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Thème

Etude de certains opérateurs elliptiques et paraboliques dégénérés

présenté par

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ABSTRACT

The aim of this thesis is to study the qualitative and quantitative behavior of the solutions of evolution equations arising from elliptic and parabolic problems on unbounded domains with unbounded coefficients. We deal with some classes of elliptic operators with unbounded coefficients of the form

$$A = Q(x)\Delta + F(x) \cdot \nabla - V(x)$$

on $\mathbb{R}^N$, $(N > 2)$. Our purpose is to investigate several properties of the realization A_p of the operator A on L^p with domain $D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N) \mid Au \in L^p(\mathbb{R}^N)\}$. In particular, under suitable conditions on the coefficients, the realization A_p generates a strongly continuous analytic semigroup $\{T_p(t)\}$ for any $p \in (1, +\infty)$. On the other hand we give an explicit description of the domain of A. Moreover this semigroup is consistent, immediately compact and ultracontractive. Furthermore we study the behaviour of the heat kernel associated to the operator A and as a consequence one obtains precise estimates for the eigenfunctions. The proof is based on the relationship between the log-Sobolev inequality and the ultracontractivity of a suitable semigroup in a weighted space.

keyword: One-parameter semigroups, Elliptic operators with unbounded coefficients

Schrödinger operator, heat kernel.

Résumé

Le but de cette thèse est d'étudier le comportement qualitatif et quantitatif des solutions d'équations d'évolution issues de problèmes elliptiques et paraboliques sur des domaines non bornés à coefficients non bornés. Nous traitons certaines classes d'opérateurs elliptiques avec des coefficients non bornés de la forme

$$Au = Q(x)\,\Delta u + F(x) \cdot \nabla u - V(x)u,$$

où tous les coefficients Q, F et V sont non bornés à l'infini. Par un argument d'approximation, on peut trouver un semigroupe solution du problème parabolique associé à l'opérateur A. Notre principal but est de montrer qu'on peut prolonger ce semi-groupe en un semi-groupe fortement continu (analytique) sur les espaces L^p. D'autre part, nous donnons une description explicite du domaine de l'operateur A. De plus, ce semigroupe est cohérent, immédiatement compact et ultracontractif. Ensuite nous étudions le comportement du noyau associé à l'opérateur A et par conséquent on obtient des estimations précises pour les fonctions propres. La preuve est basée sur la relation entre l'inégalité log-Sobolev et l'ultracontractivité d'un semigroupe approprié dans un espace avec poids.

Mots-cls: Semi-groupe, Opérateurs elliptiques à coefficients non bornés, Noyau de la chaleur.

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CHAPTER 1

General Introduction

Starting from the 1950's, linear elliptic equations arise in several models describing various phenomena in the applied sciences, the most famous being the second order stationary heat equation or, equivalently, the membrane equation. The qualitative properties of elliptic operators with unbounded coefficients in $\mathbb{R}^N$ have been investigated intensively in recent years, motivated by the important impact of these operators on stochastic processes, and their application to branches of applied sciences such as mathematical finance (where stochastic models lead to equations with unbounded coefficients, e.g., the well celebrated Black-Scholes equation, introduced in [**7**], the stochastic model for the price of a European contingent claim in the multi-factor case, considered in [**58**]), physics, biology (e.g., in study of the motion of a particle acting under a force perturbed by noise, see [**23**]). Moreover, in the study of Navier-Stokes equations in rotating domains, simple changes of variables transform operators with bounded coefficients into operators with unbounded coefficients (see e.g., [**25, 30, 32**]). Let us recall some recent results concerning elliptic operators having polynomial coefficients. We are interested in studying quantitative and qualitative properties in $L^p(\mathbb{R}^N)$, $1<p<\infty$, of the elliptic operator

$$A_{b,c}u(x) = Q(x)\Delta u(x) + F(x)\cdot\nabla u(x) - V(x)u(x), \quad x\in\mathbb{R}^N, \tag{1.1}$$

where $Q(x)=(1+|x|^\alpha)$, $F(x)=b|x|^{\alpha-2}x$, $V(x)=c|x|^\beta$, $b\in\mathbb{R}$ and $c>0$, in the case $\alpha>2$ and $\beta>\alpha-2$. Let us denote by L the operator $A_{b,c}$ with $c=0$ and illustrate the difference between the case $\alpha\in[0,2]$ and $\alpha>2$. If $\alpha\in[1,2]$ (after a modification of the drift term F near the origin, when $\alpha<2$), it is proved in [**22**] that the L^p-realization L_p of L generates an analytic semigroup in $L^p(\mathbb{R}^N)$, $1 \le p \le \infty$. Moreover, if $1 < p < \infty$, then

$$D(L_p) = \{u \in W^{2,p}(\mathbb{R}^N) : (1+|x|^\alpha)^{1/2}|\nabla u|,\ (1+|x|^\alpha)|D^2u| \in L^p(\mathbb{R}^N)\}.$$

The proof of the above result is essentially based on the a-priori estimates

$$\begin{aligned} \|(1+|x|^\alpha)^{1/2}\nabla u\|_p &\le C(\|Lu\|_p + \|u\|_p) \\ \|(1+|x|^\alpha)D^2u\|_p &\le C(\|Lu\|_p + \|u\|_p) \end{aligned}$$

for $u \in C_c^\infty(\mathbb{R}^N)$. The picture changes drastically when $\alpha > 2$. In this case G. Metafune et al. in [**46**] showed, if $\frac{N}{N-2+b} < p < \infty$, the generation of an analytic semigroup in $L^p(\mathbb{R}^N)$ which is contractive if and only if $p \ge \frac{N+\alpha-2}{N-2+b}$. Domain characterization and spectral properties have been also proved. Here the techniques are based on proving some bounds on the Green function associated to the operator L. In [**38**] (resp. [**14**]) the generation of an analytic semigroup of the L^p-realization of the Schrödinger-type operators $(1+|x|^\alpha)\Delta - |x|^\beta$ in $L^p(\mathbb{R}^N)$ for $\alpha \in [0,2]$ and $\beta > 2$ (resp. $\alpha > 2$, $\beta > \alpha - 2$) is obtained. Let us denote by A^0 the operator $A_{b,c}$ with $b = 0$, $c = 1$. In the case when $\beta = 0$ and $\alpha > 0$, generation results of analytic semigroups for suitable realizations A_p^0 of the operator A^0 in $L^p(\mathbb{R}^N)$ have been proven in [**22, 43**]. More specifically, the results in [**22**] cover the case when $\alpha \in (1,2]$ and show that the realization A_p^0 in $L^p(\mathbb{R}^N)$, with domain

$$D(A_p^0) = \{u \in W^{2,p}(\mathbb{R}^N) : (1+|\cdot|^\alpha)|D^2u|, (1+|\cdot|^\alpha)^{\frac{1}{2}}|\nabla u| \in L^p(\mathbb{R}^N)\},$$

generates a strongly continuous analytic semigroup. For $\alpha > 2$, the generation results depend upon N as it proved in [**43**]. More specifically, if $N = 1, 2$ no realization of A^0 in $L^p(\mathbb{R}^N)$ generates a strongly continuous (resp. analytic) semigroup. The same happens if $N \ge 3$ and $p \le \frac{N}{N-2}$. On the other hand, if $N \ge 3$, $p > \frac{N}{N-2}$ and $2 < \alpha \le \frac{p-1}{N-2}$, then the maximal realization

A_p^0 of the operator A^0 in $L^p(\mathbb{R}^N)$ generates a positive semigroup of contractions, which is also analytic if $\alpha < \frac{p-1}{N-2}$. Also in this case the methods for $\alpha \in [0,2]$ and $\alpha > 2$ are completely different. This is related essentially to the fact that generation of a semigroup in $L^p(\mathbb{R}^N)$ in the case $\alpha > 2$ of the operator $(1+|x|^\alpha)\Delta$ depends upon N, see [**43**], [**44**]. More recently in [**40**] the authors showed that the operator $|x|^\alpha\Delta + b|x|^{\alpha-2}x\cdot\nabla - c|x|^{\alpha-2}$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$ if and only if $s_1 + \min\{0, 2-\alpha\} < \frac{N}{p} < s_2 + \max\{0, 2-\alpha\}$, where s_i are the roots of the equation $c + s(N-2+b-s) = 0$. Moreover the domain of the generator is also characterized. At this point it is important to note that the techniques used in [**40**] are completely different from [**8**] and lead to results which are not comparable with the case $(\beta > \alpha - 2)$. Moreover, since, in the case $\alpha > 2$, the generation of a semigroup of the operator L_p depend upon N, the L^p-realization of the operator $A_{b,c}$ cannot be seen as a perturbation of L_p as one of our main results shows, see Theorem 5. We denote by A_p the realization of $A_{b,c}$ in $L^p(\mathbb{R}^N)$ endowed with the maximal domain

$$D_{p,max}(A_{b,c}) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \;:\; A_{b,c}u \in L^p(\mathbb{R}^N)\}, \tag{1.2}$$

and assume that $\alpha > 2$, $\beta > \alpha - 2$. We note that with the assumption on β and α the operator $A_{b,c}$ has unbounded coefficients at infinity and no local singularities occur. After proving a priori estimates, we deduce that the maximal domain $D_{p,max}(A_{b,c})$ of the operator $A_{b,c}$ coincides with

$$D_p(A_{b,c}) := \{u \in W^{2,p}(\mathbb{R}^N) \;:\; Vu,\; (1+|x|^{\alpha-1})\nabla u,\; (1+|x|^\alpha)D^2u \in L^p(\mathbb{R}^N)\}.$$

The core of the thesis is Chapter 3, we show in the main result of [**8**] that, for any $1 < p < \infty$, the realization A_p of $A_{b,c}$ in $L^p(\mathbb{R}^N)$, with domain $D_p(A_{b,c})$, generates a positive strongly continuous and analytic semigroup $(T_p(t))_{t\geq 0}$ for $p \in (1,\infty)$. This semigroup is also consistent, irreducible, immediately compact and ultracontractive. The article [**9**] is devoted to study the heat kernel and associated eigenfunctions. There is certainly ample evidence that the study of the heat kernel is an area of fruitful interactions between the fields of Analysis, Probability and Geometry. The simplest statement embodying these interactions is perhaps Varadhans large deviation formula [**56**]

$$\lim_{t\to 0} -4t\log p(t,x,y) = d(x,y)^2.$$

This formula relates the heat kernel $p(t,x,y)$ to the Riemannian distance function $d(x,y)$ on a complete Riemannian manifold. A remarkable generalization was given by Hino and Ramirez (see, [**4, 31**]). Namely, in the context of Dirichlet spaces,

$$\lim_{t\to 0} -4t\log \mathbb{P}(X_0\in A; X_t\in B) = d(A,B)^2.$$

Here X_t denotes the Markov process associated with the underlying local regular Dirichlet form and d is the associated intrinsic distance. Unfortunately, these elegant statements are not very easy to use in practice. For simplicity we denote by A the operator $A_{b,1}$. Since the coefficients of the operator $A_{b,c}$ are locally regular, we know that the semigroup $T_p(\cdot)$ admits an integral representation through a heat kernel $k(t,x,y)$, i.e.

$$T_p(t)f(x) = \int_{\mathbb{R}^N} k(t,x,y)f(y)\,dy, \quad t>0,\, x\in\mathbb{R}^N$$

for all $f\in L^p(\mathbb{R}^N)$, cf. [**36**], [**41**]. In [**38**] (resp. [**15**]) estimates of the kernel $k(t,x,y)$ for $b=0$, $\alpha\in[0,2)$ and $\beta>2$ (resp. $b=0$, $\alpha>2$ and $\beta>\alpha-2$) were obtained. Even in the non-autonomous case, for a large class of second order elliptic operators with unbounded coefficients, heat kernel estimates were obtained, by using the techniques of Lyapunov functions, in [**34**]. For the critical case $\beta=\alpha-2$, estimates of the heat kernel associated to the operator $(1+|x|^\alpha)\Delta - c|x|^{\alpha-2}$ for $\alpha>2$, $c>0$ and $N>2$ have been proved in [**20**]. Concerning the operator $A_{b,0}$, G. Metafune et al. in [**46**] showed that in the case where $2<\alpha\le 4+\frac{2b}{N-2}$ the kernal p with respect to the Lebesgue measure dx satisfies the following estimates

$$p(x,y,t) \leq \frac{C}{t^{\frac{N}{2}}}(1+|x|^\alpha)^{-\frac{1}{2}\frac{(N-2)(N+b-2)}{2N-4+b}}(1+|y|^\alpha)^{-\frac{1}{2}\frac{(N-2)(N+b-2)}{2N-4+b}-1+\frac{b}{\alpha}}$$

for every $t > 0$, x, $y \in \mathbb{R}^N$. The aim of chapter 4 is to show that the eigenfunction Φ of A associated to the largest eigenvalue λ_0 can be estimated from below and above by the function

$$|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}(1+|x|^\alpha)^{-\frac{b}{2\alpha}}e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr}$$

for $|x|$ and $|y|$ sufficiently large, next by means of a suitable multiplication operator $Tu = \phi u$, we rewrite the operator A in the following form

$$A = T^{-1}HT,$$

where $H = (1+|x|^\alpha)\Delta - U$ with $U = (1+|x|^\alpha)\frac{\Delta\phi}{\phi} + |x|^\beta$ and use an associated positive, closed, symmetric form $h(\cdot,\cdot)$ defined on a domain $D(h)$ in an appropriate weighted Hilbert space $L^2_\mu(\mathbb{R}^N)$. This permits us to define the associated self-adjoint operator H_μ and his corresponding semigroup (e^{tH_μ}). It can be seen that (e^{tH_μ}) is given by a heat kernel k_μ. Adapting the arguments used in [**18**] and [**38**], we prove the following intrinsic ultracontractivity

$$k_\mu(t,x,y) \leq c_1 e^{\lambda_0 t}e^{c_2 t^{-\gamma}}\Phi(x)\Phi(y), \quad t > 0,\ x,\ y \in \mathbb{R}^N,$$

where c_1, c_2 are positive constant, $\gamma = \frac{\beta-\alpha+2}{\beta+\alpha-2}$ and λ_0 is the largest eigenvalue of A, provided that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Accordingly, we obtain upper bounds of the heat kernel

$$k(t,x,y) \leq c_1 e^{\lambda_0 t + c_2 t^{-\gamma}}\left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{\frac{b}{2\alpha}}\frac{(|x||y|)^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}}{1+|y|^\alpha}e^{-\frac{\sqrt{2}}{\beta-\alpha+2}\left(|x|^{\frac{\beta-\alpha+2}{2}}+|y|^{\frac{\beta-\alpha+2}{2}}\right)}$$

for $t > 0$, $|x|, |y| \geq 1$.

Organization of the Thesis The thesis is organized as follows.

Chapter 2 The main goal of this chapter is to study general properties of the semigroup in spaces of continuous functions $C_b(\mathbb{R}^N)$ such as the existence and uniqueness of solutions to the elliptic and the parabolic equation.

Chapter 3 In this chapter we give a detailed proof of the generation of analytic C_0-semigroups on $L^p(\mathbb{R}^N)$ spaces, $1 < p < \infty$, for the operators $A_{b,c}$. Firstly, we focus on the solvability of the equation $\lambda u - A_{b,c}u = f$ for $\lambda > \lambda_0$, where λ_0 is a suitable positive constant. We show that, the operator $\lambda - A_{b,c}$ is invertible on $D_{p,max}(A_{b,c})$ for $\lambda > \lambda_0$. Next, we show via weighted a-priori estimates, that, the maximal domain coincides with the weighted Sobolev space

$$D_p(A_{b,c}) := \{u \in W^{2,p}(\mathbb{R}^N) \ : \ Vu,\ (1+|x|^{\alpha-1})\nabla u,\ (1+|x|^{\alpha})D^2u \in L^p(\mathbb{R}^N)\}.$$

Chapter 4 The last chapter is devoted to the proof of the heat kernel estimates through the relationship between the log-Sobolev inequality and the ultracontractivity of a suitable semigroup in a weighted space. The techniques consist in providing upper and lower estimates for the ground state of A_p corresponding to the largest eigenvalue λ_0 and adapting the arguments due to E. B. Davies.

Appendix In chapter A we introduce the necessary mathematical background. We start by recalling definitions and some basic properties of linear operators. Furthermore, we introduce the definition and properties of strongly continuous semigroups (C_0-Semigroups) and analytic semigroups. Next we collect a few basic classical results. We state some well-known (interior) Schauder estimates, and some maximum principle. In chapter B we recall, without proofs, some basic concepts ons sesquilinear forms and their associated operators and semigroups. We present in chapter C a few basic theory of logarithmic Sobolev inequalities, as adapted by Davies and Simon to yield ultracontractive bounds.

Comments This thesis contains material from the published article [**8**] and the submitted manuscript [**9**].

CHAPTER 2

Linear elliptic and parabolic problems in $C_b(\mathbb{R}^N)$

2.1. Introduction

In this chapter we describe the general properties of the elliptic and parabolic problems in spaces of continuous functions $C_b(\mathbb{R}^N)$. For more details of the results that we present here, we refer the reader to [**36**]. We consider the elliptic differential operator A defined by

$$\begin{aligned} Au(x) &= \sum_{i,j=1}^{N} a_{ij}(x) D_{ij}u(x) + \sum_{i=1}^{N} F_i(x) D_i u(x) + V(x)u(x) \qquad x \in \mathbb{R}^N \\ &= Tr(\mathfrak{A}(x) D^2 u(x)) + \langle F(x), \nabla_x u(x) \rangle + V(x)u(x). \end{aligned}$$

With $\mathfrak{A}(x) = (a_{ij}(x))_{1 \leq i,j \leq N}$. We assume the following hypotheses, on the coefficients of the operator A, which will be kept in the whole section:

(i) $a_{ij} = a_{ji}$ for any $i, j = 1, ..., N$ and the ellipticity condition

$$\sum_{i,j=1}^{N} a_{i,j}(x) \xi_i \xi_j \geq \theta(x) |\xi|^2,$$

for every x, $\xi \in \mathbb{R}^N$, with $\inf_K \theta(x) > 0$ for every compact $K \subset \mathbb{R}^N$, is satisfied.

(ii) a_{ij}, F_i and V are real-valued functions belonging to $C^{\nu}_{loc}(\mathbb{R}^N)$ for some $\nu \in (0,1)$;

(iii) there exists $c_0 \in \mathbb{R}$ such that $V(x) \leq c_0$ for any $x \in \mathbb{R}^N$.

Remark 1. *The operator A is locally uniformly elliptic, i.e. it is uniformly elliptic on every compact subset of $\mathbb{R}^N$; however it is not (globally) uniformly elliptic since we are assuming neither that θ is bounded away from 0, nor that the coefficients are bounded.*

We introduce the realization $\mathcal{A}$ of A in $C_b(\mathbb{R}^N)$ with domain $D_{max}(\mathcal{A})$ given by

$$D_{max}(\mathcal{A}) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad \text{for all} \quad 1 \le p < \infty : \mathcal{A}u \in C_b(\mathbb{R}^N)\},\ Au = \mathcal{A}u.$$

Section 2.2 is devoted to study the elliptic problem $\lambda u(x) - \mathcal{A}u(x) = f(x)$, $x \in \mathbb{R}^N$, and section 2.3 to study the parabolic problem $D_t u(t,x) - \mathcal{A}u(t,x) = 0$, $u(0,x) = f(x)$, $t > 0, x \in \mathbb{R}^N$. For more details see [**36**].

2.2. The elliptic problem $\lambda u(x) - Au(x) = f(x)$

The aim of this section is to prove that, for any $\lambda > c_0$ and any $f \in C_b(\mathbb{R}^N)$ the elliptic equation

$$\lambda u(x) - \mathcal{A}u(x) = f(x), \quad x \in \mathbb{R}^N, \tag{2.1}$$

admits a solution $u \in D_{max}(\mathcal{A})$.

Theorem 1 ([**36**], Theorem 2.1.1)**.** *For any $f \in C_b(\mathbb{R}^N)$ there exists $u \in D_{max}(\mathcal{A})$ which solves (2.1). Moreover, the following estimate holds:*

$$\|u\|_\infty \le \frac{1}{\lambda - c_0}\|f\|_\infty. \tag{2.2}$$

Finally, if $f \ge 0$, then $u \ge 0$.

Proof. For any $n \in \mathbb{N}$, we denote by A_n the realization of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary condition in $C(\bar{B}(n))$. By Proposition 19, the elliptic problem (2.1) admits a unique solution $u_n = R(\lambda, A_n)f$ in $\bigcap_{1 \le p < +\infty} W^{2,p}(B(n))$. Moreover, by (A.20), u_n satisfies the estimate

$$\|u_n\|_\infty \le \frac{1}{\lambda - c_0}\|f\|_\infty. \tag{2.3}$$

Let us prove that the sequence $\{u_n\}$ converges uniformly on compact sets, and in $W^{2,p}_{loc}(\mathbb{R}^N)$, to a function $u \in D_{max}(\mathcal{A})$ which satisfies the statement. For this purpose, consider first the case when $f \geq 0$. By the maximum principle, the functions u_n and $u_{n+1} - u_n$ are nonnegative in $B(n)$ for any $n \in \mathbb{N}$, that is the sequence $\{u_n\}$ is nonnegative and increasing. Therefore, using (2.3) it converges pointwise to some nonnegative function $u : \mathbb{R}^N \to \mathbb{R}$, u satisfying (2.2). Moreover, according to Theorem 13 and (2.3), the sequence $\{u_n\}$ is bounded in $W^{2,p}(B(R))$ for any $p \in [1, +\infty)$ and any fixed $R > 0$. Then, by the Sobolev embedding theorems (see [**1**, Theorem 5.4]), it is bounded in $C^1(\bar{B}(R))$ too, and the Ascoli-Arzela Theorem implies that it converges to u in $C(\bar{B}(R))$. Besides, applying again Theorem 13 to the function $u_n - u_m$, we deduce that u belongs to $W^{2,p}(B(R))$ and that u_n converges to u in $W^{2,p}(B(R))$, for any $p \in [1, +\infty)$. Since $\mathcal{A}u_n = \lambda u_n - f$ in $B(n)$, it follows that $u \in D_{max}(\mathcal{A})$ and $\mathcal{A}u = \lambda u - f$. This concludes the proof in the case when $f \geq 0$. For an arbitrary $f \in C_b(\mathbb{R}^N)$, it suffices to split $f = f^+ - f^-$ and

$$u_n = R(\lambda, A_n)(f^+) - R(\lambda, A_n)(f^-) := u_{n,1} - u_{n,2},$$

and to apply the previous arguments separately to the sequences $u_{n,1}$ and $u_{n,2}$. □

2.3. The parabolic problem $D_t u(t,x) - Au(t,x) = 0$, $u(0,x) = f(x)$

The main interest of this section is in the parabolic problem associated to the operator $\mathcal{A}$ define above. Consider

$$\begin{cases} D_t u(x,t) - \mathcal{A}u(t,x) = 0, & x \in \mathbb{R}^N, t > 0, \\ u(0,x) = f(x), & x \in \mathbb{R}^N \end{cases} \tag{2.4}$$

with $f \in C_b(\mathbb{R}^N)$. We prove the existence of a classical solution of the problem (2.4), i.e. a function $u \in C([0, +\infty) \times \mathbb{R}^N) \cap C^{1,2}((0, +\infty) \times \mathbb{R}^N)$, which is bounded in $[0, T] \times \mathbb{R}^N$ for any

$T > 0$ and satisfies $D_t u, D^2 u \in C^{\frac{\nu}{2},\nu}_{loc}((0,+\infty) \times \mathbb{R}^N)$. Since the coefficients of the operator A are not bounded, the classical theory does not give a solution of the problem. The solution is constructed through an approximation procedure as limit of solutions of Cauchy Dirichlet problems in suitable bounded domains and is given by a certain semigroup $T(t)$ applied to the initial datum f. Moreover one proves that the solution can be represented by the formula

$$u(t,x) = \int_{\mathbb{R}^N} p(x,y,t) f(y) dy, \quad t > 0, x \in \mathbb{R}^N, \tag{2.5}$$

where p is a positive function called the integral kernel. As above, p is obtained as limit of kernels of solutions in bounded domains.

Theorem 2 ([**36**], Theorem 2.2.1)**.** *For any $f \in C_b(\mathbb{R}^N)$, there exists a solution $u \in C([0,+\infty) \times \mathbb{R}^N)$ of the problem (2.4). The function u belongs to $C^{1+\frac{\nu}{2},2+\nu}_{loc}((0,+\infty) \times \mathbb{R}^N)$ and satisfies*

$$|u(t,x)| \leq e^{c_0 t} \|f\|_\infty \quad t > 0, \quad x \in \mathbb{R}^N. \tag{2.6}$$

For the proof we need the following proposition

Proposition 1 ([**36**], Proposition C.3.2)**.** *Under the following hypotheses*

(i) *Ω is an open set with a boundary which is uniformly of class $C^{2+2\nu}$ for some $\nu \in (0,1)$ (or, possibly, $\Omega = \mathbb{R}^N$);*

(ii) *a_{ij}, F_i $(i,j = 1,...,N)$ and V belong to $C^{2\nu}_b(\bar{\Omega})$;*

(iii) *$a_{ij} = a_{ji}$ for any $i,j = 1,...,N$ and*

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geq \mu_0|\xi|^2, \quad \xi \in \mathbb{R}^N, \, x \in \bar{\Omega},$$

for some positive constant μ_0.

