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Hadjer Adimi

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Les indices des algèbres de Lie et algèbres Hom-Lie

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Devant le jury

Président :	Dr Naceurdinne	Bensalem	Prof	Université Ferhat Abbas Sétif1
Raporteurs :	Dr Abdenacer	Makhlouf	M.C.H.D.R	Université de Haute Alsace France
Examinateurs :	Dr Mohand Ouamer	Hernane	Prof	USTHB Alger
	Dr Tahar	Boudjedaa	Prof	Université de Jijel
	Dr Nadir	Trabelsi	Prof	Université Ferhat Abbas Sétif1

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Introduction

The theory of Lie groups and Lie algebras began in the late 19th century with the work of the Norwegian mathematician Sophus Lie. It has many ramifications (non-Euclidean geometries, homogeneous spaces, harmonic analysis, representation theory, algebraic groups, quantum groups ...) and is still very active. Moreover, these objects appear and play an important role in many branches of mathematics : in number theory, through the "automorphic forms" and "Langlands program", and theoretical physics, specially in particle physics or general relativity. H. Weyl introduced the current terminology in 1934, the general theory of complex Lie algebras was already well developed by E. Cartan, F. Engel, and W. Killing in 1888-1894.

A Lie algebra $\mathfrak{g}$ is a $\mathbb{K}-$vector space with a bilinear and skewsymmetric map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \ \forall x, y, z \in \mathfrak{g}.$$

Different classes of Lie algebras were studied : linear Lie algebras, simple and semi-simple Lie algebras, solvable and nilpotent Lie algebras. The classification of complex nilpotent Lie algebras was completed until dimension 7. Complex nilpotent Lie algebras in dimension 7 was provided by Ancochea and Goze, according to their characteristic suite [25]. They study nilpotent Lie algebras with respect to their nilindix, starting with those with a maximum nilindex, (called filiform). Lie algebras whose nilindex is lowered by one, are called quasi-filiform Lie algebras. In 1970, Vergne has initiated the study of filiform Lie algebras [50]. She showed that over a fields having an infinite number of elements, there are only two isomorphism classes of naturally graded filiform Lie algebras of even dimension $2n$, denoted by L_{2n} and Q_{2n}, and only one odd dimension $2n+1$, denoted by L_{2n+1} with $n \in \mathbb{N}$. More recently, Snobl and Winternitz ([46]) determined Lie algebras having as nilradical the Lie algebra L_n over the complex and real fields.

Deformations of mathematical objects is one of the oldest technics used by mathematicians. they appeared in different domains such as geometry, analysis, complex manifolds, algebraic varieties, associative algebras and rings. The most popular approach was introduced by by Gerstenhaber in 1964, see [18]. It is based on formal power series and it relates the deformation to cohomology groups. The work of Gerstenhaber was extended in several directions. For example, Balavoine describes the deformations of any algebra over a quadratic operad. In the same sense, Hinich studying deformations of algebras on a differential graded operad. The case of Lie algebras was studied by Nijenhuis-Richardson. The deformation theory was revived by quantization deformation theory which was inaugurated in 1978 by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer. By studying $q-$deformations of Witt and Virasoro algebras, a modified Jacobi identity appeared. Recently in 2006, Hartwig, Larsson and Silvestrov developed a new approach to the theory of deformations of Witt and Virasoro algebra using $\sigma-$derivation in [26]. They also introduced the concept of Hom-Lie algebra. A Hom-Lie algebra structure is given by a skewsymmetric bracket on a vector space $\mathfrak{g}$, the structure is twisted by a homomorphism α and the modified Jacobi identity is defined as

$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0,\ \forall x,y,z\in\mathfrak{g}.$$

This theory has been extended by Larsson and Silvestrov to the quasi-Lie algebras in [27].

An important tool for the study of Lie algebra is the notion of representation. A representation of a Lie algebra is a way to write this algebra as a matrix algebra, or more generally of endomorphisms of a vector space with the Lie bracket given by the commutator. It is defined as follows: a representation with coefficients in $\mathbb{K}$ is a pair $(\mathbb{V},\rho)$, where $\mathbb{V}$ is a $\mathbb{K}-$vector space and $\rho:\mathfrak{g}\to gl(\mathbb{V})$ is a morphism. We often simplify the notation by setting $gv=\rho(g)(v)$ for $g\in\mathfrak{g}, v\in\mathbb{V}$. The dual representation is given by :

$$\rho^*:\mathfrak{g}\to gl(\mathbb{V}^*)$$
$$x\mapsto-\rho(x)^T.$$

A particular representation is the adjoint representation, denoted ad, and defined by

$$ad : \mathfrak{g} \rightarrow gl(\mathfrak{g})$$

$$x \mapsto ad_x : y \mapsto [x, y] \, .$$

The dual representation in this case is called the coadjoint representation.

One of the main problems in Lie algebra theory is to provide invariants and to describe them for a given Lie algebra. Invariant Theory and Lie groups theory is of great interest for both mathematician and physicists (Casimir operators, center of the enveloping algebra, Duflo isomorphism between the center and the Poisson center of the symmetric algebra...). We are interested in this work by the concept of *index* for Lie algebras and Hom-Lie algebras. The index is an important concept in the representation theory and invariant theory. It was introduced by Dixmier in [9]. Let $\mathfrak{g}$ be a Lie algebra over a fields $\mathbb{K}$ and let $f \in \mathfrak{g}^*$ be a linear functional of $\mathfrak{g}$. The stabilizer $\mathfrak{g}_f$ is the Lie subalgebra of elements of $\mathfrak{g}$ which annihilate f in the coadjoint representation. The index of the Lie algebra is

$$\chi\left(\mathfrak{g}\right) = \operatorname{ind} \mathfrak{g} = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}_f .$$

The theory of index for Lie algebras has applications in invariant theory of invariants, deformations and quantum groups. A Lie algebra is called Frobenius if its index is 0, which is equivalent to say that there is functional in the dual such that the bilinear form B_F defined by $B_F(x, y) = F([x, y])$, is non-degenerate. The Frobenius algebras were studied by Ooms in [37]. Most index studies concerne simple Lie algebras or their subalgebras. They were considered by many authors (see [8], [13]-[16], [37], [43], [47]). Note that the simple Lie algebra can never be Frobenius, but many subalgebras are. The index of a semisimple Lie algebra $\mathfrak{g}$ is equal to the rank of $\mathfrak{g}$. This can be obtained easily from the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form. There has been quite a lot of recent work on the determination of the index of certain subalgebras of a semisimple Lie algebra : parabolic subalgebras and related subalgebras ([8], [48]), centralizers of elements and related subalgebras ([12]).

The index of a semi-direct products of a Lie algebra $\mathfrak{g}$ and a vector space $\mathbb{V}$ with respect to a linear representation ρ of $\mathfrak{g}$ in $\mathbb{V}$, is given by the so-called Raïs formula.

The objective of this thesis is to study index for some classes of Lie algebras and to generalize the theory to Hom-Lie algebras. The first part deals with nilpotent Lie algebras. We study the classes of filiform and quasi-filiform Lie algebras. In the second part dealing with Hom-Lie algebras, we generalize the theory, provide some key constructions and discuss index of semi-direct products of Hom-Lie algebras.

The thesis is structured as follows:

We start with an overview in french summarizing the main results. The first chapter contains the general and necessary facts about Lie algebras illustrated by examples. The second part of this work is devoted to the index theory of Lie algebra where we present definitions and basic results. Chapter 3 and 4 include our original results. In chapter 3 we calculate the index and provide regular vectors of the two classes of nilpotent Lie algebras. We study filiform and quasi-filiform Lie algebras. We consider graded filiform Lie algebras L_n or Q_n, then $n-$dimensional filiform Lie algebras for $n < 8$, also graded quasi-filiform Lie algebras and finally Lie algebras whose nilradical is L_n or Q_{2n} denoted $\mathfrak{n}_{n,1}$. In Chapter 4 we introduce and study the notion of index for Hom-Lie algebras with respect to coadjoint and an arbitrary representation. We provide some constructions and examples. Moreover, we discuss semidirect products of Hom-Lie algebras.

Résumé de la thèse

La théorie des groupes et algèbres de Lie trouve son origine à la fin du 19ème siècle dans les travaux du mathématicien norvégien Sophus Lie. Elle a connu de nombreuses ramifications (géométries non euclidiennes, espaces homogènes, analyse harmonique, théorie des représentations, groupes algébriques, groupes quantiques...) et reste encore très active. Par ailleurs ces objets interviennent aussi dans des branches a priori plus éloignées des mathématiques : en théorie des nombres, par le truchement des "formes automorphes" et du "programme de Langlands", et en physique théorique, notamment dans la physique des particules ou la relativité générale. H. Weyl a introduit la terminologie actuelle en 1934, alors que la théorie et la structure générale des algèbres de Lie complexes étaient déjà bien développées par E. Cartan, F. Engel, et W. Killing in 1888-1894.

Une algèbre de Lie $\mathfrak{g}$ est un $\mathbb{K}-$espace vectoriel muni d'une application bilinéaire et antisymétrique $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ vérifiant l'identité de Jacobi

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \ \forall x, y, z \in \mathfrak{g}.$$

Plusieurs classes d'algèbres de Lie ont été étudiés de façon approfondie, notamment, les algèbres de Lie linéaires, les algèbres de Lie simples et semi-simples, les algèbres de Lie résolubles, et les algèbres de Lie nilpotentes. La classification des algèbres de Lie nilpotentes complexes a été complétée jusqu'en dimension 7, par Ancochea et Goze, en utilisant les suites caractéristiques [25]. Ils ont par ailleurs étudié, avec d'autres collaborateurs, la classe dont le nilindice est maximal, appelées algèbres de Lie filiformes. Les algèbres de Lie dont le nilindice maximal est abaissé d'une unité, sont appelées algèbre de Lie quasi-filiformes. Dès 1970, Vergne a initié l'étude des algèbres de Lie filiformes [50]. Elle a montré que sur un corps ayant une infinité d'éléments, qu'il n'existe que deux classes d'isomorphie d'algèbres de Lie filiformes naturellement graduées de dimension paire $2n$, notées L_{2n} et Q_{2n}, et une seule en dimension impaire $2n+1$, appelée L_{2n+1} avec $n \in \mathbb{N}$. Plus récemment, Snobl et

Winternitz ont déterminé dans [46] les algèbres de Lie ayant comme nilradical l'algèbre L_n sur le corps des complexes et des réels.

