. Then $k^{[t]-1} = e^{([t]-1)\ln k} = e^{-\mu_0([t]-1)}$ and it follows that

$$\int_{S^t_1 \times \omega} D \;\; dx\, ds \le C_5 e^{-\mu_0 t} \int_{S^t_{[t]} \times \omega} D \;\; dx\, ds + \sum_{j=1}^{[t]-1} \left(k^j \int_{S^t_{1+j} \times \omega} F^2 + \gamma |u_\infty|^{\rho+2}\, dx\, ds \right). \quad (3.41)$$

To estimate the first integral term in the right-hand side of (3.41), we write

$$\begin{aligned}
\int_{S^t_{[t]} \times \omega} D \, dx\, ds &\le \int_{Q_t} D \, dx\, ds \\
&\le \int_{Q_t} |u'|^2 + |\nabla u|^2 + |\nabla_{X_2} u_\infty|^2 + \gamma |u|^{\rho+2}\, dx\, ds \\
&\le \int_{Q_t} |u'|^2 + |\nabla u|^2 + \gamma |u|^{\rho+2}\, dx\, ds + |\nabla_{X_2} u_\infty|^2_{L^2(\omega)} \int_0^t \left(\int_{(-\ell_0 - \ell s, \ell_0 + \ell s)^{n_1}} dX_1 \right) ds.
\end{aligned}$$

Taking into account Lemma 38 and (3.30), it follows that

$$\begin{aligned}
\int_{S^t_{[t]} \times \omega} D \;\; dx\, ds &\le c_6 t (1 + |f|^2_{L^2(Q_t)}) + \frac{2^{n_1}}{\ell(n_1+1)} |f_\infty|^2_{L^2(\omega)} (\ell_0 + \ell t)^{n_1+1} \\
&\le c_7 \left(t^{n_1+1} |f_\infty|^2_{L^2(\omega)} + t |f|^2_{L^2(Q_t)} \right)
\end{aligned}$$

for large t. Substituting this in (3.41) and expending the expression of $D(t,x)$, we obtain

$$\int_{S^t_1 \times \omega} |u'|^2 + |\nabla_{X_1} u|^2 + |\nabla_{X_2} (u - u_\infty)|^2 \, \gamma |u|^{\rho+2}\, dx\, ds \le C_8 \left(t^{n_1+1} |f_\infty|^2_{L^2(\omega)} + t |f|^2_{L^2(Q_t)} \right) e^{-\mu_0 t} + g_0(t), \qquad (3.42)$$

where g_0 is the function given by (3.35). Since (3.36) and (3.37) ensure that the left-hand side of (3.42) tends to zero, as $t \to +\infty$, then the convergences (3.38) and (3.39) follow. If $f = f_\infty$ and $\gamma = 0$ then $g_0 = 0$ and $|f|^2_{L^2(Q_t)}$ grows polynomially in time, hence the claimed exponential convergences are a consequence of (3.42).

This completes the proof of theorem ■

Remark 42 *i)-The source term f satisfies (3.37) for example when $|f|_{L^2(\Omega_t)}$ is bounded or grows polynomially in time.*

ii)-The function g_0 satisfies (3.36) if the convergences $f \to f_\infty$, $\gamma(t) \to 0$, as $t \to \infty$ are strong enough. Some examples are given below.

iii)-If $f_\infty = 0$, and by consequence $u_\infty = 0$, then g_0 does not depend on γ. In this case, Theorem 41 holds without any convergence assumption of $\gamma(t)$ towards 0.

The next corollary gives the convergence on the domain Ω_1.

Corollary 43 *Under assumptions* (**??**)–(**??**), (2.38) *and* (2.39), *we have*

$$u'(t) \to 0, \quad \nabla_{X_1} u(t) \to 0, \quad \nabla_{X_2} u(t) \to \nabla_{X_2} u_\infty \quad \text{in } L^2(\Omega_1),$$
$$\gamma(t)^{\frac{1}{\rho+2}} u(t) \to 0 \quad \text{in } L^{\rho+2}(\Omega_1),$$

as $t \to +\infty$. Moreover, if $f = f_\infty$ and $\gamma = 0$, the above convergences are exponential.