For any $f \in C_b(\bar{\Omega})$ *and any* $\lambda > c_0$*, here,* $c_0 = \sup_{x\in\bar{\Omega}} V(x)$ *the problem*

$$(2.7)\qquad \begin{cases} D_t u(x,t) - Au(t,x) = 0 \quad x \in \Omega, t > 0, \\ u(t,x) = 0, \qquad x \in \partial\Omega, t > 0, \\ u(0,x) = f(x), x \in \Omega \end{cases}$$

admits a unique solution $u \in C([0,\infty) \times \bar{\Omega} \setminus (\{0\} \times \partial\Omega)) \cap C^{1,2}((0,+\infty) \times \Omega)$*, which is bounded in* $[0,T] \times \bar{\Omega}$ *for any* $T > 0$*. Moreover, it satisfies*

$$(2.8)\qquad \|u(t,.)\|_\infty \le e^{c_0 t}\|f\|_\infty \qquad t > 0;$$

and

$$t^{\frac{k}{2}}\|D^k T(t) f\|_\infty \le C\|f\|_\infty \qquad t \in (0,T),$$

for any $T > 0$*, some constant* $C = C(T)$*, independent of* f*, and any* $k = 1,2,3$*. Here,* $c_0 = \sup_{x\in\bar{\Omega}} V(x)$*. Moreover, there exists a unique function* $G_\Omega \in C((0,+\infty) \times \Omega \times \Omega)$ *such that*

$$u(t,x) = \int_\Omega G_\Omega(t,x,y) f(y) dy, \qquad t > 0, \quad x \in \Omega.$$

The function G_Ω *is called the fundamental solution of the problem (2.7), it is positive in* $(0,+\infty) \times \Omega \times \Omega$ *and satisfies*

$$(2.9)\qquad G_\Omega(t+s,x,y) = \int_\Omega G_\Omega(s,x,z) G_\Omega(t,z,y) dz,$$

for any $s,t > 0$ *and any* $x,y \in \Omega$*. Moreover, for any* $t > 0$ *the function* $G(t,\cdot,\cdot)$ *is bounded in* $\Omega\times\Omega$*. Finally, if* $f \in C_0(\Omega)$*, then* $u \in C([0,\infty)\times\bar{\Omega}) \cap C^{1+\nu,2+2\nu}_{loc}((0,+\infty)\times\bar{\Omega})$*; if* $f \in C^{2+\nu}_c(\mathbb{R}^N)$*, then* $u \in C^{1+\frac{\nu}{2},2+\nu}([0,T]\times\bar{\Omega})$ *for any* $T > 0$*, and, if* $f \in C^{2+k+\nu}_c(\Omega)$ *and* Ω *is of class* $C^{2+k+\nu}$ *for* *some $k \in \mathbb{N}$, then all the space derivatives of u up to the k-th order belong to $C^{1+\frac{\nu}{2},2+\nu}([0,T]\times\bar{\Omega})$ for any $T > 0$.*

Similar results hold for the Cauchy-Neumann problem. For more details and proofs about this type of problems and the nonhomogeneous case see the lectures of 20th Internet Seminar on Linear Parabolic Equations [**37**].

Proof. [Theorem 2]. We split the proof into two steps. First, we show that there exists a solution $u \in C_{loc}^{1+\frac{\nu}{2},2+\nu}((0,+\infty)\times\mathbb{R}^N)$ to the differential equation in (2.4), and it satisfies (2.6). Then, in Step 2, we show that u is continuous up to $t = 0$, and $u(0,\cdot) = f$.

Step 1. For any $n \in \mathbb{N}$, let $u_n \in C([0,\infty)\times\bar{B}(n)\setminus(\{0\}\times\partial B(n)))\cap C^{1,2}((0,+\infty)\times B(n))$ be the solution to the Cauchy-Dirichlet problem

$$\begin{cases} D_t u_n(x,t) - \mathcal{A}u_n(t,x) = 0, & x \in B(n),\ t > 0, \\ u_n(t,x) = 0, & x \in \partial B(n), t > 0, \\ u_n(0,x) = f(x), & x \in B(n) \end{cases} \tag{2.10}$$

in the ball $B(n)$, (see Proposition 1), which is given by $u_n(t,x) = (T_n(t)f)(x),\ t > 0,\ x \in B(n)$, where $\{T_n(t)\}$ is the semigroup in $C(\bar{B}(n))$ associated with the Cauchy-Dirichlet problem (2.10). From Proposition 1 (see (2.8)), for any $n \in \mathbb{N}$ we have

$$|u_n(t,x)| \le \exp(c_0 t)\|f\|_\infty, \quad t > 0, x \in B(n). \tag{2.11}$$

Now, fix $M \in \mathbb{N}$ and set $D(M) = (0,M)\times B(M)$ and $D'(M) = [\frac{1}{M}, M]\times B(M-1)$. From the interior Schauder estimate (A.23) we deduce that

$$\|u_n\|_{C^{1+\frac{\nu}{2},2+\nu}(D'(M))} \le C_M\|u_n\|_{L^\infty(D(M))} \le C_M(\exp(c_0 M)\vee 1)\|f\|_\infty \tag{2.12}$$

for any $n \geq M$, where $C_M > 0$ is a constant independent of $n \in \mathbb{N}$. Fix $\beta \in (0, \nu)$. By (2.12) there exists a subsequence $\{u_n^{(M)}\}$ of $\{u_n\}$ converging in $C^{1+\frac{\beta}{2}, 2+\beta}(D'(M))$ to some function $u_\infty^{(M)} \in C^{1+\frac{\nu}{2}, 2+\nu}(D'(M))$. Without loss of generality we can assume that $\{u_n^{(M+1)}\}$ is a subsequence of $\{u_n^{(M)}\}$. Thus, the functions $u_\infty^{(M)}$ and $u_\infty^{(M+1)}$ coincide in the domain $D'(M)$ and, therefore, we can define the function $u \in C_{loc}^{1+\frac{\nu}{2}, 2+\nu}((0, +\infty) \times \mathbb{R}^N)$ by setting $u = u_\infty^{(M)}$ in $D'(M)$. Moreover, the diagonal subsequence defined by $\tilde{u}_n = u_n^{(n)}$, $n \in \mathbb{N}$, converge to u in $C^{1+\frac{\beta}{2}, 2+\beta}([T_1, T_2] \times K)$ for any compact set $K \subset \mathbb{R}^N$ and any $0 < T_1 < T_2$. Hence, letting n go to $+\infty$ in the differential equation satisfied by $\tilde{u}_n$, it follows that u satisfies the equation $D_t u(t, x) - \mathcal{A}u(t, x) = 0$, $t > 0, x \in \mathbb{R}^N$. Besides, (2.6) follows from (2.11).

Step 2. To complete the proof we must show that $u \in C([0, +\infty) \times \mathbb{R}^N)$ and $u(0, x) = f(x)$. For this purpose, we take advantage of the semigroup theory. In particular, we will use the representation formula of solutions to Cauchy Dirichlet problems in bounded domains through semigroups. Fix $M \in \mathbb{N}$ and let ϑ be any smooth function such that $0 \leq \vartheta \leq 1$, $\vartheta \equiv 1 \in B(M-1)$, $\vartheta \equiv 0$ outside $B(M)$. For any $n \geq M$, let $v_n = \vartheta \tilde{u}_n$. As it is easily seen, the function v_n belongs to $C([0, +\infty) \times B(M))$ and is the solution of the Cauchy-Dirichlet problem

$$\begin{cases} D_t v_n(t, x) - \mathcal{A}v_n(t, x) = \psi_n(t, x), & x \in B(M), \ t > 0, \\ v_n(t, x) = 0, & x \in \partial B(M), t > 0, \\ v_n(0, x) = \vartheta(x) f(x), & x \in B(M), \end{cases} \tag{2.13}$$

where ψ_n is given by

$$\psi_n = -\sum_{i,j=1}^N a_{ij}(2 D_i \tilde{u}_n D_j \vartheta + \tilde{u}_n D_{ij}\vartheta) - \tilde{u}_n \sum_{i=1}^N F_i D_i \vartheta.$$

For any $t > 0$ and any $x \in B(M)$ we have

$$|\psi_n(t,x)| \le K_M \left(\exp(c_0 t)\|f\|_\infty + \sum_{i=1}^{N} \|D_i \tilde{u}_n(t,\cdot)\|_{L^\infty(B(M))} \right), \tag{2.14}$$

where $K_M > 0$ is such that

$$\sum_{i,j=1}^{N} \|a_{ij} D_{ij}\vartheta\|_{L^\infty(B(M))} + \sum_{i=1}^{N} \|F_i D_i \vartheta\|_{L^\infty(B(M))} \le K_M,$$
$$2\sum_{j=1}^{N} \|a_{ij} D_j \vartheta\|_{L^\infty(B(M))} \le K_M, \quad i = 1, ..., N.$$

We consider again the interior estimate of Theorem 15. By (A.24), the function $\tilde{u}_n$ satisfies the estimate

$$|\sqrt{t} D\tilde{u}_n(t,x)| \le C\|\tilde{u}_n\|_{L^\infty(D(M+1))} \le C(\exp(c_0) \vee 1)\|f\|_\infty,$$

for any $x \in B(M)$, any $t < 1 = dist(B(M), \partial B(M+1))$ and some positive constant C, independent of n. This gives

$$\|D_i \tilde{u}_n(t,\cdot)\|_{L^\infty(B(M))} \le t^{-\frac{1}{2}} C' \|f\|_\infty, \quad t \le 1,$$

for any $i = 1, ..., N$, where $C' = C(\exp(c_0) \vee 1)$. Then, by (2.14) it follows that

$$|\psi_n(t,x)| \le K'_M (1 + t^{-\frac{1}{2}})\|f\|_\infty, \quad t \in (0,1], x \in B(M), \tag{2.15}$$

for any $n > M$, where $K'_M > 0$ is a constant independent of n. Therefore, $\psi_n \in L^1(0,T,C_0(B(M)))$ and we can represent v_n by means of the variation of constants formula

$$v_n(t) = T_M(t)(\vartheta f) + \int_0^t T_M(t-s)\psi_n(s)ds, \quad t > 0$$

where, as usual, $\{T_M(t)\}$ is the semigroup in $C(\bar{B}(M))$ associated with the operator $\mathcal{A}$ with homogeneous Dirichlet conditions on $\partial B(M)$. Since $v_n \equiv \tilde{u}_n$ and $\vartheta \equiv 1$ in $B(M-1)$, by (2.11) and (2.15) it follows

$$|\tilde{u}_n(t,x) - f(x)| \leq \|T_M(t)(\vartheta f) - \vartheta f\|_\infty + K'_M \|f\|_\infty \int_0^t e^{c_0(t-s)}(1 + s^{-\frac{1}{2}})ds,$$

for any $t > 0$ and any $x \in B(M-1)$. Letting n go to $+\infty$ (and taking (2.8) into account) we get

$$|u(t,x) - f(x)| \leq \|T_M(t)(\vartheta f) - \vartheta f\|_\infty + K'_M \|f\|_\infty \int_0^t e^{c_0(t-s)}(1 + s^{-\frac{1}{2}})ds,$$

which shows that u is continuous at $t = 0$ for any $x \in B(M-1)$. Since $M \in \mathbb{N}$ is arbitrary, we have $u \in C([0,+\infty) \times \mathbb{R}^N)$ and $u(0,\cdot) \equiv f$.

□

Remark 2. *For further details on parabolic equations (the nonhomogeneous case) in the whole* $\mathbb{R}^N$*, or in* $\mathbb{R}^N_+$ *with homogeneous Dirichlet boundary conditions, where* $\mathbb{R}^N_+ = \{(x_1, ..., x_N) \in \mathbb{R}^N : x_N > 0\}$*, or in bounded smooth domains* Ω *with homogeneous Dirichlet boundary conditions, we refer the reader to the lectures of 20th Internet Seminar on Linear Parabolic Equations* [**37**].

CHAPTER 3

Elliptic operators with unbounded coefficients: Generation results

3.1. Introduction

Denotes by $A := A_{b,c}$, which will be kept in the whole chapter.

In the present chapter we want to study the elliptic operator A with unbounded diffusion, drift and potential terms

$$A = (1+|x|^\alpha)\Delta + b|x|^{\alpha-1}\frac{x}{|x|}\cdot\nabla - c|x|^\beta.$$

This operator has been studied in the paper [**8**]. In particular, we look for sufficient condition on $\alpha > 2, \beta > \alpha - 2$, $b \in \mathbb{R}$, $c > 0$ ensuring that A with domain $D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N) \,|\, Au \in L^p(\mathbb{R}^N)\}$ generates a strongly continuous analytic semigroup on $L^p(\mathbb{R}^N)$, $1 < p < \infty$. Furthermore, After proving a priori estimates, we deduce that the maximal domain $D_{p,max}$ of the operator A coincides with

$$D_p(A) := \{u \in W^{2,p}(\mathbb{R}^N)\ :\ Vu,\ (1+|x|^{\alpha-1})\nabla u,\ (1+|x|^\alpha)D^2u \in L^p(\mathbb{R}^N)\}.$$

This generalizes the recent results in [**14**] and in [**46**]. Moreover we show that $T(\cdot)$ is consistent, immediately compact and ultracontractive.

This Chapter is divided as follows. In section 3.2 we recall the solvability of the elliptic and parabolic problems in spaces of continuous functions. In Section 3.3 we introduce the definition of the reverse Hölder class and recall some results given in [**53**] and in [**14**] to study the solvability of the elliptic problem in $L^p(\mathbb{R}^N)$. In section 3.4 we prove that the maximal domain of the operator A coincides with the weighted Sobolev space $D_p(A)$, and we state and prove the main result of this chapter which deal with generation of semigroups for the operator A.

3.2. Solvability of $\lambda u - Au = f$ in $C_0(\mathbb{R}^N)$

In this short section we briefly recall some properties of the elliptic and parabolic problems associated with A in spaces of continuous functions.

Let us first consider the operator A on $C_b(\mathbb{R}^N)$ with its maximal domain

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad \text{for all} \quad 1 \le p < \infty : Au \in C_b(\mathbb{R}^N)\}.$$

It is known, cf. [**36**, Chapter 2, Section 2], that to the associated parabolic problem

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N,\ t > 0, \\ u(0,x) = f(x) & x \in \mathbb{R}^N\ , \end{cases} \tag{3.1}$$

where $f \in C_b(\mathbb{R}^N)$, one can associate a semigroup $(T(t))_{t\ge 0}$ of bounded operator in $C_b(\mathbb{R}^N)$ such that $u(t,x) = T(t)f(x)$ is a solution of (3.1) in the following sense:

$$u \in C([0,+\infty) \times \mathbb{R}^N) \cap C^{1+\frac{\sigma}{2},2+\sigma}_{loc}((0,+\infty) \times \mathbb{R}^N)$$

and u solves (3.1) for any $f \in C_b(\mathbb{R}^N)$ and some $\sigma \in (0,1)$. Moreover, in our case, the solution is unique. This can be seen by proving the existence of a Lyapunov function for A, i.e., a positive function $\varphi(x) \in C^2(\mathbb{R}^N)$ such that $\lim_{|x|\to\infty} \varphi(x) = +\infty$ and $A\varphi - \lambda\varphi \le 0$ for some $\lambda > 0$.

Proposition 2. *Assume that $\alpha \geq 0$ and $\beta > \max\{0, \alpha - 2\}$. Let $\psi = 1 + |x|^\gamma$ where $\gamma > 2$ then there exists a constant $C > 0$ such that*

$$A\psi \leq C\psi.$$

Proof. An easy computation gives

$$\begin{aligned} A\psi &= \gamma(N+\gamma-2)(1+|x|^\alpha)|x|^{\gamma-2} + b\gamma|x|^\alpha|x|^{\gamma-2} - c(1+|x|^\gamma)|x|^\beta \\ &\leq \{\gamma(N+\gamma-2) + |b|\gamma\}(1+|x|^\alpha)|x|^{\gamma-2} - c(1+|x|^\gamma)|x|^\beta. \end{aligned}$$

Since $\beta > \alpha - 2$, it follows that there exists $C > 0$ such that

$$\{\gamma(N+\gamma-2) + |b|\gamma\}(1+|x|^\alpha)|x|^{\gamma-2} \leq c(1+|x|^\gamma)|x|^\beta + C(1+|x|^\gamma).$$

Thus, ψ is a Lyapunov function for A. □

As in [**14**] one can prove the following result.

Proposition 3. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then the semigroup $(T(t))$ is generated by $(A, D_{max}(A) \cap C_0(\mathbb{R}^N))$ and maps $C_0(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$.*

Proof. Let $f \in C_0(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $C_0(\mathbb{R}^N)$, there is a sequence $(f_n) \subset C_c^\infty(\mathbb{R}^N)$ such that $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

On the other hand, it follows from Theorem 5 that the operator A_p with domain $D_{p,max}(A)$ generates an analytic semigroup $T_p(t)$ in $L^p(\mathbb{R}^N)$, and $D_p(A)$ is continuously embedded into $W^{2,p}(\mathbb{R}^N)$. Hence, by Theorem 4 and Sobolev's embedding theorem, $T(t)f_n = T_p(t)f_n \in D_p(A) \subset W^{2,p}(\mathbb{R}^N) \hookrightarrow C_0(\mathbb{R}^N)$ for $p > \frac{N}{2}$. Since $f_n \to f$ uniformly, it follows that $T(t)f_n \to T(t)f$ uniformly. Hence $T(t)f \in C_0(\mathbb{R}^N)$. □

Remark 3. *If $b > 2 - N$, then the semigroup $(T(t))$ generated by $(A, D_{max}(A) \cap C_0(\mathbb{R}^N))$ is compact. To prove this we recall that, by* [**46**, Proposition 2.2 (ii)], *the resolvent and the minimal semigroup $(S(t))$ generated by $(L, D_{max}(L)) \cap C_0(\mathbb{R}^N)$ map $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and are compact, where $L := q(x)\Delta + F(x) \cdot \nabla$. Set $v(t,x) = S(t)f(x)$ and $u(t,x) = T(t)f(x)$ for $t > 0, x \in \mathbb{R}^N$ and $0 \le f \in C_b(\mathbb{R}^N)$. Then the function $w(t,x) = v(t,x) - u(t,x)$ solves*

$$\begin{cases} w_t(x,t) = Lw(t,x) + V(x)u(t,x), & t > 0, \\ w(0,x) = 0 & x \in \mathbb{R}^N. \end{cases}$$

So, applying [**36**, Theorem 4.1.3]*, we have $w \ge 0$ and hence $T(t) \le S(t)$. Thus, $T(t)\mathbf{1} \in C_0(\mathbb{R}^N)$, for any $t > 0$ (see* [**43**, Proposition 2.2 (iii)]*). Therefore $T(t)$ is compact for all $t > 0$ (cf.* [**36**, Theorem 5.1.11]*).*

3.3. Solvability of $\lambda u - Au = f$ in $L^p(\mathbb{R}^N)$

In the previous section we have proved the existence and uniqueness of the elliptic and parabolic problems in $C_0(\mathbb{R}^N)$. In this section we study the solvability of the equation $\lambda u - A_p u = f$ for $\lambda > \lambda_0$, where λ_0 is a suitable positive constant.

Let $f \in L^p(\mathbb{R}^N)$ and consider the equation

$$\lambda u - Au = f. \tag{3.2}$$

Let $\phi = (1 + |x|^\alpha)^{b/\alpha}$, where $b \in \mathbb{R}$ is the coefficient of the drift term of A given by (1.1), and set $u = \frac{v}{\sqrt{\phi}}$. We note that the function ϕ is the function for which we have

$$\frac{1}{\phi}\mathrm{div}\,(\phi \nabla u) = \Delta u + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha} x \cdot \nabla u, \quad u \in C_c^\infty(\mathbb{R}^N).$$

A simple computation gives

$$\lambda u - Au = \frac{(1+|x|^\alpha)}{\sqrt{\phi}} \left[-\Delta v + Uv + \frac{V+\lambda}{1+|x|^\alpha} v \right] \tag{3.3}$$

where

$$U = -\frac{1}{4} \left| \frac{\nabla \phi}{\phi} \right|^2 + \frac{1}{2} \frac{\Delta \phi}{\phi}.$$

Then solving (3.2) is equivalent to solve

$$-Hv = \frac{\sqrt{\phi}}{1+|x|^\alpha} f, \tag{3.4}$$

where H is the Schrödinger operator defined by

$$H = \Delta - U - \frac{V+\lambda}{1+|x|^\alpha}.$$

If we denote by $G(x,y)$ the Green function of H, a solution of (3.4) is given by

$$v(x) = \int_{\mathbb{R}^N} G(x,y) \frac{\sqrt{\phi(y)}}{1+|y|^\alpha} f(y) dy,$$

and hence a solution of (3.2) should be

$$u(x) = Lf(x) := \frac{1}{\sqrt{\phi(x)}} \int_{\mathbb{R}^N} G(x,y) \frac{\sqrt{\phi(y)}}{1+|y|^\alpha} f(y) dy. \tag{3.5}$$

3.3.1. Boundedness in $L^p(\mathbb{R}^N)$ of L

First, we have to show that L is a bounded operator in $L^p(\mathbb{R}^N)$. For this purpose we need to estimate G.

We focus our attention to the operator H. Evaluating the potential $\mathcal{V} = U + \frac{V+\lambda}{1+|x|^\alpha}$, it follows that

$$\begin{aligned}\mathcal{V} &= \frac{|x|^{2\alpha-2}}{(1+|x|^\alpha)^2}\left(\frac{b^2}{4} - \frac{b\alpha}{2}\right) + \frac{|x|^{\alpha-2}}{1+|x|^\alpha}\frac{b}{2}(N+\alpha-2) + \frac{c|x|^\beta + \lambda}{1+|x|^\alpha} \\ &= \left(\frac{1}{1+|x|^\alpha} - \frac{1}{(1+|x|^\alpha)^2}\right)|x|^{\alpha-2}\left(\frac{b^2}{4} - \frac{b\alpha}{2}\right) + \frac{|x|^{\alpha-2}}{1+|x|^\alpha}\frac{b}{2}(N+\alpha-2) + \frac{c|x|^\beta + \lambda}{1+|x|^\alpha} \\ &= \frac{|x|^{\alpha-2}}{1+|x|^\alpha}\left(\frac{b^2}{4} + b\left(\frac{N-2}{2}\right)\right) + \frac{|x|^{\alpha-2}}{(1+|x|^\alpha)^2}\left(-\frac{1}{4}b^2 + \frac{1}{2}b\alpha\right) + \frac{c|x|^\beta}{1+|x|^\alpha} + \frac{\lambda}{1+|x|^\alpha}.\end{aligned}$$

We can choose $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ the potential $\mathcal{V}$ is positive. Indeed, since $\beta > \alpha - 2$ the function

$$\frac{|x|^{2\alpha-2}}{(1+|x|^\alpha)}\left(\frac{b^2}{4} - \frac{b\alpha}{2}\right) + |x|^{\alpha-2}\frac{b}{2}(N+\alpha-2) + c|x|^\beta$$

has a nonpositive minimum μ in $\mathbb{R}^N$. So, one takes $\lambda_0 > -\mu$.

On the other hand, since $\mathcal{V}(0) = \lambda > 0$ and $\mathcal{V}$ behaves like $|x|^{\beta-\alpha}$ as $|x| \to \infty$ we have the following estimates

$$C_1(1+|x|^{\beta-\alpha}) \leq \mathcal{V} \leq C_2(1+|x|^{\beta-\alpha}) \quad \text{if } \beta \geq \alpha, \tag{3.6}$$

$$C_3\frac{1}{1+|x|^{\alpha-\beta}} \leq \mathcal{V} \leq C_4\frac{1}{1+|x|^{\alpha-\beta}} \quad \text{if } \alpha - 2 < \beta < \alpha \tag{3.7}$$

for some positive constants C_1, C_2, C_3, C_4.

3.3.2. The reverse Hölder class B_q

Definition 1. *A nonnegative locally L^q-integrable function V on $\mathbb{R}^N$ is said to be in B_q, $1 < q < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality*

$$\left(\frac{1}{|B|}\int_B V^q(x)dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)dx\right)$$

holds for every ball B in $\mathbb{R}^N$. A nonnegative function $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is in B_∞ if

$$\|V\|_{L^\infty(B)} \le C \left(\frac{1}{|B|} \int_B V(x)dx \right)$$

for every ball B in $\mathbb{R}^N$.

At this point we can use bounds of G obtained by [**53**] in the case of positive potentials belonging to the reverse class B_q for some $q \ge N/2$.

As regards the potential $\mathcal{V}$ we can see that it belongs to $B_{N/2}$. Indeed, using (3.6), one has $\mathcal{V} \in B_\infty$ and hence $\mathcal{V} \in B_{N/2}$ if $\beta \ge \alpha$. If $\beta < \alpha$, then $\mathcal{V} \in B_q$ whenever $\beta - \alpha > -\frac{N}{q}$, and since $\beta - \alpha > -2$, we have that $\mathcal{V} \in B_{N/2}$. For more details on reverse Hölder classes we refer to [**55**, Chapter XI], [**29**, Chapter 9]. So, it follows from [**53**, Theorem 2.7] that for any $k \in \mathbb{N}$ there is a constant $C_k > 0$ such that

$$|G(x,y)| \le \frac{C_k}{(1+m(x)|x-y|)^k} \cdot \frac{1}{|x-y|^{N-2}}, \quad x,\, y \in \mathbb{R}^N, \tag{3.8}$$

where the auxiliary function m is defined by

$$\frac{1}{m(x)} := \sup_{r>0} \left\{ r \, : \, \frac{1}{r^{N-2}} \int_{B(x,r)} \mathcal{V}(y)dy \le 1 \right\}, \quad x \in \mathbb{R}^N. \tag{3.9}$$

In [**14**] a lower bound for the auxiliary function associated to the potential $\tilde{V} = \frac{|x|^\beta}{1+|x|^\alpha}$ was obtained.

Lemma 1. *Let $\alpha - 2 < \beta < \alpha$. There exists $C = C(\alpha, \beta, N)$ such that*

$$\tilde{m}(x) \ge C\,(1+|x|)^{\frac{\beta-\alpha}{2}}. \tag{3.10}$$

Proof. Fix $x \in \mathbb{R}^N$, and set $f_x(r) = \frac{1}{r^{N-2}} \int_{B(x,r)} \tilde{V}(y)dy$, $r > 0$. Since $\tilde{V} \in B_{N/2}$ implies $V \in B_q$ for some $q > \frac{N}{2}$, by [**53**, Lemma 1.2], we have

$$\lim_{r\to 0} f_x(r) = 0 \text{ and } \lim_{r\to\infty} f_x(r) = \infty.$$

Thus, $0 < \tilde{m}(x) < \infty$.

In order to estimate $\frac{1}{\tilde{m}(x)}$ we need to find $r_0 = r_0(x)$ such that $r \in [r_0, \infty[$ implies $f_x(r) \geq 1$. In this case we will have $\frac{1}{\tilde{m}(x)} \leq r_0$.