La déformation d'objets mathématiques est une des plus vieilles techniques utilisées par les mathématiciens, les différent domaines ou elle est apparue sont la géométrie, l'analyse, les variétés complexes, les variétés algébriques, les algèbres associatives et les anneaux. La théorie des déformations a été relancée par la quantification par déformation qui a été initiée, en 1978, par Bayen, Flato, Fronsdal, Lichnerowicz et Sternheimer. L'approche la plus populaire des déformations algébriques, a été décrite d'abord pour les algèbres associatives par Gerstenhaber en 1964, il a définit un cadre théorique pour la déformation des structures algébriques [18], utilisant les séries formelles et la cohomologie. Le travail de Gerstenhaber a été étendu à plusieurs autres structures algébriques. Par exemple, Balavoine a décrit les déformations de tout algèbre sur une opérade quadratique. Dans le même sens, Hinich a étudié les déformations des algèbres sur une opérade différentielle graduée. Un autre type de déformations d'algébres de champs de vecteurs consiste à remplacer les dérivations usuelles par des $\sigma-$dérivations. Les exemples concernant les algèbres de Witt et Virasoro ont fait apparaitre une structure d'algèbre de Lie modifiée par un homomorphisme. Une étude systématique de ce type d'algèbres a été entreprise par Hartwig, Larsson et Silvestrov dans [26] où ils ont introduit le concept d'algèbre Hom-Lie. Une algèbre Hom-Lie est donnée par un crochet antisymétrique sur un espace vectoriel et une application linéaire α modifiant la condition de Jacobi de la manière suivante

$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0,\ \forall x,y,z\in\mathfrak{g}.$$

Un outil important pour l'étude des algèbres de Lie est la notion de représentation : une représentation d'une algèbre de Lie est une façon d'écrire cette algèbre comme une algèbre de matrices, ou plus généralement d'endomorphismes d'un espace vectoriel, avec le crochet de Lie donné par le commutateur. Elle est définie comme suit : Une représentation à coefficients dans $\mathbb{K}$ est un couple $(\mathbb{V},\rho)$ où $\mathbb{V}$ est un $\mathbb{K}-$espace vectoriel et $\rho:\mathfrak{g}\rightarrow gl(\mathbb{V})$ est un morphisme, on simplifie souvent la notation en posant $gv=\rho(g)(v)$ pour $g\in\mathfrak{g}$,

$v \in \mathbb{V}$. La représentation duale est donnée par

$$\rho^* : \mathfrak{g} \to gl(\mathbb{V}^*)$$
$$x \mapsto -\rho(x)^T$$

Une représentation particulière est la représentation adjointe elle est notée ad et définie par

$$ad : \mathfrak{g} \to gl(\mathfrak{g})$$
$$x \mapsto ad_x : y \mapsto [x, y]$$

La représentation duale dans ce cas est appelée la représentation coadjointe. Depuis les travaux de Kirillov, il est bien connu que la représentation co-adjointe joue un rôle important en théorie des représentations.

La recherche des invariants dans les algèbres de Lie et la description précise de ces invariants sont un aspect important de la théorie des invariants et de la théorie des groupes de Lie, qui intéresse non seulement des mathématiciens depuis plus d'un siècle, mais également des physiciens (opérateurs de Casimir, centre de l'algèbre enveloppante d'une algèbre, isomorphisme de Duflo ...). Dans ce travail on s'intéresse à *l'indice* des algèbres de Lie. La notion d'indice est importante dans la théorie des représentations ainsi que dans la théorie des invariants. Elle a été introduite par Dixmier [9]. Une forme linéaire $f \in \mathfrak{g}^*$ est dite régulière si son orbite coadjointe $\mathfrak{g}.f$ est de dimension maximale ou si, de manière équivalente, son stabilisateur $\mathfrak{g}_f$ est de dimension minimale. L'indice d'une algèbre de Lie $\mathfrak{g}$ n'est autre que la dimension du stabilisateur d'une forme linéaire régulière et on le note $\chi(\mathfrak{g})$ i.e.

$$\chi(\mathfrak{g}) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}_f.$$

Le stabilisateur $\mathfrak{g}_f$ de f est la sous-algèbre de Lie d'éléments de $\mathfrak{g}$ qui annihilent f dans la représentation coadjointe. Qui est aussi une sous-algèbre de Lie $\mathfrak{g}$ contenant le centre $z(\mathfrak{g})$ de $\mathfrak{g}$. par [[9], 1.14.13],

$$\chi(\mathfrak{g}) = \dim \mathfrak{g} - \operatorname{rank}_{\mathbb{k}(\mathbb{V})} \left([x_i, x_j]\right)_{ij}.$$

L'indice d'une algèbre de Lie présente aussi un intérêt dans la déformation et la théorie des groupes quantique. Une algèbre de Lie est dite Frobeniusienne si l'indice est 0, ce qui est équivalent à dire qu'il existe une fonctionnelle dans le dual de telle sorte que la forme bilinéaire B_F, définie par $B_F(x,y) = F([x,y])$, est non dégénérée. Ou d'une manière équivalente s'il existe une forme linéaire $f \in \mathfrak{g}^*$ telle que $\mathfrak{g}_f = \{0\}$.

Les algèbres Frobeniusiennes ont été étudiés par Ooms dans [37]. La plupart des études de l'indice concernent les algèbres de Lie simples ou leurs sous algèbres. EIles ont été considérées par beaucoup d'auteurs ([8], [13]-[16], [37], [43], [47]). Remarquons que l'algèbre de Lie simple ne peut jamais être Frobeniusienne, mais beaucoup de sous-algèbres le sont. Il y a eu beaucoup de travaux récents sur la détermination de l'indice de certaines sous-algèbres d'une algèbre de Lie semi-simple: sous-algèbres paraboliques et les sous-algèbres associées ([8], [48]), centralisateurs d'éléments et les sous-algèbres associées ([12]). L'indice des produits semi-directs d'une algèbre de Lie $\mathfrak{g}$ par un espace vectoriel $\mathbb{V}$, relativement à une représentation linéaire ρ de $\mathfrak{g}$ dans $\mathbb{V}$, est donné par la formule de Raïs.

Cette thèse a pour objet l'étude de l'indice pour les algèbres de Lie et leur généralisation, les algèbres Hom-Lie. On étudie la classe opposée aux algèbres semi-simples qui est la classe des algèbres de Lie nilpotentes, on s'intéresse spécialement aux algèbres de Lie filiformes et quasi-filiformes. Dans la deuxième partie du travail, on établit la théorie de l'indice dans le cas des algèbres Hom-Lie. On étudie l'indice des algèbres Hom-Lie multiplicatives simples ainsi que l'indice du produit semi-direct d'algèbres Hom-Lie. Par ailleurs, on suit par déformation et par twist de Yau l'évolution de l'indice. De nombreux exemples sont aussi proposés.

La thèse est organisée de la manière suivante : Le premier chapitre contient les prélim-naires et les notions de base sur les algèbres de Lie, illustrées par des exemples. Le deuxième chapitre est consacré à la théorie des indices d'algèbre de Lie dont on présente les définitions et les résultats fondamentaux classiques. Pour les détails et les démonstrations, on renvoie aux travaux suivants ([49], [11], [38], [12], [8], [9], [36], [47], [41], [42]). Notre résultat principal dans ce chapitre est la proposition suivante concernant l'indice d'une extension centrale $\mathfrak{g}$ d'une algèbre de Lie $\mathfrak{g}_0$ par l'algèbre de Lie $\mathfrak{L} = c\mathbb{C}$:

Proposition 1 : *Soit $\mathfrak{g}_0$ une algèbre de Lie, et soit $\mathfrak{g}$ l'extension centrale de $\mathfrak{g}_0$ par* *l'algèbre de Lie* $\mathfrak{L} = c\mathbb{C}$*, alors* $\chi(\mathfrak{g}) = \chi(\mathfrak{g}_0) + 1$*. Par ailleurs si* f *est un vecteur régulier de* $\mathfrak{g}$ *alors* $f = g + \rho c^*$ *où* g *est un vecteur régulier de* $\mathfrak{g}_0$ *et* $\rho \in C$*.*

Pour établir ce résultat, on commence par définir l'algèbre de Lie $\mathfrak{g}$ par son crochet, donnant la matrice associée qui est de la forme $M = \begin{pmatrix} M_{\mathfrak{g}_0} & 0 \\ 0 & 0 \end{pmatrix}$ et donc elle a le même rang que celle de l'algèbre de Lie $\mathfrak{g}_0$ d'où l'indice est $\chi(\mathfrak{g}_0) + 1$.

On donne également dans ce chapitre la procédure pour calculer les vecteurs réguliers (Remark 2.2.7). Une autre observation dans ce chapitre établit l'évolution de l'indice d'une algèbre de Lie par déformation.

Proposition 2 : *L'indice d'une algèbre de Lie décroit par déformation formelle à un paramètre.*

Le chapitre 3 et 4 contiennent nos principales contributions à la théorie. Dans le chapitre 3, nous calculons l'indice et nous déterminons les vecteurs réguliers des classes d'algèbres de Lie nilpotentes, principalement la classe d'algèbres de Lie filiforme et quasi-filiformes. Nous considérons les deux classes d'algèbres de Lie nilpotentes L_n et Q_n, nous établissons leurs indices. Les résultats sont présentés dans la Propositions 3.3.1 pour l'indice de l'algèbre L_n.