Proof. Using Lemma **??**, we have in particular for $m = 1$,

$$\begin{aligned}\int_{\Omega_1} D(t)dx &\leq \int_{\Omega_t} D(t)\varrho_1^2(t)dx \\ &\leq C_1 \int_{S_2^t \times \omega} D\, dx\, ds + C_1 \int_{S_2^t \times \omega} F^2 + \gamma |u_\infty|^{\rho+2}\, dx\, ds.\end{aligned}$$

Then we can estimate the integral $\int_{S_2^t \times \omega} D\, dx\, ds$ by using the above iteration technique for $m = 2, \ldots, [t] - 1$. Arguing as in the proof of Theorem **??**, we end up with

$$\int_{\Omega_1} D(t)dx \leq c_9 (t^{n_1+1} |f_\infty|^2_{L^2(\omega)} + t|f|^2_{L^2(Q_t)}) e^{-\mu_0 t} + g_0(t).$$

Hence the corollary follows. ■

3.6.2 Convergence in arbitrary interior regions

The assumptions (3.36) and (3.37) can be considerably weakened to involve only the asymptotic behaviours of f and γ for large t. Moreover, we show that the above convergences hold for an arbitrary interior bounded region of t and Q_t. Let O be a bounded subset of $\mathbb{R}^{n_1} \times \omega$ and a be a positive constant. Since t is increasing in time and becomes unbounded in the X_1 direction, as $t \to +\infty$, then there exists $m_0 > a$ such that

$$(t-a,t) \times O \subset\subset (t-m_0,t) \times \Omega_{m_0}, \tag{3.43}$$

and we can check that

$$(t-m_0,t) \times \Omega_{m_0} \subset\subset S^t_{2m_0} \times \omega, \qquad \text{for } t > 2m_0.$$

Let us consider the function

$$g_{m_0}(t) = \sum_{j=2m_0+1}^{[\frac{t}{2}]} \int_{S^t_{j+1} \times \omega} \left(k^j \left|f - f_\infty\right|^2 + \gamma \left|u_\infty\right|^{\rho+2}\right) dxds, \quad t \geq 2 \tag{3.44}$$

Then, we have the following convergences on $(t-a,t) \times O$.

Theorem 44 *Under the assumptions (3.5), (3.28) and*

$$g_{m_0}(t) \to 0, \qquad t \ \left|f\right|^2_{L^2(Q_t)} = o\left(e^{\frac{\mu_0}{2}t}\right) \quad \text{as } t \to \infty, \tag{3.45}$$

we have

$$\begin{aligned} u'(t) &\to 0, \quad \nabla_{X_1} u(t) \to 0, \quad \nabla_{X_2} u(t) \to \nabla_{X_2} u_\infty \quad \text{in } L^2\left((t-a;t) \times O\right), \\ \gamma^{\frac{1}{\rho+2}} u(t) &\to 0, \qquad \text{in } L^{\rho+2}\left((t-a;t) \times O\right), \end{aligned}$$

as $t \to +\infty$. Moreover, if $f = f_\infty$ and $\gamma = 0$, the above convergences are exponential.

Proof. Let us take $t > 4m_0 + 2$, i.e., $[t/2] > 2m_0$. Since $(t-a,t) \times O \subset\subset S^t_{2m_0} \times \omega$, then iterating Inequality (3.40) for $m = 2m_0, \ldots, [t/2]-1$, we obtain

$$\begin{aligned} \int_{(t-a,t)\times O} D \ dx\, ds &\leq \int_{S^t_{2m_0}\times\omega} D \ dx\, ds \\ &\leq k^{[t/2]-2m_0} \int_{S^t_{[\frac{t}{2}]}\times\omega} D \ dx\, ds + \sum_{j=2m_0+1}^{[\frac{t}{2}]} \left(k^{j-2m_0} \int_{S^t_j \times \omega} F^2 + \gamma |u_\infty|^{\rho+2} \, dx\, ds \right) \end{aligned}$$