Since $\tilde{V} \in B_{N/2}$, there exists a constant C_1 depending only α, β, N such that

$$\left(\frac{1}{|B|}\int_B \tilde{V}^{N/2}(y)dy\right)^{2/N} \leq C_1 \left(\frac{1}{|B|}\int_B \tilde{V}(y)\, dy\right)$$

for any ball B in $\mathbb{R}^N$. Then we have

$$\begin{aligned} f_x(r) &= N^{-1}\sigma_N r^2 \frac{1}{|B(x,r)|}\int_{B(x,r)} \tilde{V}(y)dy \\ &\geq \frac{N^{-1}\sigma_N r^2}{C_1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} \tilde{V}(y)^{N/2}dy\right)^{2/N} \\ &= \frac{(N^{-1}\sigma_N)^{1-2/N}}{C_1}\left(\int_{B(x,r)} \tilde{V}(y)^{N/2}dy\right)^{2/N}, \end{aligned}$$

where σ_N is the $(N-1)$-dimensional measure of $\partial B(0,1)$. Hence, if

$$\int_{B(x,r)} \tilde{V}(y)^{N/2}dy - C_2 \geq 0\,, \tag{3.11}$$

then $f_x(r) \geq 1$, where $C_2 = C_2(\alpha,\beta,N) = \frac{C_1^{N/2}}{(N^{-1}\sigma_N)^{N/2-1}}$. Note that $\tilde{V} \geq \tilde{V}^*$ in $\mathbb{R}^N \setminus B(0,1)$ with $\tilde{V}^*(x) = \frac{1}{2}|x|^{\beta-\alpha}$. Hence,

$$\begin{aligned}
(3.12)\qquad \int_{B(x,r)} \tilde{V}(y)^{N/2}dy &\geq \int_{B(x,r)\setminus B(0,1)} \tilde{V}(y)^{N/2}dy \geq \int_{B(x,r)\setminus B(0,1)} \tilde{V}^*(y)^{N/2}dy \\
(3.13)\qquad &= \int_{B(x,r)} \tilde{V}^*(y)^{N/2}dy - \int_{B(x,r)\cap B(0,1)} \tilde{V}^*(y)^{N/2}dy \\
(3.14)\qquad &\geq \int_{B(x,r)} \tilde{V}^*(y)^{N/2}dy - \int_{B(0,1)} \tilde{V}^*(y)^{N/2}dy \\
(3.15)\qquad &= \int_{B(x,r)} \tilde{V}^*(y)^{N/2}dy - \frac{2^{1-N/2}\sigma_{N-1}}{N(2-\alpha+\beta)} \\
(3.16)\qquad &\geq N^{-1}\sigma_N r^N \inf_{B(x,r)} (\tilde{V}^*)^{N/2} - C_3(\alpha,\beta,N) \\
(3.17)\qquad &= N^{-1}\sigma_N \frac{2^{-N/2}r^N}{(|x|+r)^{\frac{\alpha-\beta}{2}N}} - C_3.
\end{aligned}$$

Let $\eta = \frac{\alpha-\beta}{2} < 1$ and $\delta > 0$ a parameter to be chosen later, and set

$$r_0 = \delta(1+|x|)^{\eta}\ .$$

By (3.12) condition (3.11) becomes

$$\begin{aligned}
\int_{B(x,r_0)} \tilde{V}(y)^{N/2}dy - C_2 &\geq N^{-1}\sigma_N \frac{2^{-N/2}r_0^N}{(|x|+r_0)^{\frac{\alpha-\beta}{2}N}} - C_2 - C_3 \\
&= N^{-1}2^{-N/2}\sigma_N \frac{\delta^N(1+|x|)^{\eta N}}{(|x|+\delta(1+|x|)^{\eta})^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&\geq N^{-1}2^{-N/2}\sigma_N \frac{\delta^N(1+|x|)^{\eta N}}{(1+|x|+\delta(1+|x|)^{\eta})^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&\geq N^{-1}2^{-N/2}\sigma_N \frac{\delta^N(1+|x|)^{\eta N}}{((\delta+1)(1+|x|))^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&= N^{-1}2^{-N/2}\sigma_N \left(\frac{\delta}{(1+\delta)^{\frac{\alpha-\beta}{2}}}\right)^N - C_4\,.
\end{aligned}$$

Since $\frac{\alpha-\beta}{2}<1$ we can choose $\delta>0$ such that $N^{-1}2^{-N/2}\sigma_N\left(\frac{\delta}{(1+\delta)^{\frac{\alpha-\beta}{2}}}\right)^N-C_4\geq 0\,.$

So, (3.11) is satisfied for $r=r_0$ and hence it is satisfied for any $r>r_0$. Thus, $f_x(r)\geq 1$ for $r>r_0$, and, hence, $\frac{1}{\tilde{m}(x)}\leq r_0=\delta(1+|x|)^{\eta}$. □

The same lower bound holds in the case $\beta\geq\alpha$ as the following lemma shows.

Lemma 2. *Let $\beta\geq\alpha$. There exists $C=C(\alpha,\beta,N)$ such that*

$$\tilde{m}(x)\geq C\,(1+|x|)^{\frac{\beta-\alpha}{2}}\,. \tag{3.18}$$

Proof. From [**53**, Lemma 1.4 (c)], there exist $C_1>0$ and $0<\eta_0<1$ such that, for $x,y\in\mathbb{R}^N$,

$$\tilde{m}(x)\geq\frac{C_1\tilde{m}(y)}{(1+|x-y|\tilde{m}(y))^{\eta_0}}\,.$$

In particular,

$$\tilde{m}(x)\geq\frac{C_1\tilde{m}(0)}{(1+|x|\tilde{m}(0))^{\eta_0}},$$

where $\frac{1}{\tilde{m}(0)}=\sup_{r>0}\{r\,:\,f_0(r)\leq 1\}$ with

$$f_0(r)=\frac{1}{r^{N-2}}\int_{B(0,r)}\frac{|z|^\beta}{1+|z|^\alpha}dz=\frac{\sigma_N}{r^{N-2}}\int_0^r\frac{\rho^{\beta+N-1}}{1+\rho^\alpha}d\rho\,.$$

We have $\frac{\sigma_N}{(\beta+N)(1+r^\alpha)}r^{\beta+2}\leq f_0(r)\leq\frac{\sigma_N}{\beta+N}r^{\beta+2}$. Since $\beta>0$ and $\beta-\alpha+2>0$ it follows that $\lim_{r\to 0}f_0(r)=0$ and $\lim_{r\to\infty}f_0(r)=\infty$. Consequently,

$$0<\sup_{r>0}\{r\,:\,f_0(r)\leq 1\}<\infty$$

and, hence, $\tilde{m}(0)=C_2$ for some constant $C_2>0$. Then

$$\tilde{m}(x)\geq\frac{C_1C_2}{(1+C_2|x|)^{\eta_0}}\geq\frac{C_3}{(1+|x|)^{\eta_0}} \tag{3.19}$$

for some constant $C_3 > 0$.

On the other hand, since $\beta \geq \alpha$, we obtain by [**14**, (3.3)] that $\tilde{V} \in B_\infty$. Then, by [**53**, Remark 2.9], we have

$$\tilde{m}(x) \geq C_5 \tilde{V}^{1/2}(x) = C_5 |x|^{\frac{\beta}{2}} (1+|x|)^{-\frac{\alpha}{2}}. \tag{3.20}$$

The thesis follows taking (3.19) and (3.20) into account. □

Since $\mathcal{V} \geq C_1 \tilde{V}$ for some positive constant C_1, we have $m(x) \geq \widetilde{m}(x)$, where $\frac{1}{\widetilde{m}(x)} := \sup_{r>0} \left\{ r : \frac{1}{r^{N-2}} \int_{B(x,r)} C_1 \tilde{V}(y) dy \leq 1 \right\}$. Replacing $\tilde{V}$ with $C_1 \tilde{V}$ in [**14**, Lemma 3.1, Lemma 3.2] we obtain $m(x) \geq C_2 (1+|x|)^{\frac{\beta-\alpha}{2}}$. So we have

Lemma 3. *Let $\alpha - 2 < \beta$. There exists $C = C(\alpha, \beta, N)$ such that*

$$m(x) \geq C (1+|x|)^{\frac{\beta-\alpha}{2}}. \tag{3.21}$$

Finally by (3.8) and the previous lemma we can estimate the Green function G

Lemma 4. *Let $G(x,y)$ denotes the Green function of the Schrödinger operator H and assume that $\beta > \alpha - 2$. Then*

$$G(x,y) \leq C_k \frac{1}{1 + |x-y|^k \, (1+|x|)^{\frac{\beta-\alpha}{2}k}} \frac{1}{|x-y|^{N-2}}, \quad x, y \in \mathbb{R}^N \tag{3.22}$$

for any $k > 0$ and some constant $C_k > 0$ depending on k.

We can prove now the boundedness in $L^p(\mathbb{R}^N)$ of the operator L given by (3.5)

Lemma 5. *Assume that $\alpha > 2$, $N > 2$ and $\beta > \alpha - 2$. Then there exists a positive constant $C = C(\lambda)$ such that for every $0 \leq \gamma \leq \beta$ and $f \in L^p(\mathbb{R}^N)$*

$$\| |x|^\gamma L f \|_p \leq C \|f\|_p. \tag{3.23}$$

Proof. Recall the function $\phi(x) = (1+|x|^\alpha)^{b/\alpha}$. Let $\Gamma(x,y) = \sqrt{\frac{\phi(y)}{\phi(x)}}\frac{G(x,y)}{1+|y|^\alpha}$, $f \in L^p(\mathbb{R}^N)$ and

$$u(x) = \int_{\mathbb{R}^N} \Gamma(x,y) f(y) dy, \quad x \in \mathbb{R}^N.$$

We have to show that

$$\||x|^\gamma u\|_p \le C\|f\|_p.$$

By setting $\Gamma_0 = \frac{G(x,y)}{1+|y|^\alpha}$, we have $\Gamma(x,y) = \left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{b/(2\alpha)} \Gamma_0(x,y)$. Moreover if we set $L_0 f(x) := \int_{\mathbb{R}^N} \Gamma_0(x,y) f(y) dy$, $x \in \mathbb{R}^N$, then [**14**, Lemma 3.4] gives

$$\||x|^\gamma L_0 f\|_p \le C\|f\|_p. \tag{3.24}$$

For $x \in \mathbb{R}^N$ let us consider the regions $E_1 := \{|x-y| \le \frac{1}{2}(1+|y|)\}$ and $E_2 := \{|x-y| > \frac{1}{2}(1+|y|)\}$ and write

$$u(x) = \int_{E_1} \Gamma(x,y) f(y) dy + \int_{E_2} \Gamma(x,y) f(y) dy =: u_1(x) + u_2(x)\,.$$

In E_1 we have $1+|y| \le 1+|x|+|x-y| \le 1+|x|+\frac{1}{2}(1+|y|)$ and hence $\frac{1}{2}(1+|y|) \le 1+|x|$. Thus,

$$\frac{1+|x|}{1+|y|} \le \frac{1+|x-y|+|y|}{1+|y|} \le \frac{3}{2} \text{ and } \frac{1+|y|}{1+|x|} \le 2\,.$$

Therefore there are constants $C, \widetilde{C} > 0$ such that $\left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{b/(2\alpha)} \le \widetilde{C}\left(\frac{1+|y|}{1+|x|}\right)^{b/2} \le C2^{|b|/2}$ and $\Gamma(x,y) \le C\Gamma_0(x,y)$ in E_1. So, we have

$$|u_1(x)| \le C\int_{\mathbb{R}^N} \Gamma_0(x,y)|f(y)|dy = CL_0(|f|)(x).$$

By (3.24) it follows that $\||x|^\gamma u_1\|_p \le C\|f\|_p$.

As regards the region E_2, we have, by Hölder's inequality,

$$||x|^\gamma u_2(x)| \le |x|^\gamma \int_{E_2} \Gamma(x,y)|f(y)|dy = \int_{E_2} \left(|x|^\gamma \Gamma(x,y)\right)^{\frac{1}{p'}} \left(|x|^\gamma \Gamma(x,y)\right)^{\frac{1}{p}} |f(y)|dy \tag{3.25}$$

$$\le \left(\int_{E_2} |x|^\gamma \Gamma(x,y)dy\right)^{\frac{1}{p'}} \left(\int_{E_2} |x|^\gamma \Gamma(x,y)|f(y)|^p dy\right)^{\frac{1}{p}} . \tag{3.26}$$

We propose to estimate first $\int_{E_2} |x|^\gamma \Gamma(x,y)dy$. In E_2 we have $1+|y| \le 2|x-y|$ and $1+|x| \le 1+|y|+|x-y| \le 3|x-y|$, then

$$\left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{b/(2\alpha)} \le \widetilde{C} \left(\frac{1+|y|}{1+|x|}\right)^{b/2} \le C|x-y|^{|b|/2}.$$

From (3.22) and by the symmetry of G it follows that

$$\begin{aligned} |x|^\gamma \Gamma(x,y) &= |x|^\gamma \left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{b/(2\alpha)} \frac{G(x,y)}{1+|y|^\alpha} \\ &\le C|x|^\gamma G(x,y)|x-y|^{|b|/2} \\ &\le C\frac{1+|x|^\beta}{|x-y|^k (1+|y|)^{k\frac{\beta-\alpha}{2}}} \frac{1}{|x-y|^{N-2-|b|/2}} \\ &\le C\frac{1}{|x-y|^{k-\beta+N-2-|b|/2}} \frac{1}{(1+|y|)^{k\frac{\beta-\alpha}{2}}}, \quad y \in E_2. \end{aligned}$$

For every $k > \beta - N + 2 + |b|/2$, taking into account that $\frac{1}{|x-y|} \le 2\frac{1}{1+|y|}$, we get

$$|x|^\gamma \Gamma(x,y) \le C\frac{1}{(1+|y|)^{k\frac{\beta-\alpha+2}{2}+N-2-\beta-|b|/2}} .$$

Since $\beta-\alpha+2>0$ we can choose k such that $\frac{k}{2}(\beta-\alpha+2)+N-2-\beta-|b|/2 > N$, then there is a constant $C_1 > 0$ such that

$$\int_{E_2} |x|^\gamma \Gamma(x,y)dy \le C\int_{\mathbb{R}^N} \frac{1}{(1+|y|)^{\frac{k}{2}(2+\beta-\alpha)+N-2-\beta-|b|/2}} dy \le C_1.$$

Moreover by (3.22) as above we have

$$\begin{aligned}|x|^\gamma \Gamma(x,y) &\le C|x|^\gamma G(x,y)|x-y|^{|b|/2}\\ &\le C\frac{1+|x|^\beta}{|x-y|^k\,(1+|x|)^{k\frac{\beta-\alpha}{2}}}\frac{1}{|x-y|^{N-2-|b|/2}}\\ &\le C\frac{1}{|x-y|^{k-\beta+N-2-|b|/2}}\frac{1}{(1+|x|)^{k\frac{\beta-\alpha}{2}}}\ .\end{aligned}$$

Taking into account that $\frac{1}{|x-y|} \le 3\frac{1}{1+|x|}$, arguing as above we obtain

$$\int_{E_2} |x|^\gamma \Gamma(x,y)dx \le C_2 \tag{3.27}$$

for some constant $C_2 > 0$. Hence (3.25) implies

$$||x|^\gamma u_2(x)|^p \le C_1^{p-1}\int_{E_2} |x|^\gamma \Gamma(x,y)|f(y)|^p dy. \tag{3.28}$$

Thus, by (3.28) and (3.27), we have

$$\begin{aligned}\||x|^\gamma u_2\|_p^p &\le C_1^{p-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x|^\gamma \Gamma(x,y)\chi_{\{|x-y|>\frac{1}{2}(1+|y|)\}}(x,y)|f(y)|^p dydx\\ &= C_1^{p-1}\int_{\mathbb{R}^N}|f(y)|^p\left(\int_{E_2}|x|^\gamma\Gamma(x,y)dx\right)dy \le C_1^{p-1}C_2\|f\|_p^p\ .\end{aligned}$$

□

Here and in Section 4 we will need the following covering result, see [**17**, Proposition 6.1].

Proposition 4. *Given a covering $\mathcal{F} = \{B(x,\rho(x))\}_{x\in\mathbb{R}^N}$ of $\mathbb{R}^N$, where $\rho : \mathbb{R}^N \to \mathbb{R}_+$ is a Lipschitz continuous function with Lipschitz constant $k < 1/2$, there exists a countable subcovering $\{B(x_n,\rho(x_n))\}_{n\in\mathbb{N}}$ of $\mathbb{R}^N$ and $\zeta = \zeta(N,k) \in \mathbb{N}$ such that at most ζ among the double balls $\{B(x_n,2\rho(x_n))\}_{n\in\mathbb{N}}$ overlap.*

Proof. We set $\rho_n = \rho(0)(1+k)^n$ for any $n \in \mathbb{N}$ and $\rho_0 = \rho(0)$. As is easily seen, the family $\{\Omega_n : n \in \mathbb{N} \cup \{0\}\}$, where $\Omega_n = B_{2\rho_n} \setminus B_{2\rho_{n-1}}$, if $n \geq 1$, and $\Omega_0 = B_{2\rho_0}$, is a covering of $\mathbb{R}^N$. By the Besicovitch covering theorem (see, e.g. [**3**, Theorem 2.18]), there exists $\xi(N) \in \mathbb{N}$, and for any $n \in \mathbb{N} \cup \{0\}$ an at most countable set $S_n \subset \Omega_n$, such that $\Omega_n \subset C_n := \bigcup_{j \in S_n} B_{\rho(x_j)}(x_j)$ for any $n \in \mathbb{N} \cup \{0\}$. Since $\rho(x) \leq \rho(0) + k|x|$ for any $x \in \mathbb{R}^N$, it follows that $C_n \subset B_{2\rho_{n+1}+\rho_0} \setminus B_{2\rho_{n-1}(1-k)-\rho_0}$ for any $n \in \mathbb{N} \cup \{0\}$.

We claim that there exists $h = h(k)$ such that $C_m \cap C_n = \varnothing$ if $n \geq m + h$. For this purpose, it suffices to show that there exists $h \in \mathbb{N}$ such that $B_{2\rho_{m+1}+\rho_0} \subset B_{2\rho_{n-1}(1-k)-\rho_0}$ if $n - m \geq h$. This is the case, if $1 < (1+k)^{m+1}((1-k)(1+k)^{n-m-2} - 1)$. Since $k < 1$, we can fix $h = h(k) \in \mathbb{N}$ such that $(1-k)(1+k)^{h-2} - 1 > 1$. With this choice of h, the chaim follows at once. We have so proved that each set C_n overlaps at most $2h$ sets from the family $\{C_m : m \in \mathbb{N} \cup \{0\}\}$.

Let us order the elements of the set $S = \bigcup_{n \in \mathbb{N} \cup \{0\}} S_n$ into a sequence $\{x_n\}$. We claim that the family $F = \{B_{\rho(x_n)}(x_n) : n \in \mathbb{N}\}$ is a subcovering of $\mathbb{R}^N$ with the properties in the statement. This family is clearly a covering of $\mathbb{R}^N$, since the union of the sets C_n covers the union of the sets Ω_n ($n \in \mathbb{N} \cup \{0\}$) which is $\mathbb{R}^N$, as has been observed. To control the overlaps of the double balls $B_{2\rho(x_n)}(x_n)$, we begin by observing that the intersection of more than $\xi(N,k) := h(k)\xi(N)$ balls of the family F is empty. Indeed, let $J \subset \mathbb{N}$ be a finite set such that $\bigcap_{j \in J} B_{\rho(x_n)}(x_n) \neq \varnothing$. Then, there exist $m \in \mathbb{N}$ and $r_1, ..., r_m \in \mathbb{N}$ such that $J_i = J \cap S_{r_i} \neq \varnothing$ and $\bigcup_{i=1} J_i = J$. Therefore, we can split $\bigcap_{j \in J} B_{\rho(x_n)}(x_n) = \bigcap_{i=1}^{m} \bigcap_{j \in J_i} B_{\rho(x_n)}(x_n)$. Since $\bigcap_{j \in J_i} B_{\rho(x_n)}(x_n) \subset C_{r_i}$, for any $i = 1, ..., m$, from the above results it follows that $m \leq h$. Moreover, since no more than $\xi(N)$ from the balls which constitute each set C_{r_i} may intersect, the cardinality of J_i does not exceed $\xi(N)$ for any $i = 1, ..., m$. Summing up, we conclude that the cardinality of J does not exceed $\xi(k, N)$. Let us fix $i, j \in \mathbb{N}$ such that $B_{2\rho(x_i)}(x_i) \cap B_{2\rho(x_j)}(x_j) \neq \varnothing$. Then, $|x_i - x_j| \leq 2(\rho(x_i) + \rho(x_j))$. Since

ρ is k-Lipschitz continuous, we deduce that $|\rho(x_i) - \rho(x_j)| \leq 2k(\rho(x_i) + \rho(x_j))$ or, equivalently,

$$\frac{1-2k}{1+2k}\rho_i \leq \rho_j \leq \frac{1+2k}{1-2k}\rho_i \tag{3.29}$$

Thus, $B_{\rho(x_j)}(x_j) \subset B_{\frac{5+2k}{1-2k}\rho_i}(x_i)$. Set $Z_i = \{j \in \mathbb{N} : B_{2\rho(x_i)}(x_i) \cap B_{2\rho(x_j)}(x_j) \neq \varnothing\}$. Then, $\bigcup_{j\in Z_i} B_{2\rho(x_j)}(x_j) \subset B_{\frac{5+2k}{1-2k}\rho_i}(x_i)$, which implies that

$$\sum_{j\in Z_i} \chi_{B_{\rho(x_j)}(x_j)} \leq \xi(N,k)\chi_{B_{\frac{5+2k}{1-2k}\rho_i}(x_i)}.$$

Integrating over $\mathbb{R}^N$ ans using the first inequality in (3.29), we obtain that

$$\left(\frac{1-2k}{1+2k}\right)^N \rho_i^N \text{card}(K_i) \leq \sum_{j\in K_i} \rho(x_j)^N \leq \xi(N,k)\left(\frac{5+2k}{1-2k}\right)^N \rho_i^N,$$

i.e.,

$$\text{card}(K_i) \leq \xi(N,k)\left(\frac{4k^2+12k+5}{(1-2k)^2}\right)^N =: \zeta(N,k) - 1.$$

This completes the proof. □

We propose now to characterize the domain $D_{p,max}(A)$.

Proposition 5. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. For $1 < p < \infty$ the following holds*

$$D_{p,max}(A) = \{u \in W^{2,p}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N)\}.$$

Proof. It suffices to prove that $D_{p,max}(A) \subset \{u \in W^{2,p}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N)\}$. Let $u \in D_{p,max}(A)$. Then $f := Au \in L^p(\mathbb{R}^N)$. This implies that

$$\widetilde{A}u := \Delta u + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha}x\cdot\nabla u - \frac{c|x|^\beta}{1+|x|^\alpha}u = \frac{f}{1+|x|^\alpha} \in L^p(\mathbb{R}^N).$$

If $\beta \leq \alpha$ then the potential $\widetilde{V}(x) := \frac{c|x|^\beta}{1+|x|^\alpha}$ is bounded and by standard regularity results for uniformly elliptic operators with bounded coefficients we deduce that $u \in W^{2,p}(\mathbb{R}^N)$.

Let us assume now that $\beta > \alpha$. Then $\widetilde{V} \in B_q$ for all $q \in (1,\infty)$. So, by [**5**, Theorem 1.1 and Corollary 1.3], we have that $D_{p,max}(\Delta - \widetilde{V}) = W^{2,p}(\mathbb{R}^N) \cap D_{p,max}(\widetilde{V})$ and the following estimate holds

$$\|\widetilde{V} f\|_p + \|\Delta f\|_p \leq C\|\Delta f - \widetilde{V} f\|_p \tag{3.30}$$

for all $f \in D_{p,max}(\Delta - \widetilde{V})$ with a constant C independent of f.

Fix now $x_0 \in \mathbb{R}^N$ and $R \geq 1$. We propose to prove the following interior estimate

$$\|\Delta u\|_{L^p(B(x_0,\frac{R}{2}))} \leq C\left(\|\widetilde{A} u\|_{L^p(B(x_0,R))} + \|u\|_{L^p(B(x_0,R))}\right) \tag{3.31}$$

with a constant C independent of u and R. To this purpose take $\sigma \in (0,1)$ and set $\sigma' := \frac{\sigma+1}{2}$. Consider a cutoff function $\vartheta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \vartheta \leq 1$, $\vartheta(x) = 1$ for $x \in B(x_0, \sigma R)$, $\vartheta(x) = 0$ for $x \in B^c(x_0, \sigma' R)$, $\|\nabla \vartheta\|_\infty \leq \frac{C}{R(1-\sigma')}$ and $\|\Delta \vartheta\|_\infty \leq \frac{C}{R^2(1-\sigma')^2}$ with a constant C independent of R.

In order to simplify the notation we write $\|\cdot\|_{p,r}$ instead of $\|\cdot\|_{L^p(B(x_0,r))}$. The function $v = u\vartheta$

belongs to $D_{p,max}(\Delta - \tilde{V})$ and so by (3.30) we have

$$
\begin{aligned}
&\|\Delta u\|_{p,\sigma R} \le \|\Delta v\|_{p,\sigma' R} \le C\|\Delta v - \widetilde{V}v\|_{p,\sigma' R} \\
\le C &\left(\|\Delta v + F\cdot\nabla v - \widetilde{V}v\|_{p,\sigma' R} + \|F\cdot\nabla v\|_{p,\sigma' R}\right) \\
\le C &\Big(\|\widetilde{A}u\|_{p,\sigma' R} + 2\|\nabla\vartheta\|_\infty\|\nabla u\|_{p,\sigma' R} + \|\Delta\vartheta\|_\infty\|u\|_{p,\sigma' R} + \|F\|_\infty\|\nabla\vartheta\|_\infty\|u\|_{p,\sigma' R} \\
&+\|F\|_\infty\|\nabla u\|_{p,\sigma' R} + \|F\|_\infty\|\nabla\vartheta\|_\infty\|u\|_{p,\sigma' R}\Big) \\
\le C &\left(\|\widetilde{A}u\|_{p,\sigma' R} + (\|\nabla\vartheta\|_\infty + \|F\|_\infty)\|\nabla u\|_{p,\sigma' R} + (\|\Delta\vartheta\|_\infty + \|\nabla\vartheta\|_\infty)\|u\|_{p,\sigma' R}\right) \\
\le C &\left(\|\widetilde{A}u\|_{p,\sigma' R} + \frac{1}{R(1-\sigma')}\|\nabla u\|_{p,\sigma' R} + \frac{1}{R^2(1-\sigma')^2}\|u\|_{p,\sigma' R}\right),
\end{aligned}
$$

where $F(x) := b\frac{|x|^{\alpha-2}}{1+|x|^\alpha}x$ and C a positive constant independent of u and R, which may change from line to line. Multiplying the above estimate by $R^2(1-\sigma')^2$ and taking into account that $1-\sigma = 2(1-\sigma')$ we obtain

$$
R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R} \le C\left(R^2\|\widetilde{A}u\|_{p,R} + R(1-\sigma')\|\nabla u\|_{p,\sigma' R} + \|u\|_{p,R}\right).
$$

So,

(3.32)

$$
\sup_{\sigma\in(0,1)}\left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\} \le C\left(\sup_{\sigma\in(0,1)}\{R(1-\sigma)\|\nabla u\|_{p,\sigma R}\} + R^2\|\widetilde{A}u\|_{p,R} + \|u\|_{p,R}\right).
$$

Thus, by [**26**, Theorem 7.28], for every $\gamma > 0$ there exists $\sigma_\gamma \in (0,1)$ such that

$$
\begin{aligned}
\sup_{\sigma\in(0,1)}\{R(1-\sigma)\|\nabla u\|_{p,\sigma R}\} &\le R(1-\sigma_\gamma)\|\nabla u\|_{p,\sigma_\gamma R} + \gamma \\
&\le \varepsilon R^2(1-\sigma_\gamma)^2\|\Delta u\|_{p,\sigma_\gamma R} + \frac{C}{\varepsilon}\|u\|_{p,R} + \gamma \\
&\le \varepsilon \sup_{\sigma\in(0,1)}\left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\} + \frac{C}{\varepsilon}\|u\|_{p,R} + \gamma.
\end{aligned}
$$

Letting $\gamma \to 0$ we deduce that

$$\sup_{\sigma\in(0,1)} \{R(1-\sigma)\|\nabla u\|_{p,\sigma R}\} \le \varepsilon \sup_{\sigma\in(0,1)} \left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\} + \frac{C}{\varepsilon}\|u\|_{p,R}. \tag{3.33}$$

Putting (3.33) into (3.32) with a suitable choice of ε we obtain

$$\sup_{\sigma\in(0,1)} \left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\} \le C\left(R^2\|\widetilde{A}u\|_{p,R} + \|u\|_{p,R}\right).$$

Hence (3.31) follows since $(1-\frac{1}{2})^2R^2\|\Delta u\|_{p,\frac{R}{2}} \le \sup_{\sigma\in(0,1)} \left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\}$.