Proposition 3 : *Pour* $n \geq 3$*, l'indice de l'algèbre de Lie filiforme* L_n *de dimension* n *est* $\chi(L_n) = n - 2$*. Les vecteurs réguliers de* L_n *sont de la forme* $f = p_1\, x_1^* + p_2\, x_2^* + p_s\, x_s^*$ *où* $s \in \{3, ..., n\}$ *et* $p_s \neq 0$*.*

La Proposition 3.3.2 établit l'indice de l'algèbre Q_n :

Proposition 4 : *Pour* $n = 2k$ *et* $k \geq 2$*, L'indice de l'algèbre de Lie filiforme* Q_n *de dimension* n *est* $\chi(Q_n) = 2$*. Les vecteurs réguliers de* Q_n *sont de la forme* $f = \sum_{i=1}^{n} p_i x_i^*$ *avec* $p_n \neq 0$*.*

Dans la suite, on détermine les indices et les vecteurs réguliers associés aux algèbres de Lie filiformes de dimension n $(n < 8)$ (Proposition 3.4.3).

Proposition 5 : Les indices d'algèbre de Lie filiforme de dimension 7 sont :

$$\chi\left(\mathcal{F}_7^i\right) = 3 \text{ pour } i = 2, 3, 5, 6, 7 \quad \chi\left(\mathcal{F}_7^4\right) = 1,$$

$$\chi\left(\mathcal{F}_7^1\right) = \begin{cases} 1 \text{ si } \alpha \neq \{0, -1\}, \\ 3 \text{ si } \alpha = 0. \end{cases}$$

$$\chi\left(\mathcal{F}_7^8\right) = 5.$$

Les vecteurs réguliers sont donnés dans le tableau suivant :

item	regular vectors
$i = 1$	$f = \sum_{i=1}^{5} p_i x_i^* + p(x_6^* + x_7^*)$, si $\alpha = 0$ $f = \sum_{i=1}^{6} p_i x_i^*$ avec $p_i \neq 0$, si $\alpha \neq 0$
$i = 2$	$f = p_1 x_1^* + p_2 x_2^* + p(x_4^* + x_5^* + x_6^*)$ avec $p \neq 0$
$i = 3$	$f = \sum_{i=1}^{4} p_i x_i^*$
$i = 4$	$f = \sum_{i=1}^{7} p_i x_i^*$ avec $p_4 = 0, p_3 = 0$
$i = 5$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p_4 x_4^* + p(x_5^* + x_6^*)$
$i = 6$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p(x_4^* + x_5^*)$
$i = 7$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p_4 x_4^* + p(x_6^* + x_7^*)$
$i = 8$	$f = \sum_{i=1}^{7} p_i x_i^*$avec un des $p_i \neq 0 \quad i \in \{3, ...7\}$

Par la suite, on s'intéresse aux algèbres de Lie quasi-filiformes graduées (Théorème 3.5.3).

Théorème 6 : Les indices d'algèbre de Lie quasi-filiformes graduées sont :

cas où n est pair

1. $\chi(L_{n-1} \oplus \mathbb{C}) = n - 2$.

2. $\chi(\mathcal{T}_{(n,n-3)}) = 2$.

3. $\chi(\mathcal{L}_{(n,r)}) = n - r - 1, \quad 3 \leq r \leq n - 3$.

cas où n est impair :

1. $\chi(L_{n-1} \oplus \mathbb{C}) = n - 2$.

2. $\chi(Q_{n-1} \oplus \mathbb{C}) = 3$.

3. $\chi(\mathcal{L}_{(n,n-2)}) = 3$.

4. $\chi(\mathcal{T}_{(n,n-4)}) = 3$.

5. $\chi(\mathcal{L}_{(n,r)}) = n - r - 1, \quad 3 \leq r \leq n-3$.

6. $\chi(Q_{(n,r)}) = 3$.

7. $\chi\left(\varepsilon_{(7,3)}\right) = 3$.

8. $\chi\left(\varepsilon^1_{(9,5)}\right) = 3$.

9. $\chi\left(\varepsilon^i_{(9,5)}\right) = 2, \quad i = 2, 3$.

La démonstration se base sur le calcul des matrices correspondantes aux algèbres de Lie graduées quasi-filiformes. On calcule leurs rang en petite dimension puis on généralise à la dimension n. En ce qui concerne les vecteurs régulìés associés aux algèbres $\mathcal{T}_{(n,n-3)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{L}_{(n,r)}$ et $Q_{(n,r)}$ les résultats sont donnés dans la Proposition 3.5.5.

Proposition 7 : *Les vecteurs réguliers des familles $T_{(n,n-3)}$, $T_{(n,n-4)}$, $L_{(n,r)}$ et $Q_{(n,r)}$ sont donnés par les fonctionnelles f suivantes, où les x_i^* sont des éléments de la base dual et les p_i sont des paramètres.*

1. $\mathcal{T}_{(n,n-3)}$:
$$f = \sum_{i=0}^{n-1} p_i \, x_i^* \text{ avec } p_{n-2} \neq 0.$$

2. $\mathcal{T}_{(n,n-4)}$:
$$f = \sum_{i=0}^{n-1} p_i x_i^* \text{ avec } p_{n-2} \neq 0.$$

3. $\mathcal{L}_{(n,r)}$: n pair ou impair et $r < n-2$:
$$f = \sum_{i=0}^{n-1} p_i x_i^* \text{ avec } p_{n-1} \neq 0 \text{ et un des } p_i \neq 0 \text{ où } i \in \{r+1, ..., n-2\}.$$

4. $Q_{(n,r)}$:
$$f = \sum_{i=0}^{n-1} p_i \, x_i^* \text{ avec } p_{n-2} \neq 0.$$

5. $\mathcal{L}_{(n,n-2)}$:

$$f=\sum_{i=0}^{n-1}p_ix_i^* \text{ avec } p_{n-1}\neq 0.$$

La démonstration se base sur le résultat donné dans la Remarque 2.2.7.

Enfin l'indice des algèbres Lie dont le nilradical est L_n noté $\mathfrak{n}_{n,1}$ et Q_{2n} est donné dans les Propositions 3.6.2 et 3.6.4 avec les vecteurs réguliers :

Proposition 8 : *Les indices des algèbres de Lie filiformes $\mathfrak{n}_{n,1}$ dont le nilradical est L_n sont*

$$\text{Si } \dim\ \tau=n+1, \text{ alors } \chi(\tau_{n+1,i})=n-1, i=1,2,3.$$

$$\text{Si } \dim\ \tau=n+2, \text{ alors } \chi(\tau_{n+2,1})=n-2.$$

Proposition 9 : *Les indices des algèbres de Lie filiformes dont le nilradical est Q_{2n} de dimension n sont*

$$\chi\left(\tau_{2n+1}\left(\lambda_2\right)\right)=1,$$

$$\chi\left(\tau_{2n+1}\left(2-n,\varepsilon\right)\right)=1,$$

$$\chi\left(\tau_{2n+1}\left(\lambda_2^5,...,\lambda_2^{2n-1}\right)\right)=1.$$

Dans le chapitre 4, nous introduisons et nous étudions la notion d'indice pour une algèbre Hom-Lie par rapport à une représentation coadjointe et une représentation arbitraire. On démontre d'abord le lemme suivant :

Lemme 10 : $\max\limits_{f\in\mathbb{V}^*}\dim\mathfrak{g}\cdot f=\dim\mathfrak{g}-\min\left\{\dim\mathfrak{g}_f, f\in\mathbb{V}^*\right\}.$

Nous constatons que la formule de l'indice ne change pas dans le cas des algèbres Hom-Lie, sauf si l'algèbre est obtenue par le principe de twisting introduit par D. Yau. Le principe de twisting de Yau consiste à associer à une algèbre de Lie, un morphisme d'algèbre α, une algèbre Hom-Lie obtenue par composition de α et du crochet et dont l'application définissant la structure est α. La formule de l'indice dans ce cas est donnée dans la Proposition 4.4.6 :

Proposition 11 *: Soit $(\mathfrak{g}_\alpha,[-,-]_\alpha,\alpha)$ une algèbre Hom-Lie et ad la représentation adjointe. Alors*

$$\chi\left(\mathfrak{g}_\alpha\right)=n-\operatorname{rank}_{\mathbb{k}(\mathbb{V})}\left(\alpha\left([e_i,e_j]\right)\right)_{ij}.$$

Ce qui nous a permis de comparer l'indice d'algèbre de Lie avec l'indice d'une algèbre Hom-Lie obtenue par le principe de twisting de Yau.

Théorème 12 : *Soit*$(\mathfrak{g}, [-,-])$ *une algèbre de Lie, et soit* $(\mathfrak{g}_\alpha, [-,-]_\alpha, \alpha)$ *l'algèbre Hom-Lie, alors on a* $\chi(\mathfrak{g}_\alpha) \geq \chi(\mathfrak{g})$. *Par ailleurs si* f *est un vecteur régulier de* $\mathfrak{g}$ *alors il est vecteur régulier de* $\mathfrak{g}_\alpha$.

Dans le cas ou α est bijective on a

$$\chi(\mathfrak{g}_\alpha) = \chi(\mathfrak{g}).$$

La démonstration se base sur des résultats d'algèbre linéaire.

On donne un exemple permettant au lecteur de voir les morphimes d'algèbre Hom-Lie obtenu par le principe de twisting, et leur indice Example 4.4.9. D'autre résultats sont obtenus dans le cas d'une algèbre Hom-Lie simple et multiplicative, les résultats sont données dans le Théorème 4.5.8

Théorème 13 : *L'indice d'une algèbre Hom-Lie multiplicative simple* $(\mathfrak{g}, [-,-], \alpha)$ *est le même que l'indice de l'algèbre de Lie induite multiplicative simple* $\left(\mathfrak{g}, \alpha^{-1}[-,-]\right)$.

On a aussi l'observation suivante (Proposition 4.5.9).

Proposition 14 : *L'indice d'une algèbre Hom-Lie multiplicative simple* $(\mathfrak{g}, [-,-], \alpha)$ *est strictement supérieur à 0.*

Finalement, nous étudions le produit semi-direct d'algèbre Hom-Lie. D'abord nous donnant la representation co-adjointe du produit semi-direct, cela revient à vérifier l'identité 4.7. On obtient le résultat suivant.