hence

$$\int_{(t-a,t)\times O} D\,dx\,ds \le c_{10}\left(\left(t^{n_1+1}|f_\infty|^2_{L^2(\omega)} + t|f|^2_{L^2(Q_t)}\right)e^{-\frac{\mu_0}{2}t} + g_{m_0}(t)\right) \tag{3.46}$$

where $c_{10} > 0$ and g_{m_0} is defined by (3.44). Under the assumption (3.45), the right-hand side tends to zero, as $t \to +\infty$, and the theorem follows ■

Remark 45 *In contrast with g_0 defined in (3.44), by a sum that involves the values of $f - f_\infty$ and γ on $S^t_{[t]} \times \omega$ (which is identical to Q_t if t is an integer), the function g_{m0} involves only the values of $f - f_\infty$ and γ on $S^t_{[\frac{t}{2}]+1} \times \omega$ included in the strip $(\frac{t}{2} - 1, t) \times \mathbb{R}^{n_1} \times \omega$.*

Corollary 46 *Under the assumptions (3.5), (3.28) and (3.45), we have*

$$\begin{aligned} u'(t) &\to 0, \quad \nabla_{X_1}u(t) \to 0, \quad \nabla_{X_2}u(t) \to \nabla_{X_2}u_\infty \quad \text{in } L^2(O), \\ \gamma^{\frac{1}{\rho+2}}u(t) &\to 0, \qquad \text{in } L^{\rho+2}(O), \end{aligned}$$

as $t \to +\infty$. Moreover, if $f = f_\infty$ and $\gamma = 0$, the above convergences are exponential.

Proof. Using Lemma 39, we have for $m = 2m_0$ and $t > 2m_0 + 1$

$$\begin{aligned} \int_O D(t)dx &\le \int_{\Omega_t} D(t)\varrho_{2m_0}(t)dx \\ &\le C_1 \int_{S^t_{2m_0+1}\times\omega} D\,dx\,ds + C_1 \int_{S^t_{2m_0+1}\times\omega} F^2 + \gamma|u_\infty|^{\rho+2}\,dx\,ds. \end{aligned}$$

The integral $\int_{S^t_{2m_0+1}\times\omega} D\,dx\,ds$ in the right-hand side can be estimated as above by iteration for $m = 2m_0 + 1, \ldots, [t/2] - 1$. The rest of the proof is similar to the proof of Theorem 41 and hence is omitted. ■

3.6.3 Exponential convergence

We give now some assumptions on the asymptotic behaviour of γ and f for large t, other than the trivial case $f = f_\infty$ and $\gamma = 0$, that ensure an exponential rate of convergences.

Theorem 47 *Assume (3.5)- (3.28), and that*

$$\gamma(t)\,, \;\; |f - f_\infty|^2_{L^2(\Omega_t)} \le K_2 e^{-\mu_1 t}, \tag{3.47}$$

for large t and some positive constants K_2 and μ_1. Then we have

$$\left|u'(t)\right|_{L^2((t-a;t)\times O)}, \quad \left|\nabla_{X_1} u(t)\right|_{L^2((t-a;t)\times O)}, \quad \left|\nabla_{X_2} u(t) - \nabla_{X_2} u_\infty\right|_{L^2((t-a;t)\times O)} \le K_2 e^{-\mu' t},$$
$$\left|\gamma^{\frac{1}{\rho+2}} u(t)\right|_{L^{\rho+2}((t-a;t)\times O)} \le M e^{\frac{-2\mu'}{\rho+2}t},$$

for some positive constants M and μ', such that $0 < \mu' < \dfrac{\min\left(\frac{\mu_0}{2}, \mu_1\right)}{2}$.