To prove that $u \in W^{2,p}(\mathbb{R}^N)$ we consider a covering $\{B(x_n, R/2) : n \in \mathbb{N}\}$ of $\mathbb{R}^N$ such that at most ζ among the doubled balls $\{B(x_n, R) : n \in \mathbb{N}\}$ overlap for some $\zeta(N) \in \mathbb{N}$, by Proposition 4. Applying (3.31) with the ball $B(x_n, R/2)$ we obtain

$$\begin{aligned}
\|\Delta u\|_p &\le \sum_{n\in\mathbb{N}} \|\Delta u\|_{L^p(B(x_n,R/2))} \\
&\le C\sum_{n\in\mathbb{N}} \left(\|\widetilde{A}u\|_{L^p(B(x_n,R))} + \|u\|_{L^p(B(x_n,R))}\right) \\
&\le C\zeta\left(\|\widetilde{A}u\|_p + \|u\|_p\right).
\end{aligned}$$

This ends the proof. □

We show now the invertibility of $\lambda - A_p$ in $D_{p,max}(A)$ for all $\lambda \ge \lambda_0$, where $\lambda_0 > 0$ is such that $\mathcal{V} \ge 0$ for all $\lambda \ge \lambda_0$.

Theorem 3. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then $[\lambda_0, \infty) \subset \rho(A_p)$ and $(\lambda - A_p)^{-1} = L$ for all $\lambda \ge \lambda_0$. Moreover there exists $C = C(\lambda) > 0$ such that, for every $0 \le \gamma \le \beta$ and $\lambda \ge \lambda_0$, the following holds*

$$\||\cdot|^\gamma u\|_p \le C\|\lambda u - A_p u\|_p, \quad \forall u \in D_{p,max}(A). \tag{3.34}$$

Proof. First we prove the injectivity of $\lambda - A_p$ for $\lambda \geq \lambda_0$. Let $u \in D_{p,max}(A)$ such that $\lambda u - A_p u = 0$. We have to distinguish two cases. The first one is when $b \leq 0$. In this case, by (3.3) we have $Hv = \Delta v - \mathcal{V} v = 0$ with $v = u\sqrt{\phi} \in D_{p,max}(H) = W^{2,p}(\mathbb{R}^N) \cap D_{p,max}(\mathcal{V})$, (see [**49**] or [**5**]). Then multiplying Hv by $v|v|^{p-2}$ and integrating by part (see [**42**]) over $\mathbb{R}^N$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} v|v|^{p-2}\Delta v\, dx - \int_{\mathbb{R}^N} \mathcal{V}|v|^p dx \\ &= -(p-1)\int_{\mathbb{R}^N} |v|^{p-2}|\nabla v|^2 dx - \int_{\mathbb{R}^N} \mathcal{V}|v|^p dx. \end{aligned}$$

Then we have $v \equiv 0$ and hence $u \equiv 0$.

The second case is when $b > 0$. For this we multiply $\frac{1}{1+|x|^\alpha}(\lambda u - A_p u)$ by $u|u|^{p-2}$ and using the fact that $u \in W^{2,p}$, by Proposition 5, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \frac{1}{1+|x|^\alpha}(\lambda u - A_p u)u|u|^{p-2}\, dx \\ &= \int_{\mathbb{R}^N} \left(\frac{\lambda + c|x|^\beta}{1+|x|^\alpha} + \frac{b(N+\alpha-2)|x|^{\alpha-2} + b(N-2)|x|^{2\alpha-2}}{p(1+|x|^\alpha)^2} \right) |u|^p\, dx \\ &\quad +(p-1)\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2}\, dx. \end{aligned}$$

Hence, $u \equiv 0$.

Let now $f \in L^p(\mathbb{R}^N)$ and $u(x) = Lf(x)$ defined by (3.5). Applying Lemma 5 with $\gamma = 0$, we have $u \in L^p(\mathbb{R}^N)$. Moreover u satisfies $\lambda u - A_p u = f$ and by elliptic regularity we deduce that $u \in D_{p,max}(A)$. Thus $\lambda - A_p$ is invertible and $(\lambda - A_p)^{-1} \in \mathcal{L}(L^p(\mathbb{R}^N))$ for all $\lambda \geq \lambda_0$. Finally, (3.34) follows from (3.23). □

The following result shows that the resolvent in $L^p(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N)$ coincides.

Theorem 4. *Assume that $N > 2$, $\beta > \alpha - 2$ and $\alpha > 2$. Then, for all $\lambda \geq \lambda_0$, $(\lambda - A_p)^{-1}$ is a positive operator on $L^p(\mathbb{R}^N)$. Moreover, if $f \in L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, then $(\lambda - A_p)^{-1} f = (\lambda - A)^{-1} f$.*

Proof. The positivity of $(\lambda - A_p)^{-1}$ follows from Theorem 3 and the positivity of L.

For the second assertion take $f \in C_c^\infty$ and set $u := (\lambda - A_p)^{-1} f$. Since the coefficients of A are Hölder continuous, by local elliptic regularity (cf. [**26**, Theorem 9.19]), we know $u \in C_{loc}^{2+\sigma}(\mathbb{R}^N)$ for some $0 < \sigma < 1$. On the other hand, $u \in W^{2,p}(\mathbb{R}^N)$ by Proposition 5.
If $p \geq \frac{N}{2}$ then, by Sobolev's inequality, $u \in L^q(\mathbb{R}^N)$ for all $q \in [p, +\infty)$. In particular, $u \in L^q(\mathbb{R}^N)$ for some $q > \frac{N}{2}$ (cf. [**12**, Corollary 9.13]) and hence $Au = -f + \lambda u \in L^q(\mathbb{R}^N)$. Moreover, since $u \in C_{loc}^{2+\sigma}(\mathbb{R}^N)$ it follows that $u \in W_{loc}^{2,q}(\mathbb{R}^N)$. So, $u \in D_{q,max}(A) \subset W^{2,q}(\mathbb{R}^N) \subset C_0(\mathbb{R}^N)$ by Proposition 5 and Sobolev's embedding theorem (cf. [**12**, Corollary 9.13]).
Let us now suppose that $p < \frac{N}{2}$. Take the sequence (r_n), defined by $r_n = 1/p - 2n/N$ and set $q_n = 1/r_n$ for $n \in \mathbb{N}$. Let n_0 be the smallest integer such that $r_{n_0} \leq 2/N$ noting that $r_{n_0} > 0$. Then, $u \in D_{p,max}(A) \subset L^{q_1}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, by the Sobolev embedding theorem. As above we obtain that $u \in D_{q_1,max}(A) \subset L^{q_2}(\mathbb{R}^N)$. Iterating this argument, we deduce that $u \in D_{q_{n_0},max}(A)$. So we can conclude that $u \in C_0(\mathbb{R}^N)$ arguing as in the previous case. Thus, $Au = -f + \lambda u \in C_b(\mathbb{R}^N)$. Again, since $u \in C_{loc}^{2+\sigma}(\mathbb{R}^N)$, it follows that $u \in W_{loc}^{2,q}(\mathbb{R}^N)$ for any $q \in (1, +\infty)$. Hence, $u \in D_{max}(A)$. So, by the uniqueness of the solution of the elliptic problem, we have $(\lambda - A_p)^{-1} f = (\lambda - A)^{-1} f$ for every $f \in C_c^\infty(\mathbb{R}^N)$. Thus the assertion follows by density. □

3.4. Characterization of the domain

The aim of this section is to characterize the domain of the operator A_p. More precisely we prove that the maximal domain $D_{p,max}(A)$ coincides with the weighted Sobolev space $D_p(A)$ defined by

$$D_p(A) := \{u \in W^{2,p}(\mathbb{R}^N) \; : \; Vu, \; (1+|x|^{\alpha-1})\nabla u, \; (1+|x|^{\alpha})D^2u \in L^p(\mathbb{R}^N)\}.$$

In the following lemma we give a complete proof of the weighted gradient and second derivative estimates.

Lemma 6. *Suppose that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then there exists a constant $C > 0$ such that for every $u \in D_p(A)$ we have*

$$\|(1+|x|^{\alpha-1})\nabla u\|_p \leq C(\|A_p u\|_p + \|u\|_p) \; , \tag{3.35}$$

$$\|(1+|x|^{\alpha})D^2 u\|_p \leq C(\|A_p u\|_p + \|u\|_p) \; . \tag{3.36}$$

Proof. Let $u \in D_p(A)$. We fix $x_0 \in \mathbb{R}^N$ and choose $\vartheta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \vartheta \leq 1$, $\vartheta(x) = 1$ for $x \in B(1)$ and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B(2)$. Moreover, we set $\vartheta_\rho(x) = \vartheta\left(\frac{x-x_0}{\rho}\right)$, where $\rho = \frac{1}{4}(1+|x_0|)$. We apply the well-known interpolation inequality (cf. [**26**, Theorem 7.27])

$$\|\nabla v\|_{L^p(B(R))} \leq C\|v\|^{1/2}_{L^p(B(R))}\|\Delta v\|^{1/2}_{L^p(B(R))}, \;\; v \in W^{2,p}(B(R)) \cap W^{1,p}_0(B(R)), \; R > 0, \tag{3.37}$$

to the function $\vartheta_\rho u$ and obtain for every $\varepsilon > 0$,

$$
\begin{aligned}
&\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))} \\
&\le \|(1+|x_0|)^{\alpha-1}\nabla(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))} \le C\|(1+|x_0|)^{\alpha}\Delta(\vartheta_\rho u)\|^{\frac{1}{2}}_{L^p(B(x_0,2\rho))}\|(1+|x_0|)^{\alpha-2}\vartheta_\rho u\|^{\frac{1}{2}}_{L^p(B(x_0,2\rho))} \\
&\le C\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))} + \frac{1}{4\varepsilon}\|(1+|x_0|)^{\alpha-2}\vartheta_\rho u\|_{L^p(B(x_0,2\rho))}\right) \\
&\le C\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + \frac{2M}{\rho}\varepsilon\|(1+|x_0|)^{\alpha}\nabla u\|_{L^p(B(x_0,2\rho))}\right. \\
&\quad\left. + \frac{\varepsilon M}{\rho^2}\|(1+|x_0|)^{\alpha}u\|_{L^p(B(x_0,2\rho))} + \frac{1}{4\varepsilon}\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right) \\
&\le C\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + 8M\varepsilon\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\right. \\
&\quad\left. + \left(16\varepsilon M + \frac{1}{4\varepsilon}\right)\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right) \\
&\le C(M)\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + \varepsilon\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\right. \\
&\quad\left. + \frac{1}{\varepsilon}\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right),
\end{aligned}
$$

where $M = \|\nabla\vartheta\|_\infty + \|\Delta\vartheta\|_\infty$. Since $2\rho = \frac{1}{2}(1+|x_0|)$ we get

$$
\frac{1}{2}(1+|x_0|) \le 1+|x| \le \frac{3}{2}(1+|x_0|), \qquad x \in B(x_0, 2\rho).
$$

Thus,

$$\|(1+|x|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))} \leq \left(\frac{3}{2}\right)^{\alpha-1}\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))} \tag{3.38}$$

$$\leq C\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))}+\varepsilon\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\right. \tag{3.39}$$

$$\left.+\frac{1}{\varepsilon}\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right) \tag{3.40}$$

$$\leq C\left(2^{\alpha}\varepsilon\|(1+|x|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))}+2^{\alpha-1}\varepsilon\|(1+|x|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\right. \tag{3.41}$$

$$\left.+\frac{2^{\alpha-2}}{\varepsilon}\|(1+|x|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right). \tag{3.42}$$

Let $\{B(x_n,\rho(x_n))\}$ be a countable covering of $\mathbb{R}^N$ as in Proposition 4 such that at most ζ among the double balls $\{B(x_n,2\rho(x_n))\}$ overlap.

We write (3.38) with x_0 replaced by x_n and sum over n, we obtain

$$\begin{aligned}\|(1+|x|)^{\alpha-1}\nabla u\|_p &\leq C\zeta\big(\varepsilon\|(1+|x|)^{\alpha}\Delta u\|_p+\varepsilon\|(1+|x|)^{\alpha-1}\nabla u\|_p+\frac{1}{\varepsilon}\|(1+|x|)^{\alpha-2}u\|_p\big)\,.\\ &\leq C\varepsilon\|A_pu\|_p+C\varepsilon(1+|b|)\|(1+|x|)^{\alpha-1}\nabla u\|_p+C(\frac{1}{\varepsilon}+\varepsilon)\|(1+|x|^{\beta})u\|_p.\end{aligned}$$

Choosing ε such that $\varepsilon C\zeta<\frac{1}{2(1+|b|)}$ we have

$$\|(1+|x|)^{\alpha-1}\nabla u\|_p\leq C\left(\|A_pu\|_p+\|(1+|x|^{\beta})u\|_p\right)$$

for some constant $C>0$. Furthermore, by (3.34), we know that $\|(1+|x|^{\beta})u\|_p\leq C(\|A_pu\|_p+\|u\|_p)$ for every $u\in D_p(A)\subset D_{p,max}(A)$ and some $C>0$. Hence,

$$\|(1+|x|)^{\alpha-1}\nabla u\|_p\leq C(\|A_pu\|_p+\|u\|_p).$$

As regards the second order derivatives we recall the classical Calderón-Zygmund inequality on $B(1)$

$$\|D^2 v\|_{L^p(B(1))} \le C\|\Delta v\|_{L^p(B(1))}, \quad v \in W^{2,p}(B(1)) \cap W_0^{1,p}(B(1)).$$

By rescaling and translating we obtain

$$\|D^2 v\|_{L^p(B(x_0,R))} \le C\|\Delta v\|_{L^p(B(x_0,R))} \tag{3.43}$$

for every $x_0 \in \mathbb{R}^N$, $R > 0$ and $v \in W^{2,p}(B(x_0,R)) \cap W_0^{1,p}(B(x_0,R))$. We observe that the constant C does not depend on R and x_0.

Then we fix $x_0 \in \mathbb{R}^N$ and choose ρ and $\vartheta_\rho \in C_c^\infty(\mathbb{R}^N)$ as above. Applying (3.43) to the function $\vartheta_\rho u$ in $B(x_0, 2\rho)$, we obtain

$$\begin{aligned}\|(1+|x_0|)^\alpha D^2 u\|_{L^p(B(x_0,\rho))} &\le \|(1+|x_0|)^\alpha D^2(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}\\ &\le C\|(1+|x_0|)^\alpha \Delta(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}.\end{aligned}$$

Arguing as above we obtain

$$\|(1+|x|)^\alpha D^2 u\|_p \le C\left(\|(1+|x|)^\alpha \Delta u\|_p + \|(1+|x|)^{\alpha-1}\nabla u\|_p + \|(1+|x|)^{\alpha-2} u\|_p\right).$$

The lemma follows from (3.34) and (3.35). □

The following result shows that $C_c^\infty(\mathbb{R}^N)$ is a core for A_p, since by Lemma 6 the norm (3.45) is equivalent to the graph norm of A_p. The proof is based on Theorem 3 and Lemma 6 and it is similar to the one given in [**14**, Lemma 4.3].

Lemma 7. *The space $C_c^\infty(\mathbb{R}^N)$ is dense in*

$$D_p(A) = \{u \in W^{2,p}(\mathbb{R}^N), Vu, (1+|x|^\alpha)D^2 u, (1+|x|^{\alpha-1})\nabla u \in L^p(\mathbb{R}^N)\} \tag{3.44}$$

endowed with the norm

$$\|u\|_{D_p(A)} := \|u\|_p + \|Vu\|_p + \|(1+|x|^{\alpha-1})|\nabla u|\|_p + \|(1+|x|^{\alpha})|D^2u|\|_p, \quad u \in D_p(A). \tag{3.45}$$

Proof. Let us first observe that $C_c^\infty(\mathbb{R}^N)$ is dense in $W_c^{2,p}(\mathbb{R}^N)$ with respect to the operator norm. Let $u \in W_c^{2,p}(\mathbb{R}^N)$ and consider $u_n = \rho_n * u$, where ρ_n are standard mollifiers. We have $u_n \in C_c^\infty(\mathbb{R}^N)$, $u_n \to u$ in $L^p(\mathbb{R}^N)$, $\nabla u_n \to \nabla u$ in $L^p(\mathbb{R}^N)$ and $D^2u_n \to D^2u$ in $L^p(\mathbb{R}^N)$. Moreover, $supp\, u_n \subset supp\, u + B(1) := K$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\|A_pu - Au_n\|_p = \|A_pu - Au_n\|_{L^p(K)} \\
&\quad \le \|(1+|x|^\alpha)\Delta(u-u_n)\|_{L^p(K)} + \|b|x|^{\alpha-1}\nabla(u-u_n)\|_{L^p(K)} + \|c|x|^\beta(u-u_n)\|_{L^p(K)} \\
&\quad \le \|(1+|x|^\alpha)\|_{L^\infty(K)}\|\Delta(u-u_n)\|_{L^p(K)} + \|b|x|^{\alpha-1}\|_{L^\infty(K)}\|\nabla(u-u_n)\|_{L^p(K)} \\
&+ \|c|x|^\beta\|_{L^\infty(K)}\|(u-u_n)\|_{L^p(K)} \to 0 \text{ as } n \to \infty \,.
\end{aligned}$$

Now, let u in $D_{p,max}(A)$ and let η be a smooth function such that $\eta = 1$ in $B(1)$, $\eta = 0$ in $\mathbb{R}^N \setminus B(2)$, $0 \le \eta \le 1$ and set $\eta_n(x) = \eta\left(\frac{x}{n}\right)$. Then consider $u_n = \eta_n u \in W_c^{2,p}(\mathbb{R}^N)$. First we have $u_n \to u$ in $L^p(\mathbb{R}^N)$ by dominated convergence. As regard A_pu_n we have

$$\begin{aligned}
&A_pu_n(x) = (1+|x|^\alpha)\Delta(\eta_n u)(x) + b|x|^{\alpha-2}x\cdot\nabla(\eta_n u)(x) - c|x|^\beta\eta_n(x)u(x) \\
&\quad = \eta_n(x)A_pu(x) + 2(1+|x|^\alpha)\nabla\eta_n(x)\nabla u(x) + b|x|^{\alpha-2}x\cdot(\nabla(\eta_n)(x))u(x) + (1+|x|^\alpha)\Delta\eta_n(x)u(x) \\
&\quad = \eta_n(x)A_pu(x) + \frac{2}{n}(1+|x|^\alpha)\nabla\eta\left(\frac{x}{n}\right)\nabla u(x) + \frac{b}{n}|x|^{\alpha-2}x\cdot\nabla\eta\left(\frac{x}{n}\right)u(x) \\
&+ \frac{1}{n^2}(1+|x|^\alpha)\Delta\eta\left(\frac{x}{n}\right)u(x) \,,
\end{aligned}$$

and

$$\eta_n A_pu \to A_pu \qquad \text{in} \qquad L^p(\mathbb{R}^N)$$

by dominated convergence. As regards the last terms we note that $\nabla\eta(x/n)$ and $\Delta\eta(x/n)$ can be different from zero only for $n \le |x| \le 2n$, then we have

$$\frac{1}{n}(1+|x|^\alpha)\left|\nabla\eta\left(\frac{x}{n}\right)\right||\nabla u| \le C(1+|x|^{\alpha-1})|\nabla u|\chi_{\{n\le|x|\le 2n\}},$$

and

$$\frac{1}{n^2}(1+|x|^\alpha)\left|\Delta\eta\left(\frac{x}{n}\right)\right||u| \le C(1+|x|^{\alpha-2})|u|\chi_{\{n\le|x|\le 2n\}}.$$

The right hand sides tend to 0 as $n \to \infty$, since by Theorem 3 and Lemma 6 we have $\|(1+|x|^{\alpha-2})u\|_p \le C\|A_p u\|_p$ and $\|(1+|x|^{\alpha-1})\nabla u\|_p \le C\|A_p u\|_p$. So, applying again the dominated convergence theorem, we obtain $A_p u_n \to A_p u$ in $L^p(\mathbb{R}^N)$. This ends the proof of the lemma. □

3.5. Generation of analytic semigroup

The aim of this section is to prove that the operator A_p generates an analytic semigroup on $L^p(\mathbb{R}^N)$, for any $p \in (1,\infty)$, provided that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$.

Theorem 5. *Suppose that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then the operator A_p with domain $D_{p,max}(A)$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$.*

Proof. Let $f \in L^p, \rho > 0$. Consider the operator $\widehat{A_p} := A_p - \omega$ where ω is a constant which will be chosen later. it is known that the elliptic problem in $L^p(B(\rho))$

$$\begin{cases} \lambda u - \widehat{A_p}u = f \quad in \quad B(\rho), \\ u = 0 \qquad on \quad \partial B(\rho). \end{cases} \tag{3.46}$$

admits a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W_0^{1,p}(B(\rho))$ for$\lambda > 0$, (cf. [**26**, Theorem 9.15]). Let us prove that $e^{\pm i\theta}\widehat{A_p}$ is dissipative in $B(\rho)$ for $0 < \theta \le \theta_\alpha$ with suitable $\theta_\alpha \in (0, \frac{\pi}{2})$. To this purpose observe that

$$\widehat{A_p} = \text{div}((1+|x|^\alpha)\nabla u_\rho) + (b-\alpha)|x|^{\alpha-1}\frac{x}{|x|}\cdot\nabla u_\rho - c|x|^\beta u_\rho - \omega u_\rho.$$

Set $u^* = \bar{u}_\rho|u_\rho|^{p-2}$ and recall that $a(x) = 1+|x|^\alpha$. Multiplying $\widehat{A_p}u_\rho$ bu u^* and integrating over $B(\rho)$, we obtain

$$\begin{aligned}
&\int_{B(\rho)} \widehat{A_p}u_\rho u^* dx = -\int_{B(\rho)} a(x)|u_\rho|^{p-4}|Re(\bar{u}_\rho\nabla u_\rho)|^2 dx - \int_{B(\rho)} a(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho\nabla u_\rho)|^2 dx \\
&-\int_{B(\rho)} \bar{u}_\rho|u_\rho|^{p-2}\nabla a(x)\nabla u_\rho dx - (p-2)\int_{B(\rho)} a(x)|u_\rho|^{p-4}\bar{u}_\rho\nabla u_\rho|Re(\bar{u}_\rho\nabla u_\rho)|^2 dx \\
&+b\int_{B(\rho)} \bar{u}_\rho|u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|}\nabla u_\rho dx - \int_{B(\rho)}\left(c|x|^\beta+\omega\right)|u_\rho|^p dx.
\end{aligned}$$

We note here that the integration by part in the singular case $1 < p < 2$ is allowed thanks to [**42**]. By taking the real part of the left and the right hand side, we have

$$\begin{aligned}
&Re\left(\int_{B(\rho)} \widehat{A_p}u_\rho u^* dx\right) \\
&= -(p-1)\int_{B(\rho)} a(x)|u_\rho|^{p-4}|Re(\bar{u}_\rho\nabla u_\rho)|^2 dx - \int_{B(\rho)} a(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho\nabla u_\rho)|^2 dx \\
&\quad -\int_{B(\rho)} |u_\rho|^{p-2}\nabla a(x)Re(\bar{u}_\rho\nabla u_\rho)dx + b\int_{B(\rho)} |u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|}Re(\bar{u}_\rho\nabla u_\rho)dx \\
&\quad -\int_{B(\rho)}\left(c|x|^\beta+\omega\right)|u_\rho|^p dx. \\
&\quad = -(p-1)\int_{B(\rho)} a(x)|u_\rho|^{p-4}|Re(\bar{u}_\rho\nabla u_\rho)|^2 dx - \int_{B(\rho)} a(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho\nabla u_\rho)|^2 dx \\
&\quad +\int_{B(\rho)}\left(\frac{(\alpha-b)(N-2+\alpha)}{p}|x|^{\alpha-2} - c|x|^\beta - \omega\right)|u_\rho|^p dx
\end{aligned}$$

Also, taking the imaginary part of the left and the right hand side, we obtain

$$
\begin{aligned}
& Im\left(\int_{B(\rho)} \widehat{A_p} u_\rho u^* dx\right) \\
&= -(p-2)\int_{B(\rho)} a(x)|u_\rho|^{p-4} Im(\bar{u}_\rho \nabla u_\rho) Re(\bar{u}_\rho \nabla u_\rho) dx \\
&\quad - \int_{B(\rho)} |u_\rho|^{p-2} \nabla a(x) Im(\bar{u}_\rho \nabla u_\rho) dx + b\int_{B(\rho)} |u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|} Im(\bar{u}_\rho \nabla u_\rho) dx
\end{aligned}
$$

We can choose $\omega > 0$ such that

$$
\frac{(\alpha-b)(N-2+\alpha)}{p}|x|^{\alpha-2} - c|x|^{\beta} - \omega \le -\frac{|\alpha-b|(N-2+\alpha)}{p}|x|^{\alpha-2}.
$$

Furthermore,

$$
\begin{aligned}
-Re\left(\int_{B(\rho)} \widehat{A_p} u_\rho\, u^\star dx\right) \quad &\geq \quad (p-1)\int_{B(\rho)} a(x)|u_\rho|^{p-4}|Re(\bar{u}_\rho \nabla u_\rho)|^2 dx \\
&\quad + \int_{B(\rho)} a(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho \nabla u_\rho)|^2 dx + \tilde{c}\int_{B(\rho)} |u_\rho|^p |x|^{\alpha-2} dx \\
&= \quad (p-1)B^2 + C^2 + \tilde{c} D^2,
\end{aligned}
$$

where $\tilde{c} = \frac{|\alpha-b|(N-2+\alpha)}{p}$ is positive constant.