Proposition 15 : *Soit* $(\mathfrak{q}, [-,-], \gamma)$ *une algèbre Hom-Lie et soit* $(\mathfrak{q}, ad, \gamma)$ *la représentation adjointe de* $\mathfrak{q}$. *Le triplet* $(\mathfrak{q}, ad^*, \gamma^*)$ *définit une représentation de* $(\mathfrak{q}, [-,-]\gamma)$ *si et seulement si :*

$$\gamma \circ ad([x+u, y+v]) = ad(x+u) \circ ad(\gamma(y+v)) - ad(y+v) \circ ad(\gamma(x+u)).$$

Ce qui nous permet de définir le stabilisateur de cette représentation.

Proposition 16 : *Pour toute* $\eta = (g, f) \in \mathfrak{q}$, *on a*

$$\begin{aligned} \mathfrak{q}_\eta &= \left\{ (x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}, ad^*_{\mathfrak{q}}(x, v)(g, f) = 0 \right\}, \\ &= \left\{ (x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}, \left(ad^*_{\mathfrak{g}}(x)(g) - v * f, x \cdot f \right) = 0 \right\}, \\ &= \left\{ (x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}, ad^*_{\mathfrak{g}}(x)(g) = v * f \text{ et } x \cdot f = 0 \right\}. \end{aligned}$$

Telle que $ad^*_{\mathfrak{g}}(x)(g) = g[x, y] = v * f : y \in \mathfrak{g}$ et $x \cdot f = 0 \Rightarrow x \in \ker(\mathcal{K}_g | \mathfrak{g}_f)$.

Dans ce cas l'indice est donné par le Lemme 4.6.4

Proposition 17 : *Soit* $(\mathfrak{q}, [-, -], \gamma)$ *une algèbre Hom-Lie,* $\chi(\mathfrak{q}) = \chi(\mathfrak{g}) + \chi(\mathfrak{g}, \rho)$.

Par contre si la représentation est adjointe, on aura : $\chi(\mathfrak{q}) = 2\chi(\mathfrak{g})$.

Chapter 1

Introduction to Lie algebras theory

The study of Lie algebras is much more elementary than that of the groups. This chapter contains the basics about Lie algebras and the definitions of some interesting classes like Nilpotent Lie algebras.

1.1 Definitions and basic properties

We will start with some elementary definitions and notions.

1.1.1 Basic definitions and examples

Definition 1.1.1 *A Lie algebra over* $\mathbb{k}$ *is a vector space* $\mathfrak{g}$ *along with an antisymmetric map* $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying the Jacobi identity :*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

We can rewrite the Jacobi identity as

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]], \tag{1.1}$$

or

$$[[x,y],z] = [x,[y,z]] - [[x,y],z]. \tag{1.2}$$

A homomorphism of Lie algebras is a $\mathbb{k}-$linear map, $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that $\alpha([x,y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in \mathfrak{g}$.

A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a $\mathbb{k}-$subspace $\mathfrak{s}$ such that $[x, y] \in \mathfrak{s}$ whenever $x, y \in \mathfrak{s}$ (i.e., such that $[\mathfrak{s}, \mathfrak{s}] \in \mathfrak{s}$). With the bracket, it becomes a Lie algebra.

A Lie algebra $\mathfrak{g}$ is said to be **commutative** (or **abelian**) if $[x, y] = 0$ whenever $x, y \in \mathfrak{g}$. Thus, to give a commutative Lie algebra amounts to giving a finite-dimensional vector space.

An injective homomorphism is sometimes called an **embedding**, and a surjective homomorphism is sometimes called a **quotient map**.

Examples of Lie algebras

1. Abelian Lie algebras. Any vector space with the zero product $[x, y] = 0$.

2. $\mathbb{R}^3$ with vector product $x \times y$. It can be defined by bilinearity and skewsymmetry, once we postulate $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$.

3. From any associative algebra A we construct a Lie algebra on the same vector space by setting
$$[x, y] = xy - yx. \tag{1.3}$$
The Lie algebra of $n \times n-$matrices is called $gl(n)$.

4. Let $sl(n)$ be the subspace of $gl(n)$ consisting of matrices with zero trace. Since $Tr(AB) = Tr(BA)$, the set $sl(n)$ is closed under $[x, y] = xy - yx$, and hence is a Lie algebra.

5. Let $o(n)$ be the subspace of $gl(n)$ consisting of skew-symmetric matrices, that is, $A^T = -A$. Then
$$\begin{aligned}(AB - BA)^T &= B^T A^T - A^T B^T \\ &= (-B)(-A) - (-A)(-B) \\ &= -(AB - BA),\end{aligned}$$
so that $o(n)$ is closed under $[x, y] = xy - yx$, and hence is a Lie algebra. If we want to emphasize the dependence of the ground field $\mathbb{k}$ we write $gl(n, k)$, $sl(n, k)$, $o(n, k)$.

To define a Lie bracket on a vector space with basis $e_1, ..., e_n$ we need to specify the structure constants c_{lm}^r, that is, elements of $\Bbbk$ such that

$$[e_l, e_m] = \sum_{r=1}^{n} c_{lm}^r e_r$$

For example, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ is a basis of the vector space $sl(2)$. One easily checks that

1. $[H, X_+] = 2X_+, [H, X_-] = -2X_-, [X_+, X_-] = H$.

Similarly,

$$R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a basis of the vector space $o(3)$. We have

$[R_x, R_y] = R_z$, $[R_y, R_z] = R_x$, $[R_z, R_x] = R_y$.

Definition 1.1.2 *An ideal in a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{a}$ such that $[x, a] \in \mathfrak{a}$ for all $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$ (i.e., such that $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$).*

Notice that, because of the skew-symmetry of the bracket

$[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a} \iff [\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a} \iff [\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$ and $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$, all left (or right) ideals are two-sided ideals.

1.1.2 Graded Lie algebras

Definition 1.1.3 *A graded Lie algebra is an ordinary Lie algebra $\mathfrak{g}$, together with a gradation of vector spaces: $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ where $\mathfrak{g}_n$ are the subalgebra of $\mathfrak{g}$. such that the Lie bracket respects this gradation :*

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

1.1.3 Derivations, the adjoint map

Definition 1.1.4 *Let A be a $\mathbb{k}-$algebra (not necessarily associative). A derivation of A is a $\mathbb{k}-$linear map $D : A \to A$ such that*

$$D(ab) = D(a)\,b + aD(b) \text{ for all } a, b \in A. \tag{1.4}$$

The composite of two derivations need not be a derivation, but their bracket

$$[D, E] = D \circ E - E \circ D \tag{1.5}$$

is, and so the set of $\mathbb{k}-$derivations $A \to A$ is a Lie subalgebra

$$Der_{\mathbb{k}} A = gl(A).$$

For example, if the product on A is trivial, then the condition 1.3 is vacuous, and so

$$Der_{\mathbb{k}}(A) = gl(A).$$

Definition 1.1.5 *Let $\mathfrak{g}$ be a Lie algebra. For a fixed x in $\mathfrak{g}$, the linear map $y :\longmapsto [x, y] : \mathfrak{g} \to \mathfrak{g}$ is called the* ***adjoint (linear) map*** *of x, and is denoted $ad_{\mathfrak{g}}(x)$ or $ad(x)$ (we sometimes omit the parentheses) .*

For each x, the map $ad_{\mathfrak{g}}(x)$ is a $\mathbb{k}-$derivation of $\mathfrak{g}$ because the condition 1.1 can be rewritten as

$$ad(x)[y, z] = [ad(x)\,y, z] + [y, ad(x)\,z].$$

Moreover, $ad_{\mathfrak{g}}$ is a homomorphism of Lie algebras $\mathfrak{g} \to Der(\mathfrak{g})$ because 1.2 can be rewritten as

$$ad([x, y]\,z) = ad(x)(ad(y)\,z) - ad(y)(ad(x)\,z)$$

The kernel of $ad_{\mathfrak{g}} : \mathfrak{g} \to Der_{\mathbb{k}}\mathfrak{g}$ is the **center** of $\mathfrak{g}$,

$$z(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, \mathfrak{g}] = 0\}.$$

The derivations of $\mathfrak{g}$ of the form adx are said to be **inner** (by analogy with the inner automorphisms of a group).

An ideal in $\mathfrak{g}$ is a subspace stable under all inner derivations of $\mathfrak{g}$. A subspace stable under all derivations is called a **characteristic ideal**. For example, the centre $z(\mathfrak{g})$ of $\mathfrak{g}$ is a characteristic ideal of $\mathfrak{g}$. An ideal $\mathfrak{a}$ in $\mathfrak{g}$ is, in particular, a subalgebra of $\mathfrak{g}$; if $\mathfrak{a}$ is characteristic, then every ideal in $\mathfrak{a}$ is also an ideal in $\mathfrak{g}$.

1.1.4 The isomorphism theorems

When $\mathfrak{a}$ is an ideal in a Lie algebra $\mathfrak{g}$, the quotient vector space $\mathfrak{g}/\mathfrak{a}$ becomes a Lie algebra with the bracket $[x+a, y+a] = [x,y]+a$.

The following statements are straightforward consequences of the similar statements for vector spaces.

Remarks

1. **(Existence of quotients)**. The kernel of a homomorphism $\mathfrak{g} \to \mathfrak{g}'$ of Lie algebras is an ideal, and every ideal $\mathfrak{a}$ is the kernel of a quotient map $\mathfrak{g} \to \mathfrak{q}$.

2. **(Homomorphism Theorem)**. The image of a homomorphism, $\alpha : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras is a Lie subalgebra, $\alpha(\mathfrak{g})$ of $\mathfrak{g}$, and α defines an isomorphism of $\mathfrak{g}/\operatorname{Ker}(\alpha)$ onto $\alpha(\mathfrak{g})$; in particular, every homomorphism of Lie algebras is the composite of a surjective homomorphism with an injective homomorphism.