Proof. On one hand, $|f|^2_{L^2(Q_t)}$ grows polynomially since (3.47) yields

$$f|^2_{L^2(Q_t)} \le 2\int_0^t f_\infty|^2_{L^2(\omega)}\left(\int_{(-\ell_0-\ell s,\ell_0+\ell s)^{n_1}} dX_1 + 2K_2 e^{-\mu_1 s}\right) ds \tag{3.48}$$

for large t. On the other hand, by Remark 40 we derive

$$\begin{aligned}\int_{S^t_{1+j}\times\omega} F^2 + \gamma|u_\infty|^{\rho+2}\,dx\,ds &\le \int_{t-(1+j)}^t \int_{\Omega_s} F^2\,dx\,ds + C_2(\ell t+\ell_0)^{n_1}\int_{t-(1+j)}^t \gamma(s)ds\\ &\le K_2(1+C_2(\ell t+\ell_0)^{n_1})\int_{t-(1+j)}^t e^{-\mu_1 s}ds\\ &\le K_2(1+C_2(\ell t+\ell_0)^{n_1})(1+j)e^{-\mu_1 t}\times e^{\mu_1(1+j)}\\ &\le c_{12}t^{n_1+1}e^{-\mu_1 t}\times e^{\mu_1 j},\end{aligned}$$

for large t. Since $k^j = e^{-\mu_0 j}$ then we have

$$k^j\int_{S^t_{1+j}\times\omega} F^2 + \gamma|u_\infty|^2\,dx\,ds \le c_{12}t^{n_1+1}e^{-\mu_1 t}\times e^{(\mu_1-\mu_0)j},$$

for $2m_0+1 \le j \le [t/2]$. Summing the above inequalities from $2m_0+1$ to $[t/2]$, we obtain

$$g_{m_0}(t) \le c_{12}t^{n_1+1}e^{-\mu_1 t}\sum_{j=2m_0+1}^{[t/2]} e^{(\mu_1-\mu_0)j}.$$

If $\mu_1 < \mu_0$, then the sum term in the right-hand is bounded independently of t. If $\mu_1 \ge \mu_0$, then

$$\sum_{j=2m_0+1}^{[t/2]} e^{(\mu_1-\mu_0)j} \le c_{13}te^{(\mu_1-\mu_0)\frac{t}{2}}.$$

Therefore, in both cases it holds that

$$g_{m_0}(t) \le c_{14}t^{n_1+2}e^{-\min\{\frac{\mu_0+\mu_1}{2},\mu_1\}t}, \tag{3.49}$$

for large t. The estimations (3.48) and (3.49) means that Assumption (3.45) is satisfied. Going back to (3.46) we derive that

$$\int_{(t-a,t)\times O} D(t,x)\,dx\,ds \le c_{10}(t^{n_1+1}|f_\infty|^2_{L^2(\omega)} + c_{11}t^{n_1+2})e^{-\frac{\mu_0}{2}t} + c_{14}t^{n_1+2}e^{-\min\{\frac{\mu_0+\mu_1}{2},\mu_1\}t}.$$

Expending the expression of $D(t,x)$, we end up with

$$\begin{aligned}&\int_{(t-a,t)\times O} |u'|^2 + |\nabla_{X_1}u|^2 + |\nabla_{X_2}(u-u_\infty)|^2 + \gamma|u|^{\rho+2}\,dx\,ds\\ &\le c_{15}t^{n_1+2}\ e^{-\min\{\frac{\mu_0}{2},\mu_1\}t}.\end{aligned}$$

This completes the proof ∎

Remark 48 *i)-Under assumption (3.47), the convergences in Corollary 46 are also exponential. ii)-Theorem 47 also holds if we replace the assumption (3.47) by the following one*

$$\int_{t-1}^{t}\gamma(s)\,ds,\quad \int_{t-1}^{t}\int_{\Omega_s}|f-f_\infty|^2\,dxds \le K_3e^{-\mu_2 t},$$

for large t and some positive constants K_3 and μ_2.