Moreover,

$$\begin{aligned}
& \left| Im \left(\int_{B(\rho)} \widehat{A_p} u_\rho \, u^\star dx \right) \right| \\
\leq \; & |p-2| \left(\int_{B(\rho)} |u_\rho|^{p-4} a(x) |Re(\bar{u}_\rho \nabla u_\rho)|^2 dx \right)^{\frac{1}{2}} \left(\int_{B(\rho)} |u_\rho|^{p-4} a(x) |Im(\bar{u}_\rho \nabla u_\rho)|^2 dx \right)^{\frac{1}{2}} \\
& \quad + |\alpha - b| \left(\int_{B(\rho)} |u_\rho|^{p-4} |x|^\alpha |Im(\bar{u}_\rho \nabla u_\rho)|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B(\rho)} |u_\rho|^p |x|^{\alpha-2} \, dx \right)^{\frac{1}{2}} \\
= \; & |p-2| BC + |\alpha - b| CD.
\end{aligned}$$

Setting

$$\begin{aligned}
B^2 &= \int_{B(\rho)} a(x) |u_\rho|^{p-4} |Re(\bar{u}_\rho \nabla u_\rho)|^2 dx \\
C^2 &= \int_{B(\rho)} a(x) |u_\rho|^{p-4} |Im(\bar{u}_\rho \nabla u_\rho)|^2 dx \\
D^2 &= \int_{B(\rho)} |x|^{\alpha-2} |u_\rho|^p dx.
\end{aligned}$$

As a result of the above estimates, we conclude

$$\left| Im \left(\int_{B(\rho)} \widehat{A_p} u_\rho u^* dx \right) \right| \leq l_\alpha^{-1} \left[-Re \left(\int_{B(\rho)} \widehat{A_p} u_\rho u^* dx \right) \right]$$

If $\tan \theta_\alpha = l_\alpha$, then $e^{\pm i\theta} \widehat{A_p}$ is dissipative in $B(\rho)$ for $0 \leq \theta \leq \theta_\alpha$. From [**51**, Theorem I.3.9] follows that the problem (3.46) has a unique solution u_ρ for every $\lambda \in \Sigma_\theta$, $0 \leq \theta < \theta_\alpha$ where

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |Arg\, \lambda| < \pi/2 + \theta\}.$$

Moreover, there exists a constant C_θ which is independent of ρ, such that

$$\|u_\rho\|_{L^p(B(\rho))} \le \frac{C_\theta}{|\lambda|}\|f\|_{L^p}, \quad \lambda \in \Sigma_\theta. \tag{3.47}$$

Let us now fix $\lambda \in \Sigma_\theta$, with $0 < \theta < \theta_\alpha$ and a radius $r > 0$. We apply the interior L^p estimates (cf. [**26**, Theorem 9.11]) to the functions u_ρ with $\rho > r + 1$. So, by (3.47) we have

$$\|u_\rho\|_{W^{2,p}(B(r))} \le C_1 \left(\|\lambda u_\rho - \widehat{A_p} u_\rho\|_{L^p(B(r+1))} + \|u_\rho\|_{L^p(B(r+1))}\right) \le C_2\|f\|_{L^p}. \tag{3.48}$$

Using a weak compactness and a diagonal argument, we can construct a sequence $(\rho_n) \to \infty$ such that the functions (u_{ρ_n}) converge weakly in $W^{2,p}_{loc}$ to a function u which satisfies $\lambda u - \widehat{A_p} u = f$ and

$$\|u\|_p \le \frac{C_\theta}{|\lambda|}\|f\|_p, \quad \lambda \in \Sigma_\theta. \tag{3.49}$$

Moreover, $u \in D_{p,max}(A_p)$. We have now only to show that $\lambda - \widehat{A_p}$ is invertible on $D_{p,max}(A_p)$ for $\lambda_0 < \lambda \in \Sigma_\theta$. Consider the set

$$E = \{r > 0 : \Sigma_\theta \cap C(r) \subset \rho(\widehat{A_p})\},$$

where $C(r) := \{\lambda \in \mathbb{C} : |\lambda| < r\}$. Since, by Theorem 3, λ_0 is in the resolvent set of $\widehat{A_p}$, then $R = \sup E > 0$. On the other hand, the norm of the resolvent is bounded by $C_\theta/|\lambda|$ in $C(R) \cap \Sigma_\theta$, consequently it cannot explode on the boundary of $C(R)$, then $R = \infty$ and this ends the proof of the theorem. □

Let us show that $D_{p,max}(A)$ and $D_p(A)$ coincide.

Theorem 6. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then maximal domain $D_{p,max}(A)$ coincides with $D_p(A)$.*

Proof. We only have to prove the inclusion $D_{p,max}(A) \subset D_p(A)$.

Let $\tilde{u} \in D_{p,max}(A)$ and set $f = \lambda\,\tilde{u} - A_p\tilde{u}$. The operator A in $B(\rho)$, $\rho > 0$, is an uniformly elliptic operator with bounded coefficients. Then the Dirichlet problem

$$\begin{cases} \lambda\, u - Au = f & \text{in } B(\rho) \\ u = 0 & \text{on } \partial B(\rho)\ , \end{cases} \tag{3.50}$$

admits a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W_0^{1,p}(B(\rho))$ (cf. [**26**, Theorem 9.15]). So, $\widetilde{u_\rho}$, the zero extension of u_ρ to the complement $B(\rho)^c$, belongs to $D_p(A)$. Thus, by Lemma 6 and (3.34), we have

$$\|(1+|x|^{\alpha-2})\widetilde{u_\rho}\|_p + \|(1+|x|^{\alpha-1})\nabla\widetilde{u_\rho}\|_p$$
$$+ \|(1+|x|^{\alpha})D^2\widetilde{u_\rho}\|_p + \|V\widetilde{u_\rho}\|_p \le C(\|A\widetilde{u_\rho}\|_p + \|\widetilde{u_\rho}\|_p)$$

with C independent of ρ.

We observe that u_ρ is solution of (3.46) with λ replaced with $\lambda - \omega$. Then arguing as in the proof of Theorem 5, by (3.47) and (3.48) for $\lambda > \omega$ we have $\|u_\rho\|_{L^p(B(\rho))} \le \frac{C_1}{\lambda-\omega}\|f\|_{L^p}$ and $\|u_\rho\|_{W^{2,p}(B(r))} \le C_2\|f\|_{L^p}$ where $r < \rho-1$ and C_1, C_2 are positive constants which do not depend on ρ.

Using a standard weak compactness argument we can construct a sequence $\widetilde{u_{\rho_n}}$ which converges to a function u in $W^{2,p}_{loc}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ such that $\lambda\, u - Au = f$. Since the estimates above are independent of ρ, also $u \in D_p(A)$. Then we have $\lambda\tilde{u} - A\tilde{u} = \lambda u - Au$ and since $D_p(A) \subset D_{p,max}(A)$ and $\lambda - A$ is invertible on $D_{p,\max}(A)$ by Theorem 3, we have $\tilde{u} = u$. □

Proposition 6. *For any $f \in L^p(\mathbb{R}^N)$, $1 < p < \infty$, and any $0 < \nu < 1$ and for all $t > 0$, the function $T_p(t)f$ belongs to $C_b^{1+\nu}(\mathbb{R}^N)$. In particular, the semigroup $(T_p(t))_{t\ge 0}$ is ultracontractive.*

Proof. In Theorem 5 we have proved that A_p generates an analytic semigroup $T_p(\cdot)$ on $L^p(\mathbb{R}^N)$ and in Theorem 4 we have obtained that for f in $L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $(\lambda - A_p)^{-1} f = (\lambda - A)^{-1} f$. Hence this shows the coherence of the resolvents on $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ by using a density argument. This will yield immediately that the semigroups are coherent in different L^p-spaces. One can deduce the result by using the same arguments as in the proof of [**38**, Proposition 2.6]. □

To end this section we study the spectrum of A_p.

Proposition 7. *Assume $N > 2$, $\alpha > 2$, $\beta > \alpha - 2$. Then, for $p \in (1, \infty)$, the resolvent operator $R(\lambda, A_p)$ is compact in $L^p(\mathbb{R}^N)$ for all $\omega_0 < \lambda \in \rho(A_p)$, where ω_0 is a suitable positive constant, and the spectrum of A_p consists of a sequence of negative real eigenvalues which accumulates at $-\infty$. Moreover, $\sigma(A_p)$ is independent of p.*

Proof. If we show that $D(A_p)$ is compactly embedded into $L^p(\mathbb{R}^N)$ for any $p \in (1, \infty)$. This will yield immediately that the resolvent operator $R(\lambda, A_p)$ is compact in $L^p(\mathbb{R}^N)$ for all $\omega_0 < \lambda \in \rho(A_p)$. Hence, its spectrum (and, consequently, the spectrum of A_p) consists of eigenvalues.

Since

$$\int_{\mathbb{R}^N} |x|^{\beta p} |u(x)|^p dx \le C_1 \left(\int_{\mathbb{R}^N} |u(x)|^p dx + \int_{\mathbb{R}^N} |A_p u(x)|^p dx \right),$$

for some positive constant C_1, independent of u, taking Lemma 6 and Lemma 7 into account, we can easily conclude that there exists a positive constant C_2, independent of u as well, such that

$$\int_{\mathbb{R}^N} |x|^{\beta p} |u(x)|^p dx \le C_2, \tag{3.51}$$

for any $u \in \mathcal{B} := \{v \in D(A_p) : ||v||_{D(A_p)} \leq 1\}$. This estimate yields the compactness of $\mathcal{B}$ in $L^p(\mathbb{R}^N)$ by a standard argument. Anyway for the reader's convenience we give some details. To prove that $\mathcal{B}$ is compact in $L^p(\mathbb{R}^N)$, we show that it is totally bounded. By estimate (3.51), we deduce that

$$\int_{\mathbb{R}^N \setminus B_M} |u(x)|^p dx \leq M^{-\beta p} \int_{\mathbb{R}^N \setminus B_M} |x|^{\beta p} |u(x)|^p dx \leq C_2 M^{-\beta p}, \quad u \in \mathcal{B}. \tag{3.52}$$

Let us fix $\varepsilon > 0$ and let M_ε be large enough such that

$$\int_{\mathbb{R}^N \setminus B_{M_\varepsilon}} |u(x)|^p dx \leq \frac{1}{2}\varepsilon^p, \quad u \in \mathcal{B}.$$

Since $D(A_p)$ is continuously embedded into $W^{2,p}(\mathbb{R}^N)$, the set $\mathcal{B}_{|B_{M_\varepsilon}}$ of the restrictions to B_{M_ε} of all the functions in $\mathcal{B}$ is continuously embedded in $W^{2,p}(B_{M_\varepsilon})$. As this latter space is compactly embedded in $L^p(B_{M_\varepsilon})$, there exist $n_\varepsilon \in \mathbb{N}$ and functions $f_1, ..., f_{n_\varepsilon}$ in $L^p(B_{M_\varepsilon})$ such that, for any $u \in \mathcal{B}$ and some $j = j(u) \in \{1, ..., n_\varepsilon\}$,

$$\int_{B_{M_\varepsilon}} |u(x) - f_j(x)|^p dx \leq \frac{1}{2}\varepsilon^p. \tag{3.53}$$

Let us now denote by $\tilde{f}_j (j = 1, ..., n_\varepsilon)$ the function which equals f_j in B_{M_ε} and identically vanishes elsewhere in $\mathbb{R}^N$. Using (3.52) and (3.53), we obtain that $||u - \tilde{f}_j||_{L^p(\mathbb{R}^N)} \leq \varepsilon$, and this shows that $\mathcal{B}$ is totally bounded in $L^p(\mathbb{R}^N)$.

The proof is similar to the one given in[**38, 46**]. □

CHAPTER 4

Elliptic operators with unbounded coefficients: Heat kernel estimates

4.1. Introduction

In Chapter 3 we have studied generation results in $L^p(\mathbb{R}^N)$ for the elliptic operator

$$A := (1+|x|^\alpha)\Delta + b|x|^{\alpha-1}\frac{x}{|x|}\cdot\nabla - c|x|^\beta.$$

In the following Chapter we use the ultracontractive bounds to obtain upper bounds on heat kernels. The chapter is organized as follows. In section 4.2 we show that the eigenfunction Φ of A associated to the largest eigenvalue λ_0 can be estimated from below and above by the function

$$|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}(1+|x|^\alpha)^{-\frac{b}{2\alpha}}e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr}$$

for $|x|$ and $|y|$ sufficiently large.

In Section 4.3, by means of a suitable multiplication operator $Tu = \phi u$, we rewrite the operator A in the following form

$$A = T^{-1}HT,$$

where $H = (1+|x|^\alpha)\Delta - U$ with $U = (1+|x|^\alpha)\frac{\Delta\phi}{\phi} + |x|^\beta$ and use an associated positive, closed, symmetric form $h(\cdot,\cdot)$ defined on a domain $D(h)$ in an appropriate weighted Hilbert space $L^2_\mu(\mathbb{R}^N)$. This permits us to define the associated self-adjoint operator H_μ and his corresponding semigroup (e^{tH_μ}). It can be seen that (e^{tH_μ}) is given by a heat kernel k_μ. Adapting the arguments used in [**18**] and [**38**], we prove the following intrinsic ultracontractivity

$$k_\mu(t,x,y) \le c_1 e^{\lambda_0 t} e^{c_2 t^{-\gamma}} \Phi(x)\Phi(y), \quad t>0,\, x,\, y \in \mathbb{R}^N,$$

where c_1, c_2 are positive constant, $\gamma = \frac{\beta-\alpha+2}{\beta+\alpha-2}$ and λ_0 is the largest eigenvalue of A, provided that $N>2$, $\alpha>2$ and $\beta>\alpha-2$.

Notation. For $x \in \mathbb{R}^N$ and $r>0$ we set $B_r = \{x \in \mathbb{R}^N : |x| < r\}$. We denote by $< \cdot, \cdot >$ the euclidean scalar product and by $|\cdot|$ the euclidean norm. We use as always a standard notation for function spaces. So, we denote by $L^p(\mathbb{R}^N)$ and $W^{2,p}(\mathbb{R}^N)$ the standard L^p and Sobolev spaces, respectively. The space of bounded and continuous functions on $\mathbb{R}^N$ is denoted by $C_b(\mathbb{R}^N)$. Finally, in the whole manuscript the notation $\phi \approx \psi$ on a set Ω means that there are positive constants C_1, C_2 such that $C_1\psi(x) \le \phi(x) \le C_2\psi(x)$ for all $x \in \Omega$.

4.2. Estimating the ground state Φ

For $\alpha>2$, $\beta>\alpha-2$ and $N>2$, we denote by A_p the realization in $L^p(\mathbb{R}^N)$, $1<p<\infty$, of the operator $A := A_{b,1}$ defined in (1.1). We recall, see [**8**, Theorem 3 and Theorem 4], that A_p with domain

$$D_p(A) = \{u \in W^{2,p}(\mathbb{R}^N) \mid (1+|x|^\alpha)|D^2u|, (1+|x|^\alpha)^{1/2}|\nabla u|, |x|^\beta u \in L^p(\mathbb{R}^N)\}$$

generates a strongly continuous and analytic semigroup $T_p(\cdot)$ in $L^p(\mathbb{R}^N)$. Moreover, for $t>0$, $T_p(t)$ maps $L^p(\mathbb{R}^N)$ into $C_b^{1+\eta}(\mathbb{R}^N)$ for any $\eta \in (0,1)$, see [**8**, Proposition 5], and the semigroup $T_p(\cdot)$ is immediately compact, see [**8**, Proposition 6]. As a consequence one obtains that the spectrum $\sigma(A_p)$ of A_p consists of a sequence of negative real eigenvalues which accumulates at $-\infty$, and $\sigma(A_p)$ is independent of p. As in [**14**] and [**38**] one can see that $T_p(\cdot)$ is irreducible, the eigenspace corresponding to the largest eigenvalue λ_0 of A_p is one-dimensional and is spanned by a strictly positive functions Φ, which is radial, belongs to $C_b^{1+\nu}(\mathbb{R}^N)\cap C^2(\mathbb{R}^N)$ for any $\nu\in(0,1)$ and tends to 0 as $|x|\to\infty$.

In this section we prove precise estimates for the eigenfunction Φ. The technique used here is inspired by the work [**15**].

Since Φ is radial, one has to analyze the asymptotic behavior of the solutions to an ordinary differential equation. In this context some ideas coming from the Wentzel-Kramers-Brillouin (or Liouville-Green) approximation will be of great help. For more details see [**48**].

Proposition 8. *Let $\lambda_0<0$ be the largest eigenvalue of A and Φ be the corresponding eigenfunction. If $N>2$, $\alpha>2$ and $\beta>\alpha-2$ then*

$$\Phi\approx|x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}(1+|x|^\alpha)^{-\frac{b}{2\alpha}}e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr}$$

on $\mathbb{R}^N\setminus B_1$.

Proof. Let $g_{\alpha,\beta,\lambda}$ be the function defined as

$$g_{\alpha,\beta,\lambda}(x)=|x|^{-\frac{N-1}{2}}(1+|x|^\alpha)^{-\frac{b}{2\alpha}}\mathfrak{h}^{-\frac{1}{4}}(|x|)\exp\left\{-\int_1^{|x|}\mathfrak{h}^{\frac{1}{2}}(s)ds-\int_1^{|x|}\mathfrak{v}_\lambda(s)ds\right\}, \tag{4.1}$$

where $\lambda\in\mathbb{R}$, $\mathfrak{h}(r)=\frac{r^\beta}{1+r^\alpha}$, and $\mathfrak{v}_\lambda$ is a smooth function to be chosen later on. If we set

$$w(r)=r^{\frac{N-1}{2}}(1+r^\alpha)^{\frac{b}{2\alpha}}g_{\alpha,\beta,\lambda}(r), \tag{4.2}$$

the calculation of w' gives us

$$\begin{aligned}w'(r)=&\left(\frac{N-1}{2}\right)r^{\frac{N-1}{2}-1}(1+r^\alpha)^{\frac{b}{2\alpha}}g_{\alpha,\beta,\lambda}(r)\\&+\frac{b}{2}r^{\frac{N-1}{2}+\alpha-1}(1+r^\alpha)^{\frac{b}{2\alpha}-1}g_{\alpha,\beta,\lambda}(r)+r^{\frac{N-1}{2}}(1+r^\alpha)^{\frac{b}{2\alpha}}g'_{\alpha,\beta,\lambda}(r).\end{aligned}$$

So, one obtains

$$w' = w\left(-\frac{\mathfrak{h}'}{4\mathfrak{h}} - \mathfrak{h}^{\frac{1}{2}} - \mathfrak{v}_\lambda\right) \qquad \text{and} \qquad w'' = w(g_1 + m + \mathfrak{h}), \tag{4.3}$$

where

$$g_1 = \frac{5}{16}\left(\frac{\mathfrak{h}'}{\mathfrak{h}}\right)^2 - \frac{\mathfrak{h}''}{4\mathfrak{h}} + \mathfrak{v}_\lambda^2 + \mathfrak{v}_\lambda\left(\frac{\mathfrak{h}'}{2\mathfrak{h}} + 2\mathfrak{h}^{\frac{1}{2}}\right) - \mathfrak{v}_\lambda' - m \tag{4.4}$$

and

$$m(r) := \frac{(N-1)(N-3)}{4r^2} + \frac{b}{2}(N-2+\alpha)\frac{r^{\alpha-2}}{1+r^\alpha} + \frac{b\alpha}{2}\left(\frac{b}{2\alpha} - 1\right)\left(\frac{r^{\alpha-1}}{1+r^\alpha}\right)^2.$$

On the other hand, by computing directly the second derivative of (4.2) one has

$$\begin{aligned} w''(r) &= r^{\frac{N-1}{2}}(1+r^\alpha)^{\frac{b}{2\alpha}}\left(g''_{\alpha,\beta,\lambda} + \left(\frac{N-1}{r} + b\frac{r^{\alpha-1}}{1+r^\alpha}\right)g'_{\alpha,\beta,\lambda}\right) \\ &+ r^{\frac{N-1}{2}}(1+r^\alpha)^{\frac{b}{2\alpha}}\left[\frac{(N-1)(N-3)}{4r^2} + \frac{b}{2}(N-2+\alpha)\frac{r^{\alpha-2}}{1+r^\alpha} + \frac{b\alpha}{2}\left(\frac{b}{2\alpha} - 1\right)\left(\frac{r^{\alpha-1}}{1+r^\alpha}\right)^2\right]g_{\alpha,\beta,\lambda}. \end{aligned}$$

So, comparing with (4.3) we get

$$g''_{\alpha,\beta,\lambda} + \left(\frac{N-1}{r} + b\frac{r^{\alpha-1}}{1+r^\alpha}\right)g'_{\alpha,\beta,\lambda} = \frac{r^\beta}{1+r^\alpha}g_{\alpha,\beta,\lambda} + g_1 g_{\alpha,\beta,\lambda}.$$

That is

$$\Delta g_{\alpha,\beta,\lambda}(x) + b\frac{|x|^{\alpha-2}x}{1+|x|^\alpha}\cdot\nabla g_{\alpha,\beta,\lambda}(x) - \frac{|x|^\beta}{1+|x|^\alpha}g_{\alpha,\beta,\lambda}(x) = g_1(|x|)g_{\alpha,\beta,\lambda}(x). \tag{4.5}$$

To evaluate the function g_1 we set $\xi = \frac{\beta-\alpha}{2} + 1$, which is positive thanks to the assumption $\beta > \alpha - 2$. We have

$$\begin{aligned}\frac{\mathfrak{h}'}{\mathfrak{h}} &= \frac{\beta r^{-1}(1+r^\alpha) - \alpha r^{\alpha-1}}{1+r^\alpha} \\ &= \frac{\beta-\alpha}{r} + \frac{1}{r}\left(\frac{\alpha}{1+r^\alpha}\right) = \frac{1}{r}(\beta-\alpha) + \frac{1}{r}O(r^{-\alpha}).\end{aligned}$$

By the same argument we obtain

$$\frac{\mathfrak{h}''}{\mathfrak{h}} = \frac{1}{r^2}(\beta-\alpha)(\beta-\alpha-1) + \frac{1}{r^2}O(r^{-\alpha}).$$

Then (4.4) is reduced to

$$\begin{aligned} g_1(r) &= -\mathfrak{v}_\lambda' + \frac{\mathfrak{v}_\lambda}{r}\left(\xi - 1 + O(r^{-\alpha}) + 2r^\xi\sqrt{\frac{r^\alpha}{1+r^\alpha}}\right) + \mathfrak{v}_\lambda^2 \\ &\quad + \frac{c_0}{r^2} + \frac{1}{r^2}\left(O(r^{-\alpha}) + O(r^{-2\alpha})\right) + \frac{b}{2}(N-2+\alpha)\frac{1}{r^2(1+r^\alpha)} \\ &\quad + \frac{b\alpha}{2}\left(\frac{b}{2\alpha} - 1\right)\frac{1+2r^\alpha}{r^2(1+r^\alpha)^2} \\ &= -\mathfrak{v}_\lambda' + \frac{\mathfrak{v}_\lambda}{r}\left(\xi - 1 + O(r^{-\alpha}) + 2r^\xi - 2r^\xi\frac{(1+r^\alpha)^{1/2} - r^{\alpha/2}}{(1+r^\alpha)^{1/2}}\right) \\ &\quad + \mathfrak{v}_\lambda^2 + \frac{c_0}{r^2} + \frac{1}{r^2}O(r^{-\alpha}) \\ &= -\mathfrak{v}_\lambda' + \frac{\mathfrak{v}_\lambda}{r}\left(\xi - 1 + 2r^\xi + (1+r^\xi)O(r^{-\alpha})\right) + \mathfrak{v}_\lambda^2 \\ &\quad + \frac{c_0}{r^2} + \frac{1}{r^2}O(r^{-\alpha}), \end{aligned}$$

(4.6)

where

$$c_0 = c_0(\xi) = \left(\frac{\xi-1}{2}\right)^2 + \frac{\xi-1}{2} - \frac{(N-1)(N-3)}{4} - \frac{b}{2}(N-2+\alpha) - \frac{b\alpha}{2}\left(\frac{b}{2\alpha} - 1\right).$$

So, if we take in (4.6)

$$\mathfrak{v}_\lambda(r) = \sum_{i=1}^{k} c_i \frac{1}{r^{i\xi+1}}, \quad r \geq 1,$$

we obtain, as in the proof of [**15**, Proposition 2.2],

$$\begin{aligned} r^2 g_1(r) &= \sum_{i=2}^{k-1} \left[c_i \xi (i+1) + 2c_{i+1} + \sum_{j+s=i} c_j c_s \right] \frac{1}{r^{i\xi}} + (2c_1\xi + 2c_2) r^{-\xi} \\ &\quad + c_k \xi (k+1) \frac{1}{r^{k\xi}} + 2c_1 + \sum_{i+j \geq k} \frac{c_i c_j}{r^{(i+j)\xi}} + c_0 + O(r^{-\alpha}), \end{aligned}$$

where $k \geq 3$ be chosen later. Again as in [**15**], we can choose $c_1, \ldots, c_k$ such that

$$2c_1 + c_0 = \lambda, \; 2c_1\xi + 2c_2 = 0 \text{ and } \left[\xi(i+1)c_i + 2c_{i+1} + \sum_{j+s=i} c_j c_s \right] = 0$$

for $i = 2, \cdots, k-1$ and obtain

$$r^2 g_1(r) = \lambda + c_k \xi (k+1) \frac{1}{r^{k\xi}} + \sum_{i+j \geq k} \frac{c_i c_j}{r^{(i+j)\xi}} + O(r^{-\alpha}).$$