3. **(Isomorphism Theorem)**. Let $\mathfrak{h}$ and $\mathfrak{a}$ be Lie subalgebras of $\mathfrak{g}$. If $[\mathfrak{h},\mathfrak{a}] \subset \mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$, $\mathfrak{h}\cap\mathfrak{a}$ is an ideal in $\mathfrak{h}$, and the map

$$x+\mathfrak{h}\cap\mathfrak{a} \mapsto x+\mathfrak{a} : \mathfrak{h}/\mathfrak{h}\cap\mathfrak{a} \to (\mathfrak{h}+\mathfrak{a})/\mathfrak{a},$$

   is an isomorphism.

4. **(Correspondence Theorem)**. Let $\mathfrak{a}$ be an ideal in a Lie algebra $\mathfrak{g}$. The map $\mathfrak{h} \mapsto \mathfrak{h}/\mathfrak{a}$ is a bijection from the set of Lie subalgebras of $\mathfrak{g}$ containing $\mathfrak{a}$ to the set of Lie subalgebras of $\mathfrak{g}/\mathfrak{a}$. A Lie subalgebra $\mathfrak{h}$ containing $\mathfrak{a}$ is an ideal if and only if $\mathfrak{h}/\mathfrak{a}$ is an ideal in $\mathfrak{g}/\mathfrak{a}$, in which case the map $\mathfrak{g}/\mathfrak{h} \to (\mathfrak{g}/\mathfrak{a})/(\mathfrak{h}/\mathfrak{a})$is an isomorphism.

1.1.5 Normalizers and Centralizers

For a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the normalizer and centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ are :

$$n_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\},$$

$$c_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] = 0\}.$$

These are both subalgebras of $\mathfrak{g}$, and $n_{\mathfrak{g}}(\mathfrak{h})$ is the largest subalgebra containing $\mathfrak{h}$ as an ideal. When $\mathfrak{h}$ is commutative, $c_{\mathfrak{g}}(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ in its centre.

1.1.6 Representations

A **representation** of a Lie algebra $\mathfrak{g}$ on a $\mathbb{k}-$vector space $\mathbb{V}$ is a homomorphism $\rho : \mathfrak{g} \to gl(\mathbb{V})$. Thus ρ sends $x \in \mathfrak{g}$ to a $\mathbb{k}-$linear endomorphism $\rho(x)$ of $\mathbb{V}$, and

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x). \tag{1.6}$$

We often call $\mathbb{V}$ a $\mathfrak{g}-$module and write xv or $x_{\mathbb{V}}v$ for $\rho(x)(v)$. With this notation

$$[x, y] v = x(yv) - y(xv).$$

A representation is said to be **faithful** if it is injective. The representation

$$x \mapsto adx$$

$$\mathfrak{g} \to gl(\mathfrak{g}),$$

is called the adjoint representation of $\mathfrak{g}$ (see Definition 1.1.5).

Let W be a subspace of $\mathbb{V}$. The **stabilizer** of W in $\mathfrak{g}$ is

$$\mathfrak{g}_W \stackrel{def}{=} \{x \in \mathfrak{g} : xW \subset W\}.$$

It is clear from 1.5 that $\mathfrak{g}_W$ is a Lie subalgebra of $\mathfrak{g}$.

Let $v \in \mathbb{V}$, **The isotropy algebra** of v in $\mathfrak{g}$ is

$$\mathfrak{g}_v \stackrel{def}{=} \{x \in \mathfrak{g} : xv = 0\}.$$

It is a Lie subalgebra of $\mathfrak{g}$. An element v of $\mathbb{V}$ is said to be fixed by $\mathfrak{g}$, or invariant under $\mathfrak{g}$, if $\mathfrak{g} = \mathfrak{g}_v$, i.e., if $\mathfrak{g}v = 0$.

Killing form

From the adjoint representation we derive the Killing form (named after W. Killing; in the literature often denoted by $\mathcal{B}$) of $\mathfrak{g}$, a symmetric bilinear form on $\mathfrak{g}$ given by

$$k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{k},$$
$$k(x, y) = k_{ad}(x, y) = tr(adx \circ ady).$$

(The trace of the composition of linear transformations adx and ady, sending $x \in \mathfrak{g}$ to $[x, [y, y]]$). Since $Tr(AB) = Tr(BA)$ the Killing form is symmetric.

For an arbitrary representation $(\rho, \mathbb{V})$ of $\mathfrak{g}$

$$k_\rho(x, y) \overset{def}{\equiv} tr(\rho\,(x) \circ \rho\,(y))$$

Proposition 1.1.6 $k_\rho([x, y], z) = k_\rho(x, [y, z]) \rightsquigarrow k_\rho(ad_y x, z) + k_\rho(x, ad_y \circ z). = 0$

If $\mathfrak{g} = span\,(e_1, e_2, ..., e_n)$ and $\mathfrak{g}^* = span\left(e^1, e^2, ..., e^n\right)$;

$$\begin{aligned} e^i\,(e_j) &= e^i.e_j \\ &= \delta^i_j k\,(x, y) \\ &= \sum_{k=1}^{n} e^k.adx \circ ady.e_k \\ &= \sum_{k=1}^{n} e^k\,[x, [y, e_k]]\,. \end{aligned}$$

Proposition 1.1.7 *1.* $\delta \in Der(\mathfrak{g}) \rightsquigarrow k(\delta(x), y) + k(x, \delta(y)) = 0.$

2. $\tau \in Aut(\mathfrak{g}) \rightsquigarrow ad_{\tau x} = \tau \circ ad_x \circ \tau^{-1} \rightsquigarrow k(\tau x, \tau y) = k(x, y).$

3. $\mathfrak{l}$ *ideal of* $\mathfrak{g} \rightsquigarrow x, y \in \mathfrak{l} \Rightarrow k(x, y) = k_{\mathfrak{l}}(x, y).$

4. $\mathfrak{l}$ *ideal of* $\mathfrak{g}, \mathfrak{l}^\perp = \{x \in \mathfrak{g} : k(x, \mathfrak{l}) = \{0\}\} \rightsquigarrow \mathfrak{l}^\perp$ *is an ideal.*

1.1.7 Extensions, semidirect products

An exact sequence of Lie algebras $0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{b} \to 0$ is called an extension of $\mathfrak{b}$ by $\mathfrak{a}$. The extension is said to be **central** if a is contained in the centre of $\mathfrak{g}$, i.e., if $[\mathfrak{g}, \mathfrak{a}] = 0$.

Let $\mathfrak{a}$ be an ideal in a Lie algebra $\mathfrak{g}$. Each element g of $\mathfrak{g}$ defines a derivation $a \longmapsto [g, a]$ of $\mathfrak{a}$, and this defines a homomorphism $\phi : \mathfrak{g} \to Der\,(\mathfrak{a})\,,\, g \to ad\,(g) \mid \mathfrak{a}$.

Definition 1.1.8 *A Lie algebra $\mathfrak{g}$ is a semidirect product of subalgebras $\mathfrak{a}$ and $\mathfrak{q}$, denoted $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{q}$, if $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ and the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ induces an isomorphism $\mathfrak{q} \to \mathfrak{g}/\mathfrak{a}$.*

We have seen that, from a semidirect product $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{q}$, we obtain a triple

$$(\mathfrak{a}, \mathfrak{q}, \phi : \mathfrak{q} \to Der_k\,(\mathfrak{a})),$$

and that the triple determines $\mathfrak{g}$. We now show that every triple $(\mathfrak{a}, \mathfrak{q}, \phi)$ consisting of two Lie algebras $\mathfrak{a}$ and $\mathfrak{q}$ and a homomorphism $\phi : \mathfrak{q} \to Der_k\,(\mathfrak{a}))$ arises from a semidirect product. As a $\mathbb{K}-$vector space, we let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{q}$, and we define :

$$\left[(a, q)\,, (a', q')\right] = \left(\left[a, a'\right] + \phi_q a' - \phi_{q'} a, \left[q, q'\right]\right) \tag{1.7}$$

Proposition 1.1.9 *The bracket 1.4 makes $\mathfrak{g}$ into a Lie algebra.*

Proof. Routine verification. ■

We denote $\mathfrak{g}$ by $\mathfrak{a} \ltimes_\phi \mathfrak{q}$. The extension

$$0 \to \mathfrak{a} \to \mathfrak{a} \ltimes_\phi \mathfrak{q} \to \mathfrak{q} \to 0,$$

is central if and only if $\mathfrak{a}$ is commutative and ϕ is the zero map.

1.1.8 The universal enveloping algebra

Recall that, for an associative algebra $\mathcal{A}$ with unity 1, a Lie algebra structure on $\mathcal{A}$ is given by the Lie bracket $[a, b] = ab - ba$. Let $\mathfrak{g}(A)$ denote this Lie algebra. Then $\mathfrak{g}$ is a functor which converts associative algebras into Lie algebras. Every Lie algebra $\mathfrak{g}$ has a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ which is an associative algebra with unity. The construction of the universal enveloping algebra of a Lie algebra is useful in order to pass from a non-associative structure to a more familiar (associative) algebra over the same field while preserving the representation theory.

1.2 Nilpotent and Solvable Lie Algebras

We define now the nilpotent and solvable Lie algebras. These play an important role in the general theory of Lie algebras, as do semisimple Lie algebras, which will neither be defined nor used here. We will assume $\mathfrak{g}$ to be a finite dimensional Lie algebra.

Solvable Lie Algebras

Recall from 1.5 that $[\mathfrak{g},\mathfrak{g}]$ is the derived subalgebra of $\mathfrak{g}$.

Definition 1.2.1 *The series* $\mathfrak{g}^{(0)}=\mathfrak{g}$, $\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}],\ldots,$ $\mathfrak{g}^{(i)}=[\mathfrak{g}^{(i-1)},\mathfrak{g}^{(i-1)}]$, *is called the derived series of* $\mathfrak{g}$.