Remark 49 *As long as the existence result of Theorem 31 holds, we can obtain the same results as in this work for more general domains, e.g.*

$$\Omega_t = \left(\prod_{i=1}^{n_1}(-\alpha_i(t),\beta_i(t))\right)\times\omega,\qquad t\ge 0,$$

where $\alpha_i(t)$ and $\beta_i(t)$ are smooth functions satisfying

$$\alpha_i(0)+\beta_i(0)>0,\qquad \alpha_i(t),\beta_i(t)\to+\infty,\qquad \text{as } t\to+\infty$$

and their derivatives satisfy

$$0<\alpha_i'(t),\quad \beta_i'(t)<1,\qquad \text{for } i=1,...,n_1.$$

Of course, the definitions of S_m^t and ϱ_m must be adapted to this case.

General conclusion and perspectives

In this thesis, we are interested in the asymptotic behaviour for the solution of linear and nonlinear wave equation in a noncylindrical domain when it becomes unbounded, in some direction as the time t goes to infinity, and we discuss some particular cases where the rate of convergence is exponential. So we obtain the following results:

- **In the second chapter:**

1-The solution $u(t,x)$ of the problem (2.5) converge to $u_\infty(X_2)$ the solution of an elliptic problem (2.31) in interior regions of the domain.

2-Using an iteration technique introduced by [9], Guesmia in [20] for we obtain the convergences.

3-In interior regions of the domain Q_t, we obtain the convergence $u(t,x) \to u_\infty(X_2)$ when the source term $f(t,x)$ converges to $f_\infty(X_2)$ faster enough.

4-if $f(t,x) = f_\infty(X_2)$, we obtain the exponential convergence, this means that in any interior bounded region of $\mathbb{R} \times \omega$, the local energy decays to 0.

5-Other than the trivial case $f(t,x) = f_\infty(X_2)$, we give some examples of source terms where the rate of convergence depends on the limit behaviour of the source terms.

- **In the last chapter:**

1-The solution $u(t,x)$ of the problem (3.1) converge to $u_\infty(X_2)$ the solution of an elliptic problem (3.29) defined in ω.

2-A local estimates of the difference between the wave and its limit is based on the use of some special cut-off functions, depending on (t, X_1).

3-the convergence $u(t, x) \to u_\infty(X_2)$, in bounded interior region of Ω_t and Q_t, given when if the source term $f(t, x)$ converges to some $f_\infty(X_2)$ and $\gamma(t)$ converges to 0, faster enough.

4-if $f(t, x) = f_\infty(X_2)$ and $\gamma(t) = 0$, we obtain the exponential convergence.

5-We give some assumptions on the asymptotic behaviour of $f(t, x)$ and $\gamma(t)$ for large t.

6-The rate of convergence depends on the limit behaviour of the source term and on the coefficient of the nonlinear term.

The results of this chapter has appeared in [1]. The best of our knowledge in the two cases, the asymptotic behaviour of such problems has not been considered before.

Finally, in the next work we will interested to the case where the equation is Cahn Hilliard. So we will study the asymptotic behaviour for the solution of Cahn Hilliard equation in a domain becoming unbounded in some directions.

Appendix

This appendix comprehends notations for estimates and some inequalities used in this thesis. Proofs and more advanced results can be found in the standard books of functional analysis and partial differential equations, see for instance Brezis [4], Chipot [7], Evans [15].

3.7 Notations for estimates

In the process of estimates, we use C to denote various constants that can be explicitly computed in terms of known quantities. The value of C may change from line to line in a given computation.

$\overset{*}{\rightharpoonup}$: weak star convergence.

$\rightharpoonup$: weakly convergence. $\quad (f(t) = o\,(g(t)) \quad \text{as } t \to \infty) \Leftrightarrow \dfrac{f(t)}{g(t)} \to 0 \quad \text{as } t \to \infty.$

$\to$: strong convergence.

3.8 Some useful inequalities

The following inequalities are often used to derive estimates in Analysis.