Hence,

$$g_1(r) = O\left(\frac{1}{r^{k\xi+2}} \right) + O\left(\frac{1}{r^{\alpha+2}} \right) + \frac{\lambda}{r^2}.$$

Since $\xi > 0$, there exists a natural number $k \geq 3$ such that $k\xi + 2 - \alpha > 0$. So, we have

$$\begin{aligned} (1+|x|^\alpha)\Delta g_{\alpha,\beta,\lambda}(x) &+ b|x|^{\alpha-2} x \cdot \nabla g_{\alpha,\beta,\lambda}(x) - |x|^\beta g_{\alpha,\beta,\lambda}(x) \\ &= o(1) g_{\alpha,\beta,\lambda}(x) + \lambda \frac{1+|x|^\alpha}{|x|^2} g_{\alpha,\beta,\lambda}(x). \end{aligned}$$

For Φ we know that

$$\Delta\Phi + b \frac{|x|^{\alpha-2}}{1+|x|^\alpha} x \cdot \nabla \Phi - \frac{|x|^\beta}{1+|x|^\alpha} \Phi - \frac{\lambda_0}{1+|x|^\alpha} \Phi = 0 \,. \tag{4.7}$$

Since $\alpha - 2 > 0$ and $\lambda_0 < 0$, for $|x|$ large enough we have

$$o(1) + 2\lambda_0 \frac{1+|x|^\alpha}{|x|^2} < \lambda_0.$$

Thus,

$$\Delta g_{\alpha,\beta,2\lambda_0}(x) + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha} x \cdot \nabla g_{\alpha,\beta,2\lambda_0}(x) - \frac{|x|^\beta}{1+|x|^\alpha} g_{\alpha,\beta,2\lambda_0}(x) - \frac{\lambda_0}{1+|x|^\alpha} g_{\alpha,\beta,2\lambda_0}(x) < 0 \tag{4.8}$$

for all $x \in \mathbb{R}^N \setminus B_R$ for some $R > 0$. Comparing (4.7) and (4.8), in $\mathbb{R}^N \setminus B_R$ we have

$$\Delta(g_{\alpha,\beta,2\lambda_0} - C\Phi) + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha} x \cdot \nabla (g_{\alpha,\beta,2\lambda_0}(x) - C\Phi) \quad < \quad \frac{\lambda_0 + |x|^\beta}{1+|x|^\alpha}(g_{\alpha,\beta,2\lambda_0} - C\Phi)$$

for any constant $C > 0$. Since $\beta > 0$, we deduce that

$$\mathcal{W}(x) := \frac{\lambda_0 + |x|^\beta}{1+|x|^\alpha} > 0$$

for $|x|$ large enough. Note that both $g_{\alpha,\beta,2\lambda_0}$ and Φ go to 0 as $|x| \to \infty$ and since there exists C_2 such that $\Phi \le C_2 g_{\alpha,\beta,2\lambda_0}$ on ∂B_R, we can apply the maximum principle to the problem

$$\begin{cases} \left(\Delta + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha} x \cdot \nabla\right) z(x) - \mathcal{W}(x) z(x) < 0 & \text{in } \mathbb{R}^N \setminus B_R, \\ z(x) \ge 0 & \text{in } \partial B_R, \\ \lim_{|x|\to\infty} z(x) = 0, \end{cases}$$

where $z := g_{\alpha,\beta,2\lambda_0} - C_2^{-1}\Phi$, to obtain that $\Phi \le C_2 g_{\alpha,\beta,2\lambda_0}$ in $\mathbb{R}^N \setminus B_R$ (and by continuity in $\mathbb{R}^N \setminus B_1$). Here we apply the classical maximum principle on bounded domains, since $\lim_{|x|\to\infty} z(x) =$

0, cf. [**26**, Theorem 3.5]. Then,

$$\Phi(x) \le C_2|x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}(1+|x|^\alpha)^{-\frac{b}{2\alpha}} \exp\left\{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}\,dr\right\}\exp\left\{-\int_1^{|x|}\mathfrak{v}_{2\lambda_0}(r)dr\right\}$$
$$\le C_3|x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}(1+|x|^\alpha)^{-\frac{b}{2\alpha}} \exp\left\{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}\,dr\right\},$$

$$\lim_{|x|\to\infty}\int_1^{|x|}\mathfrak{v}_\lambda(r)\,dr = \lim_{|x|\to\infty}\sum_{j=1}^k \frac{c_j}{j\xi}(1-|x|^{-j\xi})$$
$$= \sum_{j=1}^k \frac{c_j}{j\xi}. \tag{4.9}$$

As regards lower bounds of Φ, we observe that

$$\Delta g_{\alpha,\beta,0}(x) + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha}x\cdot\nabla g_{\alpha,\beta,0}(x) - \frac{|x|^\beta}{1+|x|^\alpha}g_{\alpha,\beta,0}(x)$$
$$= \frac{o(1)}{1+|x|^\alpha}g_{\alpha,\beta,0}(x) > \frac{\lambda_0}{1+|x|^\alpha}g_{\alpha,\beta,0}(x)$$

if $|x| \ge R$ for some suitable $R > 0$. Then,

$$\Delta g_{\alpha,\beta,0}(x) + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha}x\cdot\nabla g_{\alpha,\beta,0}(x) > \frac{|x|^\beta}{1+|x|^\alpha}g_{\alpha,\beta,0}(x) + \frac{\lambda_0}{1+|x|^\alpha}g_{\alpha,\beta,0}(x).$$

Since $\Delta\Phi + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha}x\cdot\nabla\Phi - \frac{|x|^\beta}{1+|x|^\alpha}\Phi - \frac{\lambda_0}{1+|x|^\alpha}\Phi = 0$ we obtain

$$\Delta(g_{\alpha,\beta,0}-\Phi) + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha}x\cdot\nabla(g_{\alpha,\beta,0}(x)-\Phi) > \frac{|x|^\beta+\lambda_0}{1+|x|^\alpha}(g_{\alpha,\beta,0}-\Phi).$$

Note that $|x|^\beta + \lambda_0$ is positive for $|x| \geq R$ and, arguing as above, by the maximum principle and using (4.9) we have

$$\Phi(x) \geq C_1 g_{\alpha,\beta,0}(x) \geq C_1 |x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}(1+|x|^\alpha)^{-\frac{b}{2\alpha}} \exp\left\{-\int_R^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}\,dr\right\}$$

for $|x| \geq R$. Since $0 < \Phi \in C(\mathbb{R}^N)$, by modifying the constant C_1, we can see that, the above lower estimate of Φ remain valid for $1 \leq |x| \leq R$. □

Remark 4.

1. *If in the above proposition we take $b = 0$, then we obtain exactly the upper and lower estimates for the ground state ψ associated to the operator $(1+|x|^\alpha)\Delta - |x|^\beta$ established in* [**15**, Proposition 2.2].
2. *If we denote by Φ_c the eigenfunction of the largest eigenvalue of $A_{b,c}$, then one can see that*

$$\Phi_c \approx |x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}(1+|x|^\alpha)^{-\frac{b}{2\alpha}} e^{-\int_1^{|x|}\sqrt{\frac{cr^\beta}{1+r^\alpha}}dr}$$

on $\mathbb{R}^N \setminus B_1$.

4.3. Intrinsic ultracontractivity and heat kernel estimates

In this section we prove heat kernel estimates for $T_p(\cdot)$ through the relationship between the log-Sobolev inequality and the ultracontractivity of a suitable semigroup in a weighted L^2-space.

Consider the Hilbert spaces $L^2_\mu = L^2(\mathbb{R}^N, d\mu)$ with $d\mu(x) = \frac{1}{1+|x|^\alpha}dx$. Define the function $\phi(x) = (1+|x|^\alpha)^{\frac{b}{2\alpha}}$ and the multiplication operator $T : L^2_{\phi^2\mu} \to L^2_\mu$ defined by $Tu = \phi u$. The operator A defined above can be written in the following way

$$A = T^{-1}HT,$$

where $H = (1+|x|^\alpha)\Delta - U$ and the potential $U = (1+|x|^\alpha)\frac{\Delta\phi}{\phi} + |x|^\beta$. An easy computation gives us

$$U = |x|^{\alpha-2}\frac{b}{2}\left(\frac{|x|^\alpha}{1+|x|^\alpha}\left(\frac{b}{2}-\alpha\right) + N + \alpha - 2\right) + |x|^\beta,$$

from which we can deduce that U is bounded from below, since $\beta > \alpha - 2$.

Since, for every $v \in C_c^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}\left(|\nabla v|^2 + \frac{\Delta\phi}{\phi}|v|^2\right)dx + \int_{\mathbb{R}^N}|x|^\beta|v|^2 d\mu = \int_{\mathbb{R}^N}\left|\nabla\left(\frac{v}{\phi}\right)\right|^2\phi^2 dx + \int_{\mathbb{R}^N}|x|^\beta|v|^2 d\mu \geq 0,$$

we can associate to H in L^2_μ the bilinear form h defined by

$$h(v,w) = \int_{\mathbb{R}^N}\nabla v\cdot\nabla\overline{w}dx + \int_{\mathbb{R}^N}Uv\overline{w}d\mu \tag{4.10}$$

on $D(h) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mathcal{H}}}$ with $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = \{v \in L^2_\mu \cap W^{1,2}_{loc}\left(\mathbb{R}^N\right) : |U|^{1/2}v \in L^2_\mu,\ \nabla v \in (L^2(\mathbb{R}^N))^N\}$$

endowed with the inner product

$$\langle v, w\rangle_{\mathcal{H}} = \int_{\mathbb{R}^N}(1+U)v\overline{w}\,d\mu + \int_{\mathbb{R}^N}\nabla v\cdot\nabla\overline{w}\,dx.$$

So, one can prove the following proposition

Proposition 9. *The form $(h, D(h))$ is symmetric, continuous, closed and accretive.*

Similarly to [**46**, Lemma 9.1] we have

Lemma 8. *The injection of $D(h)$ in L^2_μ is compact.*

Since the bilinear form h is densely defined, accretive, continuous, closed and symmetric, one can associate the self-adjoint, dissipative operator H_μ defined by

$$D(H_\mu) = \left\{ v \in D(h) : \exists f \in L^2_\mu \text{ s.t. } h(v,w) = -\int_{\mathbb{R}^N} f\overline{w}\, d\mu, \text{ for every } w \in D(h) \right\},$$

$$H_\mu v = f,$$

see e.g. [**50**, Prop. 1.24]. Hence, H_μ generates a positive and analytic semigroup $(e^{tH_\mu})_{t\geq 0}$ in L^2_μ, cf. [**50**, Theorem 1.52 and Theorem 2.6].

The following result gives the relationship between H_μ and A_2.

Lemma 9. *The following holds*

$$D(H_\mu) = \left\{ u \in D(h) \cap W^{2,2}_{loc}(\mathbb{R}^N) : (1+|x|^\alpha)\Delta v - Uv \in L^2_\mu \right\} \tag{4.11}$$

and $H_\mu v = (1+|x|^\alpha)\Delta v - U(x)v$ *for* $v \in D(H_\mu)$. *Moreover, if* $\lambda > \lambda'$ *where* λ' *is a suitable positive constant and* $f \in C_c^\infty(\mathbb{R}^N)$, *then*

$$T^{-1}(\lambda - H_\mu)^{-1} T f = (\lambda - A_2)^{-1} f.$$

Proof. Let us begin by proving (4.11). The first inclusion " $\subseteq$ " is obtained by local elliptic regularity and (4.10).

For the second inclusion " $\supseteq$ " let us take $v \in D(h) \cap W^{2,2}_{loc}(\mathbb{R}^N)$ such that $f := (1+|x|^\alpha)\Delta v - U(x)v \in L^2_\mu$. Integrating by parts we obtain

$$h(v,w) = -\int f\overline{w}d\mu, \quad \forall w \in C_c^\infty(\mathbb{R}^N). \tag{4.12}$$

By the density of $C_c^\infty(\mathbb{R}^N)$ in $D(h)$, (4.12) holds for every $w \in D(h)$. This implies that $v \in D(H_\mu)$.

To prove the coherence of the resolvents, we consider, for a positive function $f \in C_c^\infty(\mathbb{R}^N)$, the following elliptic problem

$$(4.13) \qquad \begin{cases} \lambda u - Au = f & x \in B_n, \\ u = 0 & x \in \partial B_n. \end{cases}$$

Since the operator A is uniformly elliptic in the ball B_n, it is known that (4.13) admits a unique solution u_n in $W^{2,2}(B_n) \cap W_0^{1,2}(B_n)$, (cf. [**26**, Theorem 9.15]). Likewise, as in [**36**], the sequence (u_n) is increasing, positive and converges to a function u in $D_{max}(A)$ satisfying $\lambda u - Au = f$, where

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W_{loc}^{2,p}(\mathbb{R}^N) \quad \text{for all} \quad 1 \leq p < \infty : Au \in C_b(\mathbb{R}^N)\}.$$

Setting $v_n = Tu_n$ and $g = Tf$ we have, by (4.13),

$$(4.14) \qquad \lambda v_n - Hv_n = g \quad \text{ on } B_n.$$

Moreover $v_n \in W^{2,2}(B_n) \cap W_0^{1,2}(B_n)$.

Multiplying in (4.14) by $\overline{w}\mu$ for $w \in W_0^{1,2}(B_n)$ and integrating by parts we obtain

$$(4.15) \qquad \lambda \int_{B_n} v_n \overline{w} d\mu + \int_{B_n} \nabla v_n \cdot \nabla \overline{w}\, dx + \int_{B_n} U v_n \overline{w}\, d\mu = \int_{B_n} g \overline{w} d\mu.$$

In particular we obtain

$$(4.16) \qquad \lambda \int_{B_n} v_n^2 d\mu + \int_{B_n} |\nabla v_n|^2\, dx + \int_{B_n} U v_n^2\, d\mu = \int_{B_n} g v_n d\mu.$$

Since $\int_{B_n} |\nabla v_n|^2\, dx + \int_{B_n} U v_n^2\, d\mu \geq 0$, it follows from (4.16) that

$$\|v_n\|_{L^2_\mu} \leq \frac{1}{\lambda} \|g\|_{L^2_\mu}$$

By the monotone convergence theorem, we deduce that $\lim_{n\to+\infty} v_n = v$ in L^2_μ. Furthermore since U is bounded from below we can choose $\tilde{\lambda}$ such that $\lambda + U \geq 0$ for $\lambda > \tilde{\lambda}$. Then, by (4.16), we have

$$\| (\lambda + U)^{\frac{1}{2}} v_n\|^2_{L^2_\mu} \leq \|g\|_{L^2_\mu}\|v_n\|_{L^2_\mu} \leq \frac{1}{\lambda}\|g\|^2_{L^2_\mu}.$$

Choosing $\lambda > \lambda' := \max\{-2\min U, \tilde{\lambda}, 0\}$, we obtain $|U| \leq |\lambda + U|$, and hence $|U|^{\frac{1}{2}} v_n \to |U|^{\frac{1}{2}} v$ in L^2_μ.

Similarly one finds

$$\|\nabla v_n\|^2_2 \leq \|g\|_{L^2_\mu}\|v_n\|_{L^2_\mu} \leq \frac{1}{\lambda}\|g\|^2_{L^2_\mu}, \quad \forall \lambda > \lambda'.$$

It follows that there exists a suitable subsequence (v_{k_n}) of (v_n) such that ∇v_{k_n} converges weakly. So, $v \in \mathcal{H}$ and v belongs to the closure in $\mathcal{H}$ of $W^{1,2}$-functions with compact support, which implies that $v \in D(h)$. Letting now $n \to +\infty$ in (4.15) we obtain

$$h(v, w) = -\langle \lambda v - g, w\rangle_{L^2_\mu}, \quad \lambda > \lambda',$$

for all $w \in W^{1,2}$ having compact support, and hence for all $w \in D(h)$. Thus, $v \in D(H_\mu)$ and $\lambda v - H_\mu v = g$ for all $\lambda > \lambda'$. Therefore, since $v = Tu$ and $g = Tf$, it follows that

$$(\lambda - H_\mu)^{-1} Tf = T(\lambda - A)^{-1} f$$

for all $f \in C_c^\infty(\mathbb{R}^N)$ and $\lambda > \lambda'$. So, the statement follows now from [**8**, Theorem 2]. □

Lemma stated above implies in particular that

$$e^{tH_\mu} f(x) = \int_{\mathbb{R}^N} k_\mu(t, x, y) f(y)\, d\mu(y), \quad f \in L^2_\mu$$

with

$$k_\mu(t,x,y) = \phi(x)k(t,x,y)\phi(y)^{-1}(1+|y|^\alpha), \quad t>0,\ x,\ y \in \mathbb{R}^N. \tag{4.17}$$

As an application of the Proposition 8, we have

Proposition 10. *If $N>2$, $\alpha>2$ and $\beta>\alpha-2$, then*

$$k(t,x,x) \geq Ce^{\lambda_0 t}\left(|x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}(1+|x|^\alpha)^{-\frac{b}{2\alpha}}e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr}\right)^2(1+|x|^\alpha)^{\frac{b}{\alpha}-1}, \quad t>0,$$

for all $x \in \mathbb{R}^N \setminus B_1$ and some constant $C>0$.

Proof. The proof is based on the semigroup law and the symmetry of $k_\mu(t,\cdot,\cdot)$ for $t>0$, and the semigroup law imply that

$$k_\mu(t,x,x) = \int_{\mathbb{R}^N} k_\mu(t/2,x,y)^2 d\mu(y), \quad t>0,\ x \in \mathbb{R}^N.$$

By Hölder's inequality and (4.17), we deduce that

$$\begin{aligned}
e^{\lambda_0 \frac{t}{2}}\Phi(x) &= T_2(t/2)\Phi(x) \\
&= \int_{\mathbb{R}^N} k(t/2,x,y)\Phi(y)\,dy \\
&= \phi(x)^{-1}\int_{\mathbb{R}^N} k_\mu(t/2,x,y)\phi(y)\Phi(y)\,d\mu(y) \\
&\leq \left(\int_{\mathbb{R}^N} k_\mu(t/2,x,y)^2 d\mu(y)\right)^{\frac{1}{2}}\|\phi\Phi\|_{L^2_\mu}\phi(x)^{-1} \\
&= k_\mu(t,x,x)^{\frac{1}{2}}\|\phi\Phi\|_{L^2_\mu}\phi(x)^{-1}
\end{aligned}$$

for all $t>0$ and $x \in \mathbb{R}^N$. Since $\frac{\beta-\alpha}{2}+1>0$, it follows from Proposition 8 that $\|\eta\Phi\|_{L^2_\mu}$ is finite. Thus, the assertion follows from (4.17).

The assertion follows now from Proposition 8. □

Now, in order to estimate k_μ we use the techniques in [**18**, Chap 4]

Proposition 11. *There exist positive constants* C_1, C_2, C_3, C_4 *such that*

$$\int_{\mathbb{R}^N} -\log(T\Phi)|v|^2 d\mu \le \varepsilon h(v,v) + (C_1\varepsilon^{-\gamma} + C_2)\|v\|^2_{L^2_\mu}, \quad v \in D(h), \tag{4.18}$$

for any $\varepsilon > 0$ *with* $\gamma = \frac{\beta-\alpha+2}{\beta+\alpha-2}$, *and*

$$\int_{\mathbb{R}^N} f|v|^2 d\mu \le C_3\|f\|_{L^{N/2}_\mu}\left(h(v) + C_4\|v\|^2_{L^2_\mu}\right), \quad v \in D(h),\ f \in L^{N/2}_\mu, \tag{4.19}$$

Proof. To prove (4.18), we apply the lower estimate of Φ obtained in Proposition 8

$$C_1|x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \le C_1|x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr} \le (T\Phi)(x)$$

we get

$$-\log(T\Phi) \quad \le \quad -\log C_1 + \left(\frac{N-1}{2} + \frac{\beta-\alpha}{4}\right)\log|x| + \frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}$$

for $|x| \ge 1$. Setting $\xi = \frac{\beta-\alpha}{2} + 1$ we have

$$-\log(T\Phi) \quad \le \quad -\log C_1 + \frac{1}{2}(N-2+\xi)\log|x| + \frac{1}{\xi}|x|^\xi$$

As a consequence, there are positive constants C_2, C_3 such that

$$-\log(T\Phi) \le C_2\left(1+\frac{1}{\xi}\right)|x|^\xi + C_3, \quad x \in \mathbb{R}^N.$$

Since $\xi < \beta$, $\gamma = \frac{\xi}{\beta-\xi}$, by using Young's inequality,[1] it follows that

$$C_2\left(1+\frac{1}{\xi}\right)|x|^{\xi} \leq \varepsilon|x|^{\beta} + C_4\varepsilon^{-\frac{\xi}{\beta-\xi}} = \varepsilon V(x) + C_4\varepsilon^{-\gamma}$$

for all $\varepsilon > 0$ where $C_4 = \left(C_2\left(1+\frac{1}{\xi}\right)\right)^{\frac{\beta}{\beta-\xi}} \frac{\beta-\xi}{\xi}\left(\frac{\beta}{\xi}\right)^{-\frac{\beta}{\beta-\xi}}$. Thus,

$$-\log(T\Phi) \leq \varepsilon|x|^{\beta} + C_4\varepsilon^{-\gamma} + C_3.$$

Taking into account that

$$0 \leq \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} \frac{\Delta\phi}{\phi}|v|^2 dx = h(v,v) - \int_{\mathbb{R}^N} |x|^{\beta} v^2 d\mu, \quad v \in D(h),$$

we obtain

$$\int_{\mathbb{R}^N} -\log(T\Phi)|v|^2 d\mu \leq \varepsilon h(v,v) + (C_4\varepsilon^{-\gamma} + C_3)\int_{\mathbb{R}^N} |v|^2 d\mu.$$

This proves (4.18).

Concerning (4.19), by density, it suffices to show it for $v \in C_c^{\infty}(\mathbb{R}^N)$. Using Hölder and Sobolev's inequalities we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} f v^2 d\mu &\leq \|f\|_{L_{\mu}^{N/2}} \|v\|^2_{L_{\mu}^{\frac{2N}{N-2}}} \leq \|f\|_{L_{\mu}^{N/2}} \|v\|^2_{L^{\frac{2N}{N-2}}} \\ &\leq \|f\|_{L_{\mu}^{N/2}} \|\nabla v\|_2^2.\end{aligned}$$

So, one obtains (4.19) by observing that

$$\|\nabla v\|_2^2 = h(v,v) - \int_{\mathbb{R}^N} U|v|^2 d\mu \leq h(v,v) + C_5\|v\|^2_{L^2_{\mu}},$$

where $C_5 = -\left(0 \wedge \min_{x\in\mathbb{R}^N} U\right)$. □

[1] $xy \leq \varepsilon x^p + C_{\varepsilon} y^q$, $\forall x \geq 0$ $\forall y \geq 0$ with $C_{\varepsilon} = \varepsilon^{-\frac{1}{p-1}}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We give now the estimate of $k_\mu(t,x,y)$.

Theorem 7. *Assume that $N>2$, $\alpha>2$ and $\beta>\alpha-2$. Then there exist C_1, C_2 positive constants such that*

$$k_\mu(t,x,y) \le C_1 e^{C_2 t^{-\gamma}} (T\Phi)(x)(T\Phi)(y), \quad 0<t\le 1,\ x,\ y\in\mathbb{R}^N. \tag{4.20}$$

Proof. It follows from Proposition 11 and Rosen's lemma, cf. [**18**, Lemma 4.4.1], that for all $0\le f\in L^2(\mathbb{R}^N,(T\Phi)^2\mu dx)$ and $\varepsilon>0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}(\log f)|v|^2 d\mu &\le \varepsilon h(v,v) + (C_2 - \frac{N}{4}\log\varepsilon + \varepsilon\frac{C_4}{2} + C_1\varepsilon^{-\gamma})\int_{\mathbb{R}^N}|v|^2 d\mu \\ &\le \varepsilon h(v,v) + (C_5 + C_6\varepsilon^{-\gamma})\int_{\mathbb{R}^N}|v|^2 d\mu, \quad v\in D(h).\end{aligned}$$

So, applying [**18**, Corollary 4.4.2], one obtains

$$\begin{aligned}\int_{\mathbb{R}^N}(f^2\log f)(T\Phi)^2 d\mu &\le \varepsilon h_{T\Phi}(f,f) + \left(C_5 + C_6\varepsilon^{-\gamma}\right)\|f\|^2_{L^2(\mathbb{R}^N,(T\Phi)^2d\mu)} \\ &\quad + \|f\|^2_{L^2(\mathbb{R}^N,(T\Phi)^2d\mu)}\log\|f\|^2_{L^2(\mathbb{R}^N,(T\Phi)^2d\mu)}\end{aligned}$$

for all $0\le f\in L^1\cap L^\infty\cap D(h_{T\Phi})$ and all $\varepsilon>0$, where $h_{T\Phi}$ is the quadratic form defined by $h_{T\Phi}(v)=h((T\Phi)v,(T\Phi)v)$ for $v\in D(h_{T\Phi}) := \{v\in L^2(\mathbb{R}^N,(T\Phi)^2d\mu) : (T\Phi)v\in D(h)\}$. We observe that $0<\gamma<1$. Then we can apply [**18**, Corollary 2.2.8] and [**18**, Lemma 2.1.2] to obtain that

$$0\le k_{T\Phi}(t,x,y)\le C_7 e^{C_8 t^{-\gamma}}, \quad 0<t\le 1,\ x,\ y\in\mathbb{R}^N,$$

where $k_{T\Phi}(\cdot,\cdot,\cdot)$ is the heat kernel of the semigroup generated by the selfadjoint operator associated to the form $h_{T\Phi}$.

The result follows now by taking into account

$$k_\mu(t,x,y) = (T\Phi)(x) k_{T\Phi}(t,x,y)(T\Phi)(y), \quad t>0,\ x,\ y \in \mathbb{R}^N.$$

□

Now, we are ready to state and give the proof of the main result of this paper.

Theorem 8. *If $N>2$, $\beta > \alpha - 2$, $\alpha > 2$ then*

$$k(t,x,y)$$

$$\leq C_1 e^{\lambda_0 t + C_2 t^{-\gamma}} \left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{\frac{b}{2\alpha}} \frac{(|x||y|)^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}}{1+|y|^\alpha} e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr} e^{-\int_1^{|y|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr} \tag{4.21}$$

for all $t>0$, $x, y \in \mathbb{R}^N \setminus B_1$, where C_1, C_2 are positive constants and $\gamma = \frac{\beta-\alpha+2}{\beta+\alpha-2}$.