Definition 1.2.2 *A Lie algebra* $\mathfrak{g}$ *is said to be solvable if there exist* $n\in\mathbb{Z}^{+}$ *such that* $\mathfrak{g}^{(n)}=0$.

Proposition 1.2.3 1. $\mathfrak{g}$ *solvable ,* $\mathfrak{l}$ *is Lie-subalgebra* $\rightsquigarrow$ $\mathfrak{l}$ *solvable.*

2. $\mathfrak{g}$ *solvable,* $\phi:\mathfrak{g}\overset{Hom-Lie}{\rightarrow}\mathfrak{g}'\rightsquigarrow\phi(\mathfrak{g})$ *solvable.*

3. $$\left\{\begin{array}{c}\mathfrak{J}\ \textit{solvable ideal}\\ \textit{and}\\ \mathfrak{g}/\mathfrak{J}\ \textit{solvable}\end{array}\right\}\Rightarrow\mathfrak{g}\ \textit{solvable.}$$

4. $\mathfrak{a}$ *and* $\mathfrak{b}$ *solvable ideals*$\Rightarrow\mathfrak{a}+\mathfrak{b}$ *solvable ideal.*

Radical

Definition 1.2.4 $\mathfrak{Rad}(\mathfrak{g})=$ *radical* $=$ *maximal solvable ideal =sum of all solvable ideals.*

Proposition 1.2.5 $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2\rightsquigarrow\mathfrak{Rad}(\mathfrak{g})=\mathfrak{Rad}(\mathfrak{g}_1)\oplus\mathfrak{Rad}(\mathfrak{g}_2)$

Nilpotent Lie Algebras

Definition 1.2.6 *The series* $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], ..., \mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}], ...$ *is called the lower or descending central series of a Lie algebra* $\mathfrak{g}$.

Definition 1.2.7 *The Lie algebra* $\mathfrak{g}$ *is said to be nilpotent if there exists* $n \in \mathbb{Z}^+$ *such that* $\mathfrak{g}^n = 0$.

Example 1.2.8 *An abelian Lie algebra is solvable and nilpotent.*

Proposition 1.2.9

1. $\mathfrak{g}$ *nilpotent* $\Rightarrow \exists\ m : adx_1 \circ adx_2 \circ ... \circ adx_m = 0 \rightsquigarrow (adx)^m = 0$.
2. $\mathfrak{g}$ *nilpotent,* $\Leftrightarrow \exists\ \mathfrak{g}_i$ *ideals* $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset ... \supset \mathfrak{g}_r = \{0\}$, $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ *and* $\dim \mathfrak{g}_i/\mathfrak{g}_{i+1} = 1$.
3. $\mathfrak{g}$ *nilpotent* $\Rightarrow \exists$ *basis* $e_1, e_2, ..., e_n$ *where* adx *is strictly upper triangular.*
4. $\mathfrak{g}$ *nilpotent* $\Rightarrow k(x, y) = 0$.
5. $\mathfrak{g}$ *nilpotent* $\rightsquigarrow z(\mathfrak{g}) \neq \{0\}$.
6. $\mathfrak{g}/z(\mathfrak{g})$ *nilpotent* $\rightsquigarrow \mathfrak{g}$ *nilpotent.*

Engel's Theorem

Theorem 1.2.10 (Engel's Theorem) $\mathfrak{g}$ *nilpotent,* $\Longleftrightarrow \exists\ n : (adx)^n = 0$

Theorem 1.2.11 $\mathfrak{g}$ *nilpotent* $\Leftrightarrow$ *exists a basis in* $\mathfrak{g}$ *such that all the matrices* adx, $x \in \mathfrak{g}$ *are strictly upper diagonal.*

Proposition 1.2.12 $\mathfrak{g}$ *nilpotent* $\Rightarrow \{x \in \mathfrak{g} \rightsquigarrow Tr\ adx = 0\} \Rightarrow k(x; y) = Tr\ (adxady) = 0$

Cartan Criteria

Lemma 1.2.13 $\left\{ \begin{array}{c} \mathfrak{g} \subset gl(\mathbb{V}) \\ x; y \in \mathfrak{g} \rightsquigarrow Tr(xy) = 0) \end{array} \right\} \Rightarrow [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^{(1)}$ *nilpotent*

Theorem 1.2.14 *(1st Cartan Criterion)*

$\mathfrak{g}$ *solvable* , $k_{[\mathfrak{g},\mathfrak{g}]} = 0$.

Corollary 1.2.15 $k(\mathfrak{g}, \mathfrak{g}) = \{0\} \rightsquigarrow \mathfrak{g}$ *solvable.*

k *is non degenerate,* $\mathfrak{a}$ *ideal of* $\mathfrak{g} \rightsquigarrow \mathfrak{a}^{\perp} = \{x \in \mathfrak{g} : k(x, \mathfrak{a}) = \{0\}\}$ *is an ideal.*

1.3 Simple and semisimple Lie Algebras

Definition 1.3.1 *A Lie algebra* $\mathfrak{g}$ *is called simple if the only ideals are* $\mathfrak{g}$ *and* $\{0\}$.

We can also defined a simple Lie algebra $\mathfrak{g}$ as :

Definition 1.3.2 *A Lie algebra is called simple if it is not abelian and has no non trivial ideal.*

Definition 1.3.3 *We define a Lie algebra* $\mathfrak{g}$ *to be semisimple if it is the finite direct sum of simple Lie algebras* $\mathfrak{g}_i$ *:* $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 ... \oplus \mathfrak{g}_n$.

Example 1.3.4 *1. The* $0-$*dimensional Lie algebra is semisimple.*

2. If $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are semisimple, the* $\mathfrak{g} \times \mathfrak{g}'$ *are also semisimple.*

Remark 1.3.5 *A simple Lie algebra is semisimple.*

Definition 1.3.6 $\mathfrak{Rad}(\mathfrak{g}) = \{0\} \rightsquigarrow \mathfrak{g}$ *semi-simple.*

Theorem 1.3.7 *(2nd Cartan Criterion)*

$\mathfrak{g}$ *semisimple* $\Longleftrightarrow k$ *non degenerate.*

Toral (Cartan) subalgebra

$\mathfrak{g}$ is semisimple.

Definition 1.3.8 *Semisimple element :*

x_s *is semisimple* $\leftrightsquigarrow x \in \mathfrak{g} : x = x_s + x_n$.

x_s *is semisimple element* $\leftrightsquigarrow adx_s$ *is diagonalizable on* $\mathfrak{g}$.

x_s *and* y_s *semisimple elements* $x_s + y_s$ *is semisimple and* $[x_s, y_s]$ *is semisimple.*

Definition 1.3.9 *Toral/ Cartan subalgebra*

Toral subalgebra $\mathfrak{h} = \{x_s\ ,\ x = x_s + x_n \in \mathfrak{g}\}$ *i.e the set of all semisimple elements.*

Theorem 1.3.10 *The toral or Cartan subalgebra is abelian.*

Roots construction

$\mathfrak{g}$ semi-simple algebra,

$\mathfrak{h} = \{x_s : x \in \mathfrak{g}\ ,\ x = x_s + x_n\}$ Toral subalgebra or Cartan subalgebra,

$\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2 + ... + \mathbb{C}h_l$.

$ad_{\mathfrak{h}}$ is a matrix Lie algebra of commuting matrices all the matrices have common eigenvectors.

$\sum_i$ = eigenvalues of $h_i \rightsquigarrow \mathfrak{g} = \bigsqcup\limits_{\lambda_i \in \sum_i} \mathfrak{g}\lambda_i$, $\mathfrak{g}\lambda_i$ linear vector space.

$x \in \mathfrak{g}_{\lambda_i} \rightsquigarrow ad_{h_i}x = \lambda_i x$ and $\lambda_i \neq \nu_i \rightsquigarrow \mathfrak{g}_{\lambda_i} \cap \mathfrak{g}_{\nu_i} = \{0\}$,

$x \in \mathfrak{g}_{\lambda_1} \cap \mathfrak{g}_{\lambda_2} \cap ... \cap \mathfrak{g}_{\lambda_l}$, $h = c_1h_1 + c_2h_2 + ... + c_lh_l$,

$ad_h x = (c_1\lambda_1 + c_2\lambda_2 + ... + c_l\lambda_l)x$,

$\mathfrak{h}^* = \mathbb{C}\mu_1 + \mathbb{C}\mu_2 + ... + \mathbb{C}\mu_l, \qquad \mu_i \in \mathfrak{h}^*, \qquad \mu_i(h_j) = \delta_{ij}.$

Definition 1.3.11 *Of the roots* $\mathfrak{h}^* \ni \lambda = \lambda_1\mu_1 + \lambda_2\mu_2 + ... + \lambda_l\mu_l \rightsquigarrow c_1\lambda_1 + c_2\lambda_2 + ... + c_l\lambda_l = \lambda(h)$ *is a root.* $x \in \mathfrak{g}_\lambda = \mathfrak{g}_{\lambda_1} \cap \mathfrak{g}_{\lambda_2} \cap ... \cap \mathfrak{g}_{\lambda_l} \rightsquigarrow [h, x] = ad_h x = \lambda(h)x$,

$\mathfrak{g} = \bigsqcup\limits_{\lambda} \mathfrak{g}\lambda, \qquad \lambda \neq \mu \rightsquigarrow \mathfrak{g}_\lambda \cap \mathfrak{g}_\mu = \{0\}$.

Roots

$\mathfrak{g}$ semisimple algebra, $\mathfrak{h} = \{x_s : x \in \mathfrak{g}\ ,\ x = x_s + x_n\}$ Toral subalgebra, $ad_{\mathfrak{h}}$ is a matrix Lie algebra of commuting matrices $\rightsquigarrow$ all the matrices have common eigenvectors.