A polynomial inequality

Let $1 < p < \infty$ and a, $b > 0$, then

$$a^p + b^p \le (a+b)^p \le 2^{p-1}\,(a^p + b^p)\,.$$

Young's inequality

Assume $1 < p,\, p' < \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$,Then for any $a, b > 0$, it holds

$$a\ b \le \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

It is sometimes convenient to use the form

$$a\ b \le \lambda a^p + C_\lambda b^{p'}, \qquad C_\lambda = \frac{1}{\lambda^{p-1}}.$$

The Cauchy-Schwartz inequality

$$|\langle x, y\rangle| \le |x| \ \ |y|\,, \quad \forall x, y \in \mathbb{R}^n.$$

Poincare's inequality for $H^1(\Omega)$

Let Ω be abounded connected open set in $\mathbb{R}^n$ sufficiently smooth such that $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Then we have, for some constant C depending on Ω

$$|u|^2_{L^2(\Omega)} \leq C \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2_{L^2(\Omega)},$$

for any $u \in H^1(\Omega)$ such that $\int_\Omega u(x)dx = 0$.

Green's formula

Let u, $v \in \mathcal{C}^2\left(\overline{\Omega}\right)$, then we have

$$\begin{aligned} &i)\ \int_\Omega \triangle u\ dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma, \\ &ii)\ \int_\Omega Du \cdot Dv\ dx = -\int_\Omega u \triangle v\ dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u\ d\sigma, \\ &iii)\ \int_\Omega (u \triangle v - v \triangle u)\ dx = -\int_\Omega u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}\ d\sigma, \end{aligned}$$

where $v = (\nu_1, \nu_2, ..., \nu_n)$ is the unit outward normal on $\partial\Omega$, and D is the gradient of the function.

Gronwall's inequality

Let ζ be a function of $L^\infty(0,T)$ and η of $L^1(0,T)$ where $\zeta(t) \geqslant 0$ and $\eta(t) \geqslant 0$, for almost any $t \in (0,T)$, $(T > 0)$. We assume that there is a constant C, such that

$$\zeta(t) \leqslant \int_0^t \zeta(\sigma)\eta(\sigma)d\sigma + C, \qquad \text{a.e } t \in [0,T].$$

Then

$$\zeta(t) \leqslant C \exp \int_0^t \eta(\sigma)d\sigma \quad \text{a.e } t \in [0,T].$$

Assuming that C is an increasing function of t, and that we have

$$\zeta(t) \leqslant \int_0^t \zeta(\sigma)\eta(\sigma)d\sigma + C(t), \qquad \text{a.e } t \in [0,T].$$

Then

$$\zeta(t) \leqslant C(t) \exp \int_0^t \eta(\sigma)d\sigma \quad \text{a.e } t \in [0,T].$$

Bibliography

[1] A. Aibeche and S. Hadi and A. Sengouga. Asymptotic behaviour of nonlinear wave equations in a noncylindrical domain becoming unbounded. Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 288, pp. 1-15. ISSN: 1072-6691, 2017.

[2] S. Alinhac. Hyperbolic partial differential equations, Springer, 2009.

[3] C. Bardos and G. Chen. Control and stabilization for the wave equation III: Domain with moving boundary. SIAM J. Cntrol Optim, 19:123-138, 1981.

[4] H. Brezis. Functional Analysis,sobolev spaces and partial differential equations. Springer, 2011.

[5] B. Brighi and S. Guesmia. Asymptotic behavior of solutions of hyperbolic problems on a cylindrical domain. Discrete Contin. Dyn. Syst. suppl. (Special): 160-169, 2007.

[6] M. Chipot. l goes plus infinity, Birkhäuser, 2002.

[7] M. Chipot. Elliptic Equations:An introductory course. Birkhäuser, 2009.

[8] M. Chipot and A. Rougirel. On the asymptotic behaviour of the solution of parabolic problems in cylindrical domains of large size in some directions. Discrete Contin. Dyn. Syst. Ser. B 1(3):319-338, 2001.

[9] M. Chiopt and K. Yerssain. Exponential rates of convergence by an iteration technique C. R. Math. Acad. Sci. Paris. Sér. I, 346;21-26, 2008.

[10] M. Chipot and S. Mardare. Asymptotic behavior of the stokes problem in cylinders becoming unbounded in one direction. J. Math. Pures Appl, 90(2):133-159. 2008.