Proof. By (4.17), (4.20) and Proposition 8 we have

$$\begin{aligned}
k(t,x,y) &= \frac{\phi(y)}{\phi(x)} \frac{1}{1+|y|^\alpha} k_\mu(t,x,y) \\
&\leq C_1 \frac{\phi^2(y)}{1+|y|^\alpha} e^{-C_2 t^{-\gamma}} \Phi(x)\Phi(y) \\
&= C_1 \frac{(1+|y|^\alpha)^{\frac{b}{\alpha}}}{1+|y|^\alpha} e^{-C_2 t^{-\gamma}} \Phi(x)\Phi(y) \\
&\leq C_1 e^{-C_2 t^{-\gamma}} \left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{\frac{b}{2\alpha}} \frac{(|x||y|)^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}}{1+|y|^\alpha} \\
&\quad e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr} e^{-\int_1^{|y|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr}
\end{aligned}$$

for $0 < t \leq 1$, x, $y \in \mathbb{R}^N \setminus B_1$ and $\gamma = \frac{\beta-\alpha+2}{\beta+\alpha-2}$.

To end the proof let us consider the case $t > 1$.

The semigroup law and the symmetry of $k_\mu(t,\cdot,\cdot)$ imply that

$$k_\mu(t,x,y)=\int_{\mathbb{R}^N} k_\mu(t-1/2,x,z)k_\mu(1/2,y,z)d\mu(z), \qquad t>1/2,\ \ x,y\in\mathbb{R}^N.$$

On the other hand, thanks to (4.20), and since $\beta>\alpha-2$, one deduces that the function $k_\mu(1/2,y,\cdot)$ belongs to L^2_μ. Hence,

$$k_\mu(t,x,y)=(e^{(t-\frac{1}{2})H_\mu}k_\mu(1/2,y,\cdot))(x), \qquad t>1/2,\ \ x,y\in\mathbb{R}^N.$$

Using again the semigroup law and the symmetry we have

$$\begin{aligned} k_\mu(t,x,x)=&\int_{\mathbb{R}^N}|k_\mu(t/2,x,y)|^2d\mu(y)\\ \le& Me^{\lambda_0(t-1)}\|k_\mu(1/2,x,\cdot)\|^2_{L^2_\mu}\\ =& Me^{\lambda_0(t-1)}k_\mu(1,x,x), \quad t>1,\ x\in\mathbb{R}^N. \end{aligned}$$

So, by applying (4.20) to $k_\mu(1,x,x)$ and using the inequality

$$k_\mu(t,x,y)\le (k_\mu(t,x,x))^{1/2}(k_\mu(t,y,y))^{1/2}, \quad t>0,\ x,\ y\in\mathbb{R}^N,$$

one obtains (4.21). □

Remark 5.

1. *The heat kernel estimates $k(\cdot,\cdot,\cdot)$ in Theorem 8 could be sharp in the space variables. This can be seen from Proposition 10.*

2. *Since, for* $r \geq 1$, $\sqrt{\frac{2r^\beta}{1+r^\alpha}} \geq r^{\frac{\beta-\alpha}{2}}$, *it follows from Theorem 8 that*

$$k(t,x,y) \leq c_1 e^{\lambda_0 t + c_2 t^{-\gamma}} \left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{\frac{b}{2\alpha}} \frac{(|x||y|)^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)}}{1+|y|^\alpha} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}\left(|x|^{\frac{\beta-\alpha+2}{2}}+|y|^{\frac{\beta-\alpha+2}{2}}\right)}$$

for $t>0$, $|x|$, $|y| \geq 1$.

If we denote by Φ_j the eigenfunction of A_2 associated to the eigenvalue λ_j, then $T\Phi_j$ is the eigenfunction of H_μ associated to λ_j. Hence, for any $t>0$ and any $x \in \mathbb{R}^N$, we have

$$\begin{aligned} e^{\lambda_j t}|T\Phi_j(x)| &= \left|\int_{\mathbb{R}^N} k_\mu(t,x,y)(T\Phi_j)(y)d\mu(y)\right| \\ &\leq \left(\int_{\mathbb{R}^N} k_\mu(t,x,y)^2 d\mu(y)\right)^{\frac{1}{2}} \|T\Phi_j\|_{L^2_\mu} \\ &= (k_\mu(2t,x,x))^{\frac{1}{2}} \|T\Phi_j\|_{L^2_\mu}. \end{aligned}$$

So, by (4.20), we obtain the following estimates.

Corollary 1. *Suppose that the assumptions of Theorem 8 hold. Then all eigenfunctions* Φ_j *of* A *with* $\|T\Phi_j\|_{L^2_\mu} = 1$ *satisfy*

$$|\Phi_j(x)| \leq C_j |x|^{\frac{\alpha-\beta}{4}-\frac{N-1}{2}} e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr},$$

for all $j \in \mathbb{N}$, $x \in \mathbb{R}^N \setminus B_1$ *and some constant* $C_j > 0$.

Remark 6. *In the case* $b > 2-N$, *we can obtain better estimates of the kernels* k *with respect to the time variable* t *for small* t. *In fact if we denote by* $S(\cdot)$ *the semigroup generated by* $(1+|x|^\alpha)\Delta + b|x|^{\alpha-2}x\cdot\nabla$ *in* $C_b(\mathbb{R}^N)$, *which is given by a kernel* p, *then by domination we* *have* $0 < k(t,x,y) \le p(t,x,y)$ *for* $t > 0$ *and* $x,\ y \in \mathbb{R}^N$*. So, by* [**46**, Remark 9.12]*, it follows that*

$$\begin{aligned} k(t,x,y) &\le Ct^{-\frac{N+b+\alpha-4}{\alpha-2}}(1+|x|)^{2-N-b}(1+|y|)^{2-N-\alpha}, \quad 2 < \alpha \le 4 + \frac{2b}{N-2}, \\ k(t,x,y) &\le Ct^{-N/2}(1+|x|^{\alpha})^{2-N-b}(1+|y|^{\alpha})^{2-N-\alpha}, \quad \alpha \ge 4 + \frac{2b}{N-2} \end{aligned}$$

for $0 < t \le 1,\ x,\ y \in \mathbb{R}^N$*.*

Appendices

CHAPTER A

Preliminary facts on semigroups of linear operators

Semigroups of linear operators have been widely studied during the last years and there are many monographs dealing with them. We give some definitions and recall several important results and properties of unbounded operators and semigroups. For more details and proofs of the classical results given below see the excellent graduate texts by, K-J. Engel, R. Nagel [**21**], and references as E.M. Ouhabaz [**50**], L. Lorenzi, M. Bertoldi [**36**], J.A. Goldstein [**28**], A. Pazy [**51**] and [**33, 39**]. Sections A.1, A.2 are devoted to basic properties of linear operators and strongly continuous semigroups in Banach spaces. Section A.3 deals with analytic semigroups acting in general Banach spaces.

Notation: Let X, Y be real or complex Banach spaces. $\mathbb{K}$ will denote the underlying scalar field, $\mathbb{K} = \mathbb{R}$, the real numbers, or $\mathbb{K} = \mathbb{C}$, the complex numbers. $\mathcal{B}(X, Y)$ is the space of all bounded linear operators from X to Y. $\mathcal{B}(X) = \mathcal{B}(X, X)$. A is an operator on X to Y means A is a linear operator from its domain $\mathcal{D}(A) \subset X$ to Y. A is an operator on X means A is an operator on X to X.

A.1. Definition and some basic properties

Definition 2. *An operator A on X to Y is closed if its graph $\mathcal{G}(A) = \{(f, Af) : f \in \mathcal{D}(A)\}$ is a closed subspace of $X \times Y$, or equivalently, if $f_n \in \mathcal{D}(A)$, $f_n \to f$, and $Af_n \to g$ imply $f \in \mathcal{D}(A)$ and $Af = g$.*

Note also that A is a closed operator if and only if $\mathcal{D}(A)$ endowed with the graph norm $\|\cdot\| + \|A\cdot\|$ is a complete space.

Definition 3. *An operator A on X to Y is closable if the closure of its graph $\overline{\mathcal{G}(A)}$ is a graph, i.e. $(0, y) \in \overline{\mathcal{G}(A)}$ implies $y = 0$. Then $\overline{\mathcal{G}(A)}$ is the graph of a closed operator, which is called the closure of A and is denoted by $\overline{A}$.*

Definition 4. *Let A be an operator on X. The resolvent set of A is $\rho(A) = \{\lambda \in \mathbb{K} : \lambda I - A : \mathcal{D}(A) \to X : (\lambda I - A)^{-1} \in \mathcal{B}(X)\}$. Here I is the identity operator on X and $(\lambda - A)^{-1}$ is called the resolvent (operator) of A.*

The complement of $\rho(A)$ in $\mathbb{K}$

$$\sigma(A) := \mathbb{K} \backslash \rho(A)$$

is called the spectrum of A.

Proposition 12. (1) *Assume that A is a closed operator on a Banach space X. Then $\lambda \in \rho(A)$ if and only if $\lambda I - A$ is invertible (from $\mathcal{D}(A)$ into X).*

(2) *If the resolvent set $\rho(A)$ is not empty, then A is a closed operator.*

Proof. (i) Follows from the closed graph theorem.

(ii) Suppose that the sequence $\{x_n\}$, contained in the domain of A, is such that $x_n \to x$ and $Ax_n \to y$. Given $\lambda \in \rho(A)$, we get $\lambda x_n - Ax_n \to \lambda x - y$ and then $x_n \to R_\lambda(\lambda x - y)$. Because of the uniqueness of the limit, we find $x = R_\lambda(\lambda x - y)$. This shows that $x \in D(\lambda I - A) = D(A)$ and $(\lambda I - A)x = \lambda x - y$, i.e. $Ax = y$ and the proposition is proved.

□

Definition 5. *Let A be an operator with domain $\mathcal{D}(A)$ on a Banach space X. A linear subspace of $\mathcal{D}(A)$ is called a core of A if it is dense in $\mathcal{D}(A)$, endowed with the graph norm $\|\cdot\| + \|A\cdot\|$.*

A.2. Strongly continuous semigroups

Let X be a Banach space. A semigroup of linear operators on X is a family of linear and continuous operators $T(t)\,(0 \leq t < \infty)$ from X into itself such that

$$T(0) = I$$

$$T(t+s) = T(t)T(s) \quad (s, t \geq 0).$$

From the semigroup $T(t)_{t\geq 0}$ we define the linear operator

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} \tag{A.1}$$

with domain $\mathcal{D}(A)$, the set of $x \in X$ such that the following limit exists:

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

We remark that the linear operator A does not need to be continuous. We say that $T(t)$ is a strongly continuous semigroup (briefly, a C_0-semigroup) if

$$\lim_{t\to 0^+} T(t)x = x, \quad \forall x \in X.$$

The operator A is said to be the generator of the C_0-semigroup if (A.1) holds for any $x \in \mathcal{D}(A)$.

The theory of C_0-semigroups was developed in order to study existence and uniqueness of solutions to the evolution equations (or the abstract Cauchy problem)

$$\frac{d}{dt}u(t) = Au(t), \qquad u(0) = x, \quad t \geq 0. \tag{A.2}$$

Note that the property of strong continuity is precisely the strong continuity at $t_0 = 0$. From this and the semigroup property it follows that $(T(t))_{t\geq 0}$ is strongly continuous at each $t_0 \in [0, \infty)$.

Remarks 1. *(a) If T is defined on $\mathbb{R}$ instead of $[0, \infty)$, and $T(t+s) = T(t)T(s)$ holds for all $t, s \in \mathbb{R}$, then T is called a one-parameter group, and if additionally the strong continuity holds, then T is called a C_0-**group**.*

(b) Property $T(t+s) = T(t)T(s)$ implies that for $t, s \geq 0$ the operators $T(t)$ and $T(s)$ commute.

(d) If T is a C_0-semigroup, then $T(0)f = \lim_{t\to 0^+} T(t)T(0)f = \lim_{t\to 0^+} T(t)f = f$ for all $f \in X$, i.e., $T(0) = I$.

The following estimate holds for any C_0-semigroup.

Proposition 13. *Let $T(t)$ be a C_0-semigroup. There exist two constants $\omega \in \mathbb{R}$, $M \geq 1$ such that*

$$||T(t)|| \leq Me^{\omega t} \quad 0 \leq t < \infty. \tag{A.3}$$

Proof. First let us show that there exist constants M and $\eta > 0$ such that

$$||T(t)|| \leq M \quad \forall\, t \in [0, \eta]. \tag{A.4}$$

If (A.4) is false, we can find a sequence of real numbers $t_n > 0$ such that $||T(t_n)|| > n, t_n \to 0$. It follows that there exists $x \in X$ such that

$$\sup_{n\in\mathbb{N}} ||T(t_n)x|| = \infty.$$

If not, we would have

$$\sup_{n\in\mathbb{N}} ||T(t_n)x|| < \infty \quad \forall x \in X;$$

in view of the Banach-Steinhaus Theorem, this implies

$$\sup_{n\in\mathbb{N}} ||T(t_n)|| < \infty.$$

and this is absurd. Formula (A.4) is proved.

Since $||T(0)|| = 1$, we have $M \geq 1$. Let now t be a nonnegative number; we can write $t = n\eta + \delta$, where n is a natural number and $0 \leq \delta < \eta$. Therefore

$$||T(t)|| = ||T(\delta)T(\eta)^n|| \leq M^{n+1} = M^{1+\frac{t-\delta}{\eta}} \leq M^{1+\frac{t}{\eta}} = Me^{\omega t}$$

where $\omega = \frac{(\log M)}{\eta}$. □

A first consequence of the above proposition is that $t \to T(t)x$ is continuous.

Proposition 14. *Let $T(t)$ be a C_0-semigroup on a Banach space X. For any $x \in X$, the function $0 \leq t \to T(t)x \in X$ is continuous.*

Proof. The continuity from the right at $t = 0$ is obvious. Let us fix $t > 0$ and take $h \geq 0$; we have

$$||T(t+h)x - T(t)x|| \leq ||T(t)||||T(h)x - x|| \leq Me^{\omega t}||T(h)x - x||$$

and then

$$\lim_{h\to 0^+} ||T(t+h)x - T(t)x|| = 0.$$

On the other hand, if $t-h \geq 0$, we have also

$$||T(t-h)x - T(t)x|| \leq ||T(t-h)|| ||x - T(h)x|| \leq Me^{\omega(t-h)}||x - T(h)x||.$$

It follows that

$$\lim_{h\to 0^-} ||T(t+h)x - T(t)x|| = 0$$

and the result is proved. □

The next proposition shows some properties of C_0-semigroups.

Proposition 15. *Let $T(t)$ be a C_0-semigroup and A its generator on a Banach space X. Then*

a- $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x \quad \forall x \in X$;

b- $x \in X \Longrightarrow \int_0^t T(s)xds \in \mathcal{D}(A)$ *and*

(A.5) $$A\left(\int_0^t T(s)xds\right) = T(t)x - x;$$

c- $x \in \mathcal{D}(A) \Longrightarrow T(s)x \in \mathcal{D}(A)$ *and*

(A.6) $$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax;$$

d- *for any $x \in \mathcal{D}(A)$ we have*

(A.7) $$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

Proof. Cf. Engel-Nagel [**21**, Lemma.1.3]. □

Proposition 16. *Let A be the generator of a C_0-semigroup $T(t)$ on a Banach space X. Then A is a densely defined closed operator.*

Proof. We start by proving that $\mathcal{D}(A)$ is dense in X. Let $x \in X$ and define

$$x_t = \frac{1}{t}\int_0^t T(s)x ds.$$

From b- of Proposition 15, $x_t \in \mathcal{D}(A)$ and from a- $x_t \to x$. This implies that $\overline{\mathcal{D}(A)} = X$.

In order to prove that A is a closed operator, we have to show that this implication holds

$$\text{(A.8)} \qquad \begin{cases} x_n \in \mathcal{D}(A) \\ x_n \to x \\ Ax_n \to y \end{cases} \implies \begin{cases} x \in \mathcal{D}(A) \\ Ax = y. \end{cases}$$

Since $x_n \in \mathcal{D}(A)$, (A.7) implies

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds$$

from which

$$\frac{T(t)x - x}{t} = \frac{1}{t}\int_0^t T(s)y ds.$$

As $t \to 0^+$, the right-hand side tends to y and thus $x \in \mathcal{D}(A), Ax = y$. □

The next result shows that a C_0-semigroup is uniquely determined by its generator.

Proposition 17. *Let A and B two generators of the C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ respectively. If $A = B$ then the two semigroups coincide, i.e., $T(t) = S(t)$ for any $t \geq 0$.*

Proof. Let $x \in D(A) = D(B)$. From (A.6) it follows that

$$\begin{aligned}\frac{d}{ds}T(t-s)S(s)x &= -AT(t-s)S(s)x + T(t-s)BS(s)x \\ &= -T(t-s)AS(s)x + T(t-s)BS(s)x = 0 \quad (0 < s < t)\end{aligned}$$

and then the function $T(t-s)S(s)x$ of the real variable s is constant. In particular $T(t)x = S(t)x$, i.e., $T(t) = S(t)$ on $\mathcal{D}(A)$. The domain $\mathcal{D}(A)$ being dense in X (see Proposition 16), it follows that $T(t) = S(t)$. □

From (A.6) and above Proposition we deduce that for any given $u_0 \in \mathcal{D}(A)$ the function $u(t) = T(t)u_0$ is the only classical solution of the abstract Cauchy problem

(A.9)
$$\begin{cases} \frac{du}{dt} = Au, \quad (t > 0) \\ u(0) = u_0. \end{cases}$$

Remark 7. *It is still possible to solve the Cauchy problem (A.9) in the case where u_0 is an arbitrary element of X. In order to do that, it is necessary to introduce a concept of generalized solution. For this we refer to* [**51**, Chap. 4].

Exemple 1. *Let $X = C^0([0, \infty])$, where this symbol means the space of the complex-valued functions defined in $[0, \infty)$ such that there exists the limit*

$$\lim_{x \to +\infty} f(x).$$

The space X, equipped with the norm

$$||f||_\infty = \sup_{x \in [0, +\infty)} |f(x)|,$$

is a Banach space. Define the family of operators $T(t)(t \geq 0)$ by

$$[T(t)f](x) = f(x+t).$$

Obviously, for any $t \geq 0$, it makes sense to consider $f(x+t)$. Moreover $T(t)f$ is a continuous function and since

$$\lim_{x \to +\infty} [T(t)f](x) = \lim_{x \to +\infty} f(x),$$

$T(t)$ maps X into itself. Let us remark that

$$||T(t)f||_\infty \leq ||f||_\infty.$$

It is clear that $T(t)$ is a semigroup. Let us prove that it is a C_0-semigroup, i.e.

$$\lim_{t \to 0^+} ||T(t)f - f||_\infty = 0. \tag{A.10}$$

By hypothesis, there exists $\alpha \in \mathbb{C}$ to which $f(x)$ tends as $x \to +\infty$. Given $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$|f(x) - \alpha| < \varepsilon \qquad \forall x \geq K_\varepsilon.$$

This implies

$$|f(x+t) - f(x)| \leq |f(x+t) - \alpha| + |\alpha - f(x)| < 2\varepsilon \quad \forall x \geq K_\varepsilon, t \geq 0. \tag{A.11}$$

On the other hand f is uniformly continuous on $[0, K_\varepsilon + 1]$ and then there exists $\delta_\varepsilon > 0$ (which can be supposed to be less than 1) such that

$$|f(x+t) - f(x)| < \varepsilon \quad \forall x \in [0, K_\varepsilon], 0 \leq t < \delta_\varepsilon.$$

Keeping in mind (A.11), we find

$$|f(x+t) - f(x)| < 2\varepsilon \quad \forall x \in [0,\infty), 0 \le t < \delta_\varepsilon$$

and (A.10) is proved.

What is the generator A of $T(t)$ and its domain $D(A)$?

The function f belongs to $D(A)$ if and only if there exists in X the limit

$$Af = \lim_{t\to 0^+} \frac{T(t)f - f}{t} = \lim_{t\to 0^+} \frac{f(t+\cdot) - f(\cdot)}{t}.$$

In particular

$$Af(x) = \lim_{t\to 0^+} \frac{f(t+x) - f(x)}{t} \quad \forall x \in [0,\infty)$$

and then f admits the right derivative for any $x \ge 0$ and the right derivative, Af, is continuous everywhere. But then, in view of a well-known result in the theory of functions of one real variable (see, e.g., [**51**, pp. 42-43]*), f is differentiable for any $x > 0$.*

Moreover, since $Af \in X$, there exists also

$$\lim_{x\to+\infty} f'(x).$$

Therefore $D(A)$ is contained in the space of the functions $f \in C^1([0,\infty))$ such that $f' \in X$ and $Af = f'$.

Viceversa, if $f \in C^1([0,\infty))$ and $f' \in X$, then $f \in D(A)$. In fact, we have

$$\frac{f(x+t) - f(x)}{t} - f'(x) = \frac{1}{t}\int_x^{x+t} [f'(u) - f'(x)]du.$$

But, since $f' \in X$. Thus

$$\left|\frac{f(x+t) - f(x)}{t} - f'(x)\right| \le \frac{1}{t}\int_0^t |T(\tau)f' - f'(x)|d\tau.$$

So $\lim_{\tau\to 0} T(\tau)f' - f' = 0$ *in* X *we deduce that*

$$\lim_{x\to 0^+} \left\| \frac{f(\cdot+t)-f(\cdot)}{t} - f' \right\|_\infty = 0,$$

i.e. $f \in D(A)$ *and* $Af = f'$. *We have thus proved that*

$$D(A) = \{f \in X \,|\, f' \in X\}, \quad Af = f'.$$

The case when $\omega = 0$ and $M = 1$ in the inequality (A.3) is of particular interest. We have

$$||T(t)|| \leq 1$$

and the semigroup is said to be a contraction semigroup or a semigroup of contractions. If the operator A is the generator of a C_0-semigroup of contractions, the solution of the Cauchy problem (A.9) satisfies the estimate

$$||u(t)|| \leq ||u_0|| \quad for\ \ all\ \ (t \geq 0).$$

Remark 8. *If* $(T(t))_{t\geq 0}$ *is* C_0*-semigroup, then the mapping* $(t,x) \to T(t)x$ *is locally uniformly continuous in* $[0,+\infty) \times X$. *See, e.g.,* [**21**, Chapter 1, Lemma 5.2]. *As a consequence, for any compact set* $K \subset X$ *and any* $T > 0$, *the mapping* $t \to T(t)x$ *is continuous in* $[0,T]$, *uniformly with respect to* $x \in K$.

The Hille-Yosida theorem is a central theorem in semigroup theory, since it allows us to give a complete characterization of the infinitesimal generators of strongly continuous semigroups.

Theorem 9. *(Hille-Yosida) Let* A *be a densely defined operator on* X. *The following assertions are equivalent:*

(i) *A is the generator of a* C_0*-semigroup.*

(ii) There exists a constant w such that $(w,\infty) \subseteq \rho(A)$ and

$$\sup_{\lambda > w, n \in \mathbb{N}} \|(\lambda - w)^n (\lambda I - A)^{-n}\|_{\mathcal{B}(X)} < \infty.$$

If the operator A is bounded on X, then it generates a strongly continuous semigroup. In addition, this semigroup is given by

$$e^{tA} = \Sigma_{k\geq 0} \frac{t^k A^k}{k!}.$$

By analogy to this case, we will denote by $(e^{tA})_{t\geq 0}$ the strongly continuous semigroup generated by the operator A, even when A is not bounded. The resolvent of the generator A coincides with the Laplace transform of the semigroup, that is,

$$(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} e^{tA} dt \; for\, all \; \lambda > w.$$

Conversely, the semigroup can be written in terms of the resolvent. This is given by the exponential formula

$$e^{tA} u = \lim_n \left(I - \frac{t}{n} A \right)^{-n} u \; for\, all \; u \in X.$$

Another important Theorem in the theory of C_0-semigroups is the Lumer-Phillips theorem.

Definition 6. *Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator. A is called dissipative if*

$$\|\lambda x - Ax\|_X \geq \lambda \|x\|_X, \qquad \lambda > 0, \quad x \in \mathcal{D}(A).$$

Theorem 10. *(Lumer-Phillips) Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator with dense domain. Then, the following properties are hold:*

(*i*) *if* A *is dissipative and there exists* $\lambda_0 > 0$ *such that the range of the operator* $\lambda_0 I - A$ *is* X*, then* A *is the infinitesimal generator of a strongly continuous semigroup of contractions;*

(*ii*) *if* A *is the infinitesimal generator of a strongly continuous semigroup of contractions in* X*, then the range of the operator* $\lambda I - A$ *is* X *for any* $\lambda > 0$*. Moreover,* A *is dissipative.*

A consequence of the Lumer-Phillips theorem is the following proposition.

Proposition 18. *Let* $A : \mathcal{D}(A) \subset X \to X$ *be a dissipative operator. Then, the following properties are satisfied:*

(*i*) *if* A *is closable, then the closure* $\bar{A}$ *is also dissipative;*

(*ii*) *if* $\mathcal{D}(A)$ *is dense in* X *and* $(\lambda I - A)(\mathcal{D}(A))$ *is dense in* X *for some (and hence all* $\lambda > 0$*), then the closure* $\bar{A}$ *is the infinitesimal generator of a strongly continuous semigroup of contractions.*

A.3. Analytic Semigroups

Throughout this section, X is a complex Banach space. Here we introduce an important class of closed operators, the so-called, sectorial operators, to which we can associate a semigroup.