Theorem 1.3.12 *(Root space)*

Exists root space $\Delta \subset \mathfrak{h}^*$:

1. $\mathfrak{g} = \mathfrak{h} \oplus \bigsqcup\limits_{\alpha \in \Delta} \mathfrak{g}_\alpha$,

2. $x \in \mathfrak{g}_\alpha, \qquad h \in \mathfrak{h} \rightsquigarrow ad_h x = [h, x] = \alpha(h)x$,

3. $\mathfrak{h}$ *is a Lie subalgebra,* $\mathfrak{g}_\alpha$ *are vector spaces.*

Proposition 1.3.13 $\lambda, \mu \in \Delta \rightsquigarrow \left\{ \begin{array}{ll} [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu} & if : \lambda + \mu \in \Delta \\ [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = \{0\} & if : \lambda + \mu \notin \Delta \end{array} \right\}$

Reductive Lie algebra

Definition 1.3.14 *A Lie algebra is reductive if its adjoint representation is completely reducible, whence the name. More concretely, a Lie algebra is reductive if it is a direct sum of a semisimple Lie algebra* $\mathfrak{s}$ *and an abelian Lie algebra* $\mathfrak{a}$ *:* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$.

Definition 1.3.15 *A Lie algebra* $\mathfrak{g}$ *over a field of characteristic* 0 *is called reductive if any of the following equivalent conditions are satisfied:*

1. *The adjoint representation of* $\mathfrak{g}$ *is completely reducible (a direct sum of irreducible representations).*

2. $\mathfrak{g}$ *admits a faithful, completely reducible, finite-dimensional representation.*

3. *The radical of* $\mathfrak{g}$ *equals the center:* $\mathfrak{Rad}(\mathfrak{g}) = z(\mathfrak{g})$, *The radical always contains the center, but need not equal it.*

4. $\mathfrak{g}$ *is the direct sum of a semisimple ideal* $\mathfrak{s}_0$ *and its center* $z(\mathfrak{g}) : \mathfrak{g} = \mathfrak{s}_0 \oplus z(\mathfrak{g})$.

   Compare to the Levi decomposition, which decomposes a Lie algebra as its radical (which is solvable, not abelian in general) and a Levi subalgebra (which is semisimple).

5. $\mathfrak{g}$ *is a direct sum of prime ideals:* $\mathfrak{g} = \sum_i \mathfrak{g}_i$.

Some of these equivalences are easily seen. For example, the center and radical of $\mathfrak{s} \oplus \mathfrak{a}$ is $\mathfrak{a}$, while if the radical equals the center, the Levi decomposition yields a decomposition $\mathfrak{g} = \mathfrak{s}_0 \oplus z(\mathfrak{g})$. Further, simple Lie algebras and the trivial $1-$dimensional Lie algebra $\mathfrak{k}$ are prime ideals.

Definition 1.3.16 *Reductive Lie algebras are a generalization of semisimple Lie algebras, and share many properties with them : many properties of semisimple Lie algebras depend only on the fact that they are reductive.*

Corollary 1.3.17 *1. Let* $\mathfrak{g}$ *be a Lie algebra, and* $\mathfrak{a}$ *a semisimple Lie subalgebra of* $\mathfrak{g}$*. Then* $\mathfrak{a}^{\perp}$ *the orthogonal for the Killing form is a supplementary for* $\mathfrak{a}$ *and we have* $[\mathfrak{a}, \mathfrak{a}^{\perp}] \subset \mathfrak{a}^{\perp}$*.*

2. If furthermore $\mathfrak{a}$ *is an ideal in* $\mathfrak{g}$*, then* $\mathfrak{a}^{\perp}$ *is also an ideal and* $\mathfrak{g} = \mathfrak{a} \times \mathfrak{a}^{\perp}$*. Furthermore* $\mathfrak{a}^{\perp} = z_{\mathfrak{g}}(\mathfrak{a})$.

1.3.1 Parabolic Lie algebra

Definition 1.3.18 *In algebra, a parabolic Lie algebra* $\mathfrak{p}$ *is a subalgebra of a semisimple Lie algebra* $\mathfrak{g}$ *satisfying one of the following two conditions:*

1. $\mathfrak{p}$ *contains a maximal solvable subalgebra (a Borel subalgebra) of* $\mathfrak{g}$*;*

2. *The Killing form of* $\mathfrak{p}$ *in* $\mathfrak{g}$ *is the nilradical of* $\mathfrak{p}$*.*

Biparabolic subalgebras of semisimple Lie algebras were introduced by V. Dergachev and A. Kirillov (see [8]) under the name of Lie algebras of seaweed type.

Definition 1.3.19 *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra. A biparabolic subalgebra of* $\mathfrak{g}$ *is defined to be the intersection of two parabolic subalgebras whose sum is* $\mathfrak{g}$*. Such algebras are natural generalizations of parabolic subalgebras.*

Chapter 2

Lie algebras Index

2.1 Introduction

The definition of the index goes back to Dixmier. It is a very important notion in representation theory and in invariant theory. By definition, the index of a Lie algebra $\mathfrak{g}$ noted $ind\mathfrak{g}$ or $\chi_{\mathfrak{g}}$, is the minimum dimension of the stabilizers in a $\mathfrak{g}$ element $\mathfrak{g}^*$, for the coadjoint action. The index theory of Lie algebras was intensively studied by Elashvili (see [13],[16]), in the particular case of semi-simple Lie algebras and Frobenius Lie algebras. He classified all the algebraic Frobenius algebras up to dimension 6. In [8], the authors connect the computation of the index to combinatorial theory of meanders and evaluate the index of a Lie algebra of seaweed type, which is equal to the number of cycles in an associated permutation. The index of semi-simple Lie algebras was also studied in [48]. The authors of that paper consider a semi-simple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$, R its corresponding root system, π a base of R, and S, T subsets of π. They provide an upper bound for the index of $\mathfrak{g}_{S,T}$, the direct sum of $\mathfrak{h}$, and the sum of the root spaces for the positive roots in the space spanned by S and the sum of the root spaces for the negative roots in the space spanned by T. They then verify that this inequality is actually an equality in a number of special cases and conjecture that equality holds in all cases. See also [47], where the index of a Borel subalgebra of a semi-simple Lie algebra is determined.

In this chapter we summarize the index theory of the index of Lie algebras. Let $\mathfrak{g}$ be an n-dimensional Lie algebra. Let $x \in \mathfrak{g}$, we denote by adx the endomorphism of $\mathfrak{g}$ defined by $adx\,(y) = [x, y]$ for all $y \in \mathfrak{g}$.

2.2 Index of Lie algebra

Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{K}$ provided with the Zariski topology, $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual. Then $\mathfrak{g}$ actes on $\mathfrak{g}^*$ as follows :

$$\mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{g}^*$$
$$(x, f) \mapsto x \cdot f$$

where $\forall \in \mathfrak{g} : (x \cdot f)(y) = f([x, y])$.

Let $f \in \mathfrak{g}^*$ and Φ_f be a skew-symmetric bilinear form defined by

$$\Phi_f : \mathfrak{g}\times\mathfrak{g} \to \mathbb{K}$$
$$(x, y) \mapsto \Phi_f(x, y) = f([x, y]).$$

We denote the kernel of the map Φ_f by $\mathfrak{g}_f$,

$$\mathfrak{g}_f = \{x \in \mathfrak{g} : f([x, y]) = 0 \quad \forall y \in \mathfrak{g}\}. \tag{2.1}$$

Definition 2.2.1 *The index of a Lie algebra $\mathfrak{g}$ is the integer*

$$\chi(\mathfrak{g}) = \inf\{\dim \ \mathfrak{g}_f ; f \in \mathfrak{g}^*\}.$$

A linear functional $f \in \mathfrak{g}^*$ is called regular if $\dim \mathfrak{g}_f = \chi(\mathfrak{g})$. *The set of all regular linear functionals is denoted by $\mathfrak{g}_r^*$.*

Remark 2.2.2 *The set $\mathfrak{g}_r^*$ of all regular linear functionals is a nonempty Zariski open set.*

Let $\{x_1, \cdots, x_n\}$ be a basis of $\mathfrak{g}$. We can express the index using the matrix $([x_i, x_j])_{1\leq i<j\leq n}$ as a matrix over the ring $S(\mathfrak{g})$, (see [9]). We has the following proposition.

Proposition 2.2.3 *The index of an n-dimensional Lie algebra $\mathfrak{g}$ is the integer*

$$\chi(\mathfrak{g}) = \dim \mathfrak{g} - \operatorname{rank}_{\mathbb{k}(\mathbb{V})}([x_i, x_j])_{ij}$$

where $\mathbb{K}(\mathbb{V})$ is the quotient field of the symmetric algebra $S(\mathfrak{g})$.

Remark 2.2.4 *The index of an n-dimensional abelian Lie algebra is n.*

2.2.1 Representation Index

Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(\mathbb{V})$ a finite-dimensional representation of $\mathfrak{g}$, i.e., $\mathbb{V}$ is a $\mathfrak{g}-$module. Abusing notation, we write $g.v$ in place of $\rho(g)v$, if $g \in \mathfrak{g}$ and $v \in \mathbb{V}$. An element $v \in \mathbb{V}$ is called regular or $\mathfrak{g}-$regular whenever its stationary subalgebra

$$\mathfrak{g}.v = \{g \in \mathfrak{g} | \quad g.v = 0\}$$

has minimal dimension. Because the function

$$v \mapsto \dim \mathfrak{g}.v \quad (v \in \mathbb{V})$$

is upper semicontinuous, the set of all $\mathfrak{g}-$regular elements is open and dense in $\mathbb{V}$.

Definition 2.2.5 *The nonnegative integer* $\dim \mathbb{V} - \max_{f\in\mathbb{V}^*}(\dim \mathfrak{g}{\cdot}x) = \dim \mathbb{V} - \dim \mathfrak{g} + \min_{f\in\mathbb{V}^*}(\dim \mathfrak{g}{\cdot}f)$ *is called the index of (the* $\mathfrak{g}-$*module)* $\mathbb{V}$*. It will be denoted by* $\chi(\mathfrak{g}, \mathbb{V})$*.*

This is the index of the $\mathfrak{g}-$module $\mathbb{V}$. In the special case of the adjoint representation ($\mathbb{V} = \mathfrak{g}$), we simply write $\chi(\mathfrak{g})$, and speak about the index of a Lie algebra $\mathfrak{g}$.

Proposition 2.2.6 *Let* $\mathfrak{g}_0$ *be a Lie algebra, and* $\mathfrak{g}$ *be a central extension of* $\mathfrak{g}_0$ *by a* $1-$*dimensional Lie algebra* $\mathcal{L} = c\mathbb{C}$*, then* $\chi(\mathfrak{g}) = \chi(\mathfrak{g}_0) + 1$*. Moreover* f *is a regular vector of* $\mathfrak{g}$ *then* $f = g + \rho c^*$ *where* g *is a regular vector of* $\mathfrak{g}_0$ *and* $\rho \in \mathbb{C}$*.*

Proof. Indeed, we have

$$\begin{cases} [x, c] = 0 \quad \forall x \in \mathfrak{g}_0, \\ \qquad [c, c] = 0. \end{cases}$$

Then the matrix associated to $\mathfrak{g}$ is of the form $M = \begin{pmatrix} M_{\mathfrak{g}_0} & 0 \\ 0 & 0 \end{pmatrix}$. It follows that $\operatorname{rank}(\mathfrak{g}) = \operatorname{rank}(\mathfrak{g}_0)$. Therefore

$$\chi(\mathfrak{g}) = \chi(\mathfrak{g}_0) + 1.$$

Let g be a regular vector of $\mathfrak{g}_0$. Then $\dim \mathfrak{g}_0^g = \chi(\mathfrak{g}_0)$ and $f = g + \rho c^*$ is a regular vector of $\mathfrak{g}$.We know that $\mathfrak{g}_f = \{x \in \mathfrak{g}, \quad f([x, y]) = 0, \forall y \in \mathfrak{g}\}$.We set

$$x = x_0 + \lambda c \text{ and } y = y_{0+}\mu c.$$

Then

$$f([x,y]) = g([x,y]) + \rho c^*([x,y]) = g([x_0,y_0]).$$

We have

$$g([x_0,y_0]) = 0 \text{ if } x_0 \in \mathfrak{g}_0, \forall y \in \mathfrak{g}_0.$$

Therefore $\mathfrak{g}_f = \mathfrak{g}_0^g + c\mathbb{C}$. ■

Remark 2.2.7 *In the sequel, we use the following procedure to compute regular vectors. We recall that if* $\dim \mathfrak{g}_f = \chi(\mathfrak{g})$ *then* f *is a regular vector of* $\mathfrak{g}$, *where* $\chi(\mathfrak{g}) = \min\{\dim \mathfrak{g}_f, f \in \mathfrak{g}^*\}$ *and* $\mathfrak{g}_f = \{x \in \mathfrak{g} : f([x,y]) = 0 \ \forall y \in \mathfrak{g}\}$.

The equation $f([x,y]) = 0$ *implies* $\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^n a_i b_j p_s x_s^*([x_i,x_j]) = 0$.

It is equivalent to

$\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^n a_i b_j p_s C_{ij}^s = 0$, *where* C_{ij}^s *are the structure constants with respect to the basis* $\{x_i\}_i$. *Then for all* j, *we have* $\sum_{s=1}^n \sum_{i=1}^n a_i p_s C_{ij}^s = 0$. *It leads to*

$$\left(\sum_{s=1}^n p_s C_{ij}^s\right)_{ij} \begin{pmatrix} a_1 \\ . \\ . \\ . \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

We denote by $M = \left(\sum_{s=1}^n p_s C_{ij}^s\right)_{ij}$ *and assume* $C_{ij}^s = -C_{ji}^s$. *We search the minors of order* $n - \chi(\mathfrak{g})$ *of non-zero determinant of the matrix* M. *The matrix* $M = \left(\sum_{s=1}^n p_s C_{ij}^s\right)_{ij}$ *is the same matrix as the multiplication table in which we replace* x_s *by* p_s.

Definition 2.2.8 *A Lie algebra* $\mathfrak{g}$ *over an algebraically closed field of characteristic* 0 *is said to be Frobenius if there exists a linear form* $f \in \mathfrak{g}^*$ *such that the bilinear form* Φ_f *on* $\mathfrak{g}$ *is nondegenerate.*

In [15], the author described all the Frobenius algebraic Lie algebras $\mathfrak{g} = R + N$ whose nilpotent radical N is abelian in the following two cases: the reductive Levi subalgebra R

acts on N irreducibly and R is simple. He classified all the algebraic Frobenius algebras up to dimension 6. See also [37] and [38] for further computations.

We discuss now the evolution by deformation of the index of a Lie algebra. About deformation theory we refer to [18, 35, 30]. Let $\mathbb{V}$ be a $\mathbb{K}-$vector space and $\mathfrak{g}_0 = (\mathbb{V}, [\ ,\ ]_0)$ be a Lie algebra. Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{K}$ and $\mathbb{V}[[t]]$ be the set of formal power series whose coefficients are elements of $\mathbb{V}$. A *formal Lie deformation* of $\mathfrak{g}_0$ is given by the $\mathbb{K}[[t]]$-bilinear map $[\ ,\ ]_t : \mathbb{V}[[t]] \times \mathbb{V}[[t]] \to \mathbb{V}[[t]]$ of the form $[\ ,\ ]_t = \sum_{i \geq 0} [\ ,\ ]_i t^i$, where each $[\ ,\ ]_i$ is a $\mathbb{K}$-bilinear map $[\ ,\ ]_i : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, satisfying the skew-symmetry and the Jacobi identity.

Proposition 2.2.9 *The index of a Lie algebra decreases by one parameter formal deformation.*

Proof. The rank of the matrix $([x_i, x_j])_{ij}$ increases by deformation, consequently the index decreases. ■

2.3 Some useful inequalities about the index

Corollary 2.3.1 *([49]) Let $\mathfrak{g}$ be a Lie algebra and $f \in \mathfrak{g}^*$. Then $\chi(\mathfrak{g}) \leq \chi(\mathfrak{g}_f)$.*

Remark 2.3.2 *([49]) If f is regularin $\mathfrak{g}^*$, then the prvious corollary implies that*

$$\dim \mathfrak{g}_f = \chi(\mathfrak{g}) \leq \chi(\mathfrak{g}_f) \leq \dim \mathfrak{g}_f .$$

Thus $\chi(\mathfrak{g}_f) = \dim \mathfrak{g}_f$, which proves that $\mathfrak{g}_f$ is a commutative Lie algebra.

Theorem 2.3.3 *([38])Let $\mathfrak{g}$ be Frobenius and $\mathfrak{h}$ an ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{h}$ is nilpotent. Then $\chi(\mathfrak{h}) = \dim \mathfrak{g} - \dim \mathfrak{h}$.*

Proposition 2.3.4 *([11])*

1. *If $\mathfrak{g} = \mathfrak{g}_1 \dot{+} \mathfrak{g}_2$, then $\chi(\mathfrak{g}) = \chi(\mathfrak{g}_1) \dot{+} \chi(\mathfrak{g}_2)$;*

2. *$\chi(\mathfrak{g}) \geq \dim z(\mathfrak{g})$, where $z(\mathfrak{g})$ is the centre of $\mathfrak{g}$.*

3. *If* $\mathfrak{g}$ *is reductive, then* $\chi(\mathfrak{g}) = \operatorname{rank}\mathfrak{g}$.

Proposition 2.3.5 *[8] Consider* $\Omega = \Phi_f(x, y)$ *as a skew-symmetric form over* $\mathfrak{g}$ *with coefficients in polynomials over* $\mathfrak{g}^*$. *Let* r *be the maximal number such that* $\Lambda^r\Omega$ *is non-zero. Then*

$$2r + \chi(\mathfrak{g}) = \dim\mathfrak{g}$$

Definition 2.3.6 *([9], 1.12.7) A Lie subalgebra* P *of* $\mathfrak{g}$ *is called a* ***polarization*** *if* $f([P, P]) = 0$ *and* $\dim P = \frac{1}{2}(\dim\mathfrak{g} + \dim\mathfrak{g}_f)$, *in other words* P *is a maximal totally isotropic subspace of* $\mathfrak{g}$ *(equipped with* Φ_f *). If in addition* P *is commutative then* f *is regular by the following observation.*

Proposition 2.3.7 *(see Theorem 14 of [36]). Let* P *be a commutative Lie subalgebra of* $\mathfrak{g}$: $h_1, ..., h_m$ *a basis of* P *and* $x_1, ..., x_n$ *a basis of* $\mathfrak{g}$. *Then the following conditions are equivalent:*

1. $\dim P = \frac{1}{2}(\dim\mathfrak{g} + \chi(\mathfrak{g}))$.

2. $P = P_f$ *(such an* f *is necessarily regular)*

3. $\operatorname{rank}_{\mathbb{K}(\mathbb{V})}([h_i, x_j]) = \dim\mathfrak{g} - \dim P$

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{u}$ (resp.$\mathfrak{l}$) the nilpotent radical (resp. a Levi factor) of $\mathfrak{p}$. In [[12], Corollary 1.5(i)], Panyushev showed that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) \leq \dim\mathfrak{l}$$

He then suggested [[12], Remark (ii) of Section 6] that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) \geq \operatorname{rank}\mathfrak{g}. \tag{2.2}$$

For example, it is well known that if $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ is its nilpotent radical, then $\chi(\mathfrak{b}) + \chi(\mathfrak{n}) = \operatorname{rank}\mathfrak{g}$ (see for example [47], [[49], Chapter 40]).

It is therefore also interesting to characterise parabolic subalgebras where equality holds in 2.2. Indeed, in [41], Raïs ıs looked for examples of direct sum decompositions $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}$ verifying the "index additivity condition", namely $\mathfrak{m}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{g}$ and

$$\chi(\mathfrak{g}) = \chi(\mathfrak{m}) + \chi(\mathfrak{n})$$

Parabolic subalgebras and their generalizations form one of the most interesting classes of non-reductive Lie algebras. The seaweed subalgebras were introduced in [8] as the intersections of two parabolic