[11] M. R. Clark. Existence of solutions for a nonlinear hyperbolic-parabolic equation in a non-cylindrical domain. Internal. J. Math. Sci., 19(1):151-160, 1996.

[12] J. Cooper and C. Bardos. A nonlinear wave equation in a dependent domain. J. Math. Anal. Appl., 42:29-60, 1973.

[13] J. Cooper and L. A. Medeiros. The Cauchy problem for non linear wave equations in domain with moving boundary. Ann. Super. Pisa, Sci. Fis. mat., III Ser., 26:829-838, 1972.

[14] R. F. Dautray and J. L. Lions. Analyse Mathématique et Calcule Numérique pour les Sciences et les Techniques, volume 1-6. Masson,Paris, 1984.

[15] L. C. Evans. Partial Differential Equations, volume 19 of Grad. Stud.Math. Am. Math.Soc., Providence,RI, 1998.

[16] J. Ferreira. Nonlinear hyperbolic-parabolic partial differential equation in non-cylindrical domain. Rend. Circ. Mat. Palermo (2), 44(1):135-146, 1995.

[17] J. Ferreira and N. A. Lar'kin. Global solvability of a mixed problem for a nonlinear hyperbolic-parabolic equation in a non-cylindrical domain. Portugal. Math., 53(4): 381-395, 1996.

[18] S. Guesmia. Some results on the asymptotic behavior for hyperbolic problems in cylindrical domains becoming unbounded. J. Math. Anal. Appl., 341(2):1190-1212, 2008.

[19] S. Guesmia. Some convergence results for quasilinear parabolic boundary value problems in cylindrical domain of large size. Nonlinear Anal., 70(9):3320-3331, 2009.

[20] S. Guesmia. Large time and space size behaviour of the heat equation in non-cylindrical domains. Archiv der Mathematik; 101(3)293-299, 2013.

[21] S. Guesmia Etude du comportement asymptotique de certaines équations aux dérivées partielles dans des domaines cylindriques, Thèse Université de Haute Alsace, December 2006.

[22] T. G. Ha and J. Y. Park. Global existence and uniform decay of a damped Klein-Gordon equation in a non-cylindrical domain. Nonlinear. Anal. Theory Methods Appl., Ser. A, Theory Methods, 74(2):577-584, 2011.

[23] R. Gell. Sungular perturbations of hyperbolic type. ISBN 90 6196 166 1. Amsterdam 1978.

[24] E. Knobloch, R. Kerchetnikov. Problems on time-varying domains: Formulation, dynamics and challenges. Acta Applicandae Math., 137(1):123{157, Dec. 2015.

[25] N. A. Lar'kin and M. H. Simôes. Nonlinear wave equation with a nonlinear boundary damping in a noncylindrical domain. Matematica Contempornen, 2:19-34., 2002.

[26] J. L. Lions. Quelque méthodes de résolution des problèmes aux limites non linéaires. Dunod-Gautier Villars, 1969.

[27] J. L. Lions and E. Magenes. problème aux limite non homogénes et applications, volume 1-3 of Trav.et Rech. Math., No. 17. Dunod,Paris, 1968.

[28] J. L. Lions and W. A. Strauss. Some nonlinear evolution equations. Bull. Soc.math.France, 93:43–96, 1965.

[29] L. A. Medeiros. Non-linear wave equations in domains with variable boundary. Arch. Ration. Mech. Anal., 47:47-58, 1972.

[30] M. Nakao. A difference inequality and its application to nonlinear evolution equations. J. Math. Soc. Japan. Vol. 30, No. 4, 1978.

[31] M. L. Nakao and T. Narazaki. Existence and decay of soluyions of some nonlinear wave equations in non-cylindrical domains. Math. Reports. Kyushu. Univ., 11(2):117-125, 1978.

[32] T. N. Rabello. Decay of solutions of a nonlinear hyperbolic system in noncylindrical domain. Int. J. Math. Math. Sci., 17(3): 561-570, 1994.