Definition 7. *For* $0 < \theta \leq \pi$*, let* Σ_θ *denote the sector*

$$\Sigma_\theta = \{z \in \mathbb{C} : z \neq 0, |arg\, z| < \theta\}.$$

If A *is an operator on* X*, we say that* $A \in \mathcal{SA}_b(\theta, M)$*(where* $\frac{\pi}{2} < \theta \leq \pi$*,* $M \geq 1$*) if* A *is closed, densely defined, and for all* $\lambda \in \Sigma_\theta$*,* $\lambda \in \rho(A)$ *and*

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda|}.$$

We say that $A \in \mathcal{SA}_b(\theta)$ *(where* $\frac{\pi}{2} < \theta \le \pi$*) if for each* $\varepsilon > 0$ *(with* $\varepsilon < \theta - \frac{\pi}{2}$*), there exists an* $M_\varepsilon \ge 1$ *such that* $A \in \mathcal{SA}_b(\theta - \varepsilon, M_\varepsilon)$*. The notation* $A \in \mathcal{SA}_b$ *(sectorial) means that* A *generates a* (C_0) *uniformly bounded semigroup which has an analytic extension into a sector of the complex plane.*

Definition 8. *An analytic semigroup of type* (α, M) *(where* $0 < \alpha \le \frac{\pi}{2}, M \ge 1$*) is a family of operators* $T = \{T(t) : t \in \Sigma_\alpha \cup \{0\}\}$ *satisfying*

(i) $T(t)T(s) = T(t+s)$ *for all* $t, s \in \Sigma_\alpha$, $T(0) = I$*;*

(ii) *for each* $f \in X$ *and each* $\psi \in X^*$*, the dual space of* X*, the complex-valued function* $\langle T(\cdot)f, \psi\rangle$ *is analytic on* Σ_α*;*

(iii) $\lim_{t\to 0} T(t)f = f$, $t \in \Sigma_{\alpha-\varepsilon}$ *for each* $f \in X$ *and each* $\varepsilon \in (0, \alpha)$*;*

Theorem 11. $A \in \mathcal{SA}_b(\theta)$*(where* $\frac{\pi}{2} < \theta \le \pi$*) iff* A *generates a bounded analytic semigroup of type* $(\theta - \frac{\pi}{2})$*.*

Theorem 12. *For an operator* $(A, \mathcal{D}(A))$ *on a Banach space* X*, the following statements are equivalent.*

(a) A *generates a bounded analytic semigroup* $(T(z))_{z\in\Sigma_\alpha\cup\{0\}}$ *on* X*.*

(b) *There exists* $\mathcal{V} \in (0, \frac{\pi}{2})$ *such that the operators* $e^{\pm i\mathcal{V}}A$ *generate bounded strongly continuous semigroups on* X*.*

(c) A *generates a bounded stongly continuous semigroup* $(T(t))_{t\ge 0}$ *on* X *such that* $rg(T(t)) \subset \mathcal{D}(A)$ *for all* $t > 0$*, and*

$$M := \sup_{t>0} \|tAT(t)\| < \infty. \tag{A.12}$$

(*d*) *A generates a bounded strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on* X*, and there exists a constant* $C>0$ *such that*

$$\|R(r+is,A)\| \leq \frac{C}{|s|} \tag{A.13}$$

for all $r>0$ *and* $0\neq s\in\mathbb{R}$.

(*e*) *A is sectorial.*

Proof. See. Engel-Nagel [**21**]. Page 101. □

A.4. A priori estimates

In this section we state some well-known (interior) Schauder estimates, and some maximum principles. We consider the elliptic differential operator A defined by

$$Au(x)=\sum_{i,j=1}^{N} a_{ij}(x)D_{ij}u(x)+\sum_{i=1}^{N} F_i(x)D_iu(x)+V(x)u(x) \qquad x\in\mathbb{R}^N,$$

Theorem 13 ([**27**], Theorems 9.11, 9.19)**.** *Let* Ω *be an open set and* p *any real number in the interval* $(1,+\infty)$*. Then, the following properties hold:*

(*i*) *if the coefficients of the operator* A *are bounded and continuous in* Ω*, then, for any open set* $\Omega'\subset\subset\Omega$*, there exists a positive constant* C*, depending on* p,Ω,Ω',μ_0 *and the moduli of continuity of the coefficients* a_{ij} *in* Ω' $(i,j=1,...,N)$*, such that*

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|u\|_{L^p(\Omega)}+\|Au\|_{L^p(\Omega)}), \tag{A.14}$$

for any $u\in L^p(\Omega)\cap W^{2,p}_{loc}(\Omega)$ *such that* $Au\in L^p(\Omega)$

Proposition 19 ([**36**], Proposition.C.3.4)**.** *Under the following hypotheses* (*i*) Ω *is an open set with a boundary which is uniformly of class* $C^{2+2\nu}$ *for some* $\nu \in (0,1)$ *(or, possibly,* $\Omega = \mathbb{R}^N$*);*

(*ii*) a_{ij}, $F_i\,(i,j = 1,...,N)$ *and* V *belong to* $C_b^{2\nu}(\bar{\Omega})$*;*

(*iii*) $a_{ij} = a_{ji}$ *for any* $i,j = 1,...,N$ *and*

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geq \mu_0|\xi|^2, \quad \xi \in \mathbb{R}^N,\, x \in \bar{\Omega},$$

for some positive constant μ_0.

For any $f \in C_b(\bar{\Omega})$ *and any* $\lambda > c_0$, *where* $c_0 = \sup_{x\in\bar{\Omega}} V(x)$, *there exists a unique solution* $u \in \bigcap_{1\leq p<+\infty} W^{2,p}(\Omega)$ *to the Dirichlet problem*

$$\text{(A.15)} \qquad \begin{cases} \lambda u(x) - Au(x) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

The function u *can be represented by*

$$\text{(A.16)} \qquad u(x) = \int_{\Omega} K_\lambda^\Omega(x,y)f(y)dy, \qquad x \in \Omega,$$

where K_λ^Ω, *the so-called Green's function, is given by*

$$\text{(A.17)} \qquad K_\lambda^\Omega(x,y) = \int_0^\infty e^{-\lambda t}G_\Omega(t,x,y)dt, \qquad x,y \in \Omega,$$

where The function G_Ω *is called the fundamental solution.*

Let us recall now the classical maximum principle for continuous solutions to both Dirichlet and Neumann elliptic problems

Theorem 14 ([**36**], Theorem. C.2.2)**.** *Let* $\lambda > c_0$ *and suppose that* $u \in W^{2,p}_{loc}(\Omega)$ *for any* $p \in (1, +\infty)$ *satisfies the differential inequality* $\lambda u - Au \geq 0$*. Then, the following properties are met:*

(*i*) *if* $u \geq 0$ *on* $\partial\Omega$*, then* $u \geq 0$ *in* Ω*;*

(*ii*) *if* $\frac{\partial u}{\partial \nu} \geq 0$ *on* $\partial\Omega$*, then* $u \geq 0$ *on* Ω*.*

Moreover, if $\lambda > c_0$*,* $f \in C_b(\bar{\Omega})$ *and* $u \in \bigcap_{1 \leq p < +\infty} W^{2,p}_{loc}(\Omega)$ *solves the problem*

$$\begin{cases} \lambda u(x) - Au(x) = f(x) & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{A.18}$$

or

$$\begin{cases} \lambda u(x) - Au(x) = f(x) & x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{A.19}$$

then

$$\|u\|_\infty \leq \frac{1}{\lambda - c_0} \|f\|_\infty \tag{A.20}$$

Concerning the parabolic equation $D_t u - Au = 0$ we have the following result.

Proposition 20 ([**36**], Proposition.C.2.3)**.** *Fix* $T > 0$*. Then, the following properties are met:*

If $z \in C_b([0,T]\times\bar{\Omega}) \cap C^{1,2}((0,T]\times\Omega)$ *satisfies*

$$
\text{(A.21)} \qquad \begin{cases} D_t z(x,t) - \mathcal{A}z(t,x) = 0 & x \in \Omega, t \in (0,T], \\ z(t,x) \le 0, & x \in \partial\Omega, t \in (0,T], \\ z(0,x) \le 0, & x \in \bar{\Omega}, \end{cases}
$$

Then $z \le 0$ *in* $[0,T]\times\bar{\Omega}$.

A.4.1. Schauder interior estimates

In this subsection we recall some classical L^p-and Schauder interior estimates for solutions to elliptic and parabolic problems in bounded domains Ω or in the whole of $\mathbb{R}^N$. For more details and proofs about this we refer to [**36**], and [**24, 35**].

We now consider the parabolic problems associated with the operator A. For this purpose, we denote by $d : [0,+\infty)\times\bar{\Omega} \to \mathbb{R}$ the function defined by $d(t,x) = dist(x,\partial\Omega)\wedge\sqrt{t}, \quad t > 0, x \in \bar{\Omega}$.

Theorem 15 ([**24**], Theorems. 3.5, 3.10 and [**35**])**.** *Assume that the coefficients* a_{ij}, $F_i(i,j = 1,...,N)$ *and* V *of the operator* A *belong to* $C_b^{\nu}(\bar{\Omega})$ *for some* $\nu \in (0,1)$. *Further assume that* $u \in C_{loc}^{1+\frac{\nu}{2},2+\nu}((0,T)\times\Omega)$ *is a bounded (with respect to the sup-norm) solution of the equation* $D_t u(t,x) - Au(t,x) = 0,\ t \in (0,T), x \in \Omega$. *Then, the following properties are met:*

(i) *there exists a positive constant* C_1, *depending only on the coefficients of* A, *such that*

$$
\text{(A.22)} \qquad |d^0,u|_\nu + \sum_{i=1}^{N} |d, D_i u|_\nu + \sum_{i,j=1}^{N} |d^2, D_i D_j u|_\nu + |d^2, D_t u|_\nu \le C_1 \sup_{(0,T)\times\Omega} |u|,
$$

where

$$|d^m, u|_\nu = \sup_{(t,x)\in(0,T)\times\Omega} |(d(t,x))^m u(t,x)| + \sup_{(t,x),(s,y)\in(0,T)\times\Omega_{(t,x)\neq(s,y)}} (d(t,x)\wedge d(s,y))^{m+\nu} \frac{|u(t,x)-u(s,y)|}{(|x-y|^2+|t-s|)^{\frac{\nu}{2}}}$$

In particular, for any open set $\Omega' \subset\subset \Omega$ and any $s \in (0,T)$, there exists a positive constant C_2 depending on s, the coefficients of the operator A, Ω, Ω' and T, such that

$$\|u\|_{C^{1+\frac{\nu}{2},2+\nu}([s,T)\times\Omega')} \leq C_2 \sup_{(0,T)\times\Omega} |u|. \tag{A.23}$$

Moreover, if dist$(\Omega, \Omega') > \sqrt{T}$, then

$$\sup_{(t,x)\in(0,T)\times\Omega'} \left(t^{\frac{1}{2}}|Du(t,x)| + t|D^2u(t,x)|\right) \leq C_3 \sup_{(t,x)\in(0,T)\times\Omega} |u(t,x)|, \tag{A.24}$$

for some positive constant C_3 , depending on the coefficients of the operator A, Ω, Ω' and T;

(ii) let the coefficients a_{ij}, F_i and V belong to $C^{k+\nu}(\Omega)$ for any $i,j = 1,...,N$ and some $k \in \mathbb{N}$. Then, u is continuously differentiable in Ω_T up to the $(k+2)$-th-order, with respect to the space variables. Moreover, $D^k u \in C^{1+\frac{\nu}{2},2+\nu}_{loc}((0,T)\times\Omega)$ and

$$\|D^k u\|_{C^{1+\frac{\nu}{2},2+\nu}([\varepsilon',T]\times\Omega')} \leq C_4 \sup_{(t,x)\in[\varepsilon,T]\times\Omega} |u(t,x)|. \tag{A.25}$$

for any $\varepsilon, \varepsilon' > 0$ $(\varepsilon < \varepsilon' < T)$, any open set $\Omega' \subset\subset \Omega$ and some positive constant C_4, depending on $\varepsilon, \varepsilon', T, \Omega, \Omega'$, $\|a_{ij}\|_{C^{1+\nu}(\Omega)}$ and $\|F_i\|_{C^{1+\nu}(\Omega)}$, $(i,j = 1,...,N)$ and $\|V\|_{C^{1+\nu}(\Omega)}$. Finally, if dist$(\Omega, \Omega') \geq \sqrt{T}$, then

$$\sup_{(t,x)\in(0,T)\times\Omega'} \left(t^{\frac{3}{2}}|D^3u(t,x)|\right) \leq C \sup_{(t,x)\in(0,T)\times\Omega} |u(t,x)|, \tag{A.26}$$

CHAPTER B

Sesquilinear forms and associated operators

In the present appendix we shall recall without proofs some basic concepts on sesquilinear forms and their associated operators and semigroups. For a detailed presentation of all the results that have been stated here without proof, we refer to Ouhabaz, [**50**].

Let X be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $D(a)$ a linear subspace of X. We denote by $(\cdot,\cdot)$ the inner product of X and by $\|\cdot\|$ the corresponding norm. An application

$$a : D(a) \times D(a) \to \mathbb{K}$$

such that for every $\alpha \in \mathbb{K}$ and $u, v, w \in D(a)$ satisfies

$$a(\alpha u + v, w) = \alpha a(u, w) + a(v, w) \quad \text{and} \quad a(u, \alpha v + w) = \overline{\alpha} a(u, v) + a(u, w)$$

is called *unbounded sesquilinear form.* The space $D(a)$ is called the domain of a and $a(u) := a(u, u)$ the associated quadratic form.

Definition 9. *Let $a : D(a) \times D(a) \to \mathbb{K}$ be a sesquilinear form. We say that*

(i) *a is* densely defined *if*

$$D(a) \textit{ is dense in } X. \tag{B.1}$$

(ii) *a is* accretive *if*

$$Re a(u) \geq 0 \textit{ for all } u \in D(a). \tag{B.2}$$

(iii) *a is* continuous *if there exists a non-negative constant M such that*

(B.3) $$|a(u,v)| \leq M\|u\|_a\|v\|_a \text{ for all } u,v \in D(a)$$

where $\|u\|_a := \sqrt{Re a(u) + \|u\|^2}$.

(iv) *a is* closed *if*

(B.4) $$(D(a), \|\cdot\|_a) \text{ is a complete space.}$$

If the form a satisfies conditions (B.1)-(B.4) one can easily check that $\|\cdot\|_a$ is a norm on $D(a)$, the norm associated with the form a, and $D(a)$ is a Hilbert space.

A stronger assumption than continuity of a form is sectoriality. It is defined as follows.

Definition 10. *A sesquilinear form a acting on a complex Hilbert space X is called* sectorial *if there exists a non-negative constant C such that*

$$|Im a(u)| \leq C Re a(u) \text{ for all } u \in D(a).$$

A relation between continuity and sectoriality is given by the following lemma.

Lemma 10. *If a is an accretive and continuous sesquilinear form on a complex Hilbert space X, then* $1 + a$ *is sectorial. More precisely, if a satisfies* (B.3) *with some constant M, then*

$$|Im[(u,u) + a(u)]| \leq M Re[(u,u) + a(u)] \text{ for all } u \in D(a).$$

The sum $a + b$ of two sesquilinear forms a and b on X is defined by

$$[a+b](u,v) := a(u,v) + b(u,v), \ \ D(a+b) = D(a) \cap D(b).$$

The natural question that arises is that if the properties of the forms carry over in the sum. In particular, if one of the two forms, say a, satisfies (B.2)-(B.4), the following theorem shows that under some additional assumptions these properties are preserved.

Theorem 16. *Let a be an accretive and continuous sesquilinear form on a complex Hilbert space X. Assume that a' is a sesquilinear form such that $D(a) \subseteq D(a')$ and, for some α, β non-negative constant with $\alpha < 1$, the following inequality holds*

$$|a'(u)| \leq a(u) + \beta\|u\|^2 \text{ for all } u \in D(a).$$

Then, the form sum $\mathfrak{t} := a + a' + \beta$ with domain $D(\mathfrak{t}) = D(a)$ is accretive and continuous. Moreover, $\mathfrak{t}$ is closed if and only if a is closed.

As for operators, if a form is not closed we can ask if it admits a closed extension. We define the smallest closed extension by $\overline{a}$, the closure of a.

Definition 11. *A densely defined accretive sesquilinear form a is closable if there exists a closed accretive form $\mathfrak{c} : D(\mathfrak{c}) \subseteq X \to X$ such that $D(a) \subseteq D(\mathfrak{c})$ and $a(u, v) = \mathfrak{c}(u, v)$ for all $(u, v) \in D(a)$.*

If a is a closable form we define the smallest closed extension $\overline{a}$ as follows

$$D(\overline{a}) = \{u \in X \text{ s.t. } \exists u_n \in D(a) : \lim_{n\to\infty} u_n = u, \lim_{n,m\to\infty} a(u_n - u_m) = 0\},$$

and

$$\overline{a}(u, v) := \lim_{n\to\infty} a(u_n, v_n),$$

for $u, v \in D(\overline{a})$, where $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are any sequences of elements of $D(a)$ which converge respectively to u and v and satisfy $a(u_n - u_m) \to 0$ and $a(v_n - v_m) \to 0$ as $n, m \to \infty$. The limits are taken with respect to the norm of X.

One can show the following.

Proposition 21. *Let a be a densely defined, accretive, and continuous sesquilinear form. If a is closable, then $\overline{a}$ is well defined and satisfies* (B.1)-(B.4)*. In addition, every closed extension of a is also an extention of $\overline{a}$.*

Now we want to define the operator associated to a form. Let a be a densely defined, accretive, continuous and closed sesquilinear form on X. We can define an unbounded operator A on a linear subspace $D(A)$ of X, which is called the *operator associated to the form a*, as follows

$$D(A) = \{u \in X \text{ s.t. } \exists v \in X : a(u, \phi) = (v, \phi) \ \forall \phi \in D(a)\}, \ Au := v.$$

Therefore, we can study the properties of A as an operator on X through the form a and viceversa. For example, the operator associated to a sectorial form is a sectorial operator and the converse is also true.

Proposition 22. *Let a be a densely defined, accretive, continuous and closed sesquilinear form acting on a complex Hilbert space X. Denote by A the associated operator. The following assertions are equivalent:*

- *a is a sectorial form;*
- *A is a sectorial operator.*

What we are interested in are generation results for the operator A in terms of the form a as the following.

Theorem 17. *Let a be a densely defined, accretive, continuous and closed sesquilinear form on a Hilbert space X. Denote by A the operator associated with a. Then $-A$ is the generator of a strongly continuous contraction semigroup on X. Moreover, the semigroup is holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan M)$ where M is the constant in the continuity assumption* (B.3).

In particular, the following result deals with generation of holomorphic semigroups for sectorial operators.

Theorem 18. *Let A be a densely defined operator on a complex Hilbert space X. Assume that A is sectorial, that is,*

$$|Im(Au, u)| \leq CRe(Au, u) \text{ for all } u \in D(A),$$

where $C \geq 0$ is a constant. Assume also that there exists $\lambda_0 \in \rho(A)$ with $dist(\lambda_0, \Sigma(\arctan C)) > 0$. Then, $-A$ generates a strongly continuous semigroup which is holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan C)$ and such that e^{-zB} is a contraction operator on X for every $z \in \Sigma(\frac{\pi}{2} - \arctan C)$.

CHAPTER C

Log-Sobolev inequality and ultracontractivity

For the sake of completeness, we include in this appendix all statements of the results needed to prove Theorem 7 which can be found in the book by E.B. Davies [**18**].

Let us consider a locally compact, second countable, Hausdorff space Ω and a Borel measure $d\mu$. On $L^2(\Omega)$ we consider a nonnegative and continuous bilinear form h with a densely defined domain $D(h)$ such that h is closed. To h we can associate a self-adjoint dissipative and densely defined operator H. So, H generates a symmetric C_0-semigroup of contractions $(e^{tH})_{t\geq 0}$ on $L^2(\Omega)$. Here and in the sequel we assume that $(e^{tH})_{t\geq 0}$ is positivity preserving and L^∞-contractive, i.e. e^{tH} maps $L^2(\Omega)\cap L^\infty(\Omega)$ into $L^2(\Omega)\cap L^\infty(\Omega)$ and can be extended to a contraction on $L^\infty(\Omega)$.

We say that $(e^{tH})_{t\geq 0}$ is ultracontractive if e^{tH} is bounded from $L^2(\Omega)$ to $L^\infty(\Omega)$ for all $t>0$. If it is the case we denote the norm of e^{tH} from $L^2(\Omega)$ to $L^\infty(\Omega)$ by $c_t = \|e^{tH}\|_{\infty,2}$.

The following result states that every ultracontractive semigroup can be given by an integral kernel, cf. [**18**, Lemma 2.1.2].

Proposition 1. *If $(e^{tH})_{t\geq 0}$ is ultracontractive then*

$$e^{tH}f(x) = \int_\Omega k(t,x,y)f(y)\,d\mu(y)$$

for an integral kernel k which satisfies

$$0 \leq k(t,x,y) \leq c_{\frac{t}{2}}^2$$

for all $t > 0$ *and a.e.* $x, y \in \Omega$.

Ultracontractivy can be characterize by a log-Sobolev inequality as the following result shows, cf. [**18**, Corollary 2.2.8].

Proposition 2. *Assume that there is a monotonically decreasing function* $\beta : (0, \infty) \to (0, \infty)$ *such that*

$$\int_\Omega f^2 \log f \, d\mu \leq \varepsilon h(f, f) + \beta(\varepsilon)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2$$

for all $\varepsilon > 0$ *and* $0 \leq f \in D(h) \cap L^1(\Omega) \cap L^\infty(\Omega)$. *If* $M(t) := \frac{1}{t}\int_0^t \beta(\varepsilon)\, d\varepsilon$ *is finite for all* $t > 0$, *then* $(e^{tH})_{t\geq 0}$ *is ultracontractive and*

$$\|e^{tH}\|_{\infty,2} \leq e^{M(t)}$$

for all $t > 0$.

A useful tool to verify the above log-Sobolev inequality is the so-called Rosen lemma, cf. [**18**, Lemma 4.4.1].

From now on we assume that there is a positive C^2 function ϕ such that $H\phi = \lambda_0\phi$ for some $\lambda_0 < 0$.

Lemma 3. *Assume that there are constants* a *and* c *such that*

$$\int_\Omega g f^2 \, d\mu \leq a\|g\|_{L^{N/2}(\Omega, d\mu)}(h(f, f) + c\|f\|_2^2)$$

for some $N \in (2, \infty)$ *and all* $g \in L^{N/2}(\Omega, d\mu)$ *and* $f \in L^2(\Omega)$. *If the inequality*

$$-\int_\Omega f^2 \log\phi \, d\mu \leq \varepsilon h(f, f) + \gamma(\varepsilon)\|f\|_2^2$$

holds for some $\gamma(\varepsilon)$ *and all* $\varepsilon > 0$, $f \in L^2(\Omega)$, *then*

$$\int_\Omega f^2 \log u \, d\mu \le \varepsilon h(f,f) + \int_\Omega (k - \frac{N}{4} \log \varepsilon + \frac{\varepsilon c}{2} + \gamma(\varepsilon/2)) f^2 \, d\mu$$

for all $0 \le u \in L^2(\Omega, \phi^2 d\mu)$ *of norm one, and all* $f \in L^2(\Omega)$, $\varepsilon > 0$.

As a consequence on obtain a weighted log-Sobolev inequality, cf. [**18**, Corollary 4.4.2].

Corollary 4. *Let us assume the same conditions as in Lemma 3. Then*

$$\int_\Omega (f^2 \log f) \phi^2 \, d\mu \le \varepsilon h(\phi f, \phi f) + \beta(\varepsilon) \|f\|^2_{L^2(\Omega,\phi^2 d\mu)} + \|f\|^2_{L^2(\Omega,\phi^2 d\mu)} \log \|f\|^2_{L^2(\Omega,\phi^2 d\mu)}$$

for all $0 \le f \in L^1 \cap L^\infty \cap \{u \in L^2(\Omega, \phi^2 d\mu) : \phi u \in D(h)\}$ *and all* $\varepsilon > 0$, *where*

$$\beta(\varepsilon) = k - \frac{N}{4} \log \varepsilon + \frac{\varepsilon c}{2} + \gamma(\varepsilon/2).$$

List of Symbols and Abbreviations

$\mathbb{N}$	set of all positive natural numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}^N$	set of all real N-tuples
$B(R)$	open disk in $\mathbb{R}^N$ with centre at 0 and radius $R > 0$
$x + B(R)$	open disk in $\mathbb{R}^N$ with centre at x and radius $R > 0$
$L(X, Y)$	the set of all the bounded linear operators from X to Y
$L(X)$	$:= L(X, X)$
$\text{div}f$	the divergence of $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$, i.e., $\text{div}f = \sum_{i=1}^N D_i f$
$\|\|f\|\|_\infty$	the sup-norm of $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$, i.e., $\|\|f\|\|_\infty := \sup_\Omega \|f\|$ (whenever is finite)
$D^\alpha f$	the derivative $\frac{\partial^r f}{\partial x_{i_1},...,\partial x_{i_r}}$ of the function f, where $\alpha = (\alpha_1, ..., \alpha_N)$
χ_A	the characteristic function of the set A, i.e., the function defined by $\chi_A(x) = 1$ for any $x \in \mathbb{R}^N$ and $\chi_A(x) = 0$ for any $x \in C^A$
$D(A)$	the domain of the (linear) operator A
$\rho(A)$	resolvent set of a linear operator A
$\sigma(A)$	spectrum of a linear operator A
I	in a Banach space X the identity operator
dx	the Lebesgue measure in $\mathbb{R}^N (N \geq 1)$
$\text{Re}\lambda$	the real part of the complex number λ
$\text{Im}\lambda$	the imaginary part of the complex number λ
(x, y)	inner euclidean product between the vectors $x, y \in \mathbb{R}^N$

supp$u$	support of a given function u
$(L^p(\mathbb{R}^N), \|\cdot\|_p)$	space of Lebesgue measurable functions u in $\mathbb{R}^N$, with $\|u\|_p := \int_{\mathbb{R}^N} \|u(x)\|^p \, dx < \infty$
$\|\cdot\|_{p\to q}$	the norm of operators acting from $L^p(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$
$W^{k,p}(\mathbb{R}^N)$	space of functions $u \in L^p(\mathbb{R}^N)$ with weak derivatives up to order k in $L^p(\mathbb{R}^N)$
$W^{k,p}_{loc}(\mathbb{R}^N)$	space of functions of $W^{k,p}(\Omega)$, for all bounded open set Ω with $\overline{\Omega} \subset \mathbb{R}^N$
$C_b(\mathbb{R}^N)$	space of continuous bounded functions of $\mathbb{R}^N$
$C^\alpha(\mathbb{R}^N)$	space of α-hölderian functions u in $\mathbb{R}^N$, i.e., $u \in C_b(\mathbb{R}^N)$ with $[u]_\alpha := \sup_{x,y\in\mathbb{R}^N; x\neq y} \frac{\|u(x)-u(y)\|}{\|x-y\|^\alpha} < +\infty$, with norm $\| u \|_\alpha := \| u \|_\infty + [u]_\alpha$
$C^\alpha_{loc}(\mathbb{R}^N)$	space of functions in $C^\alpha(\Omega')$ for all Ω' bounded open subset of $\mathbb{R}^N$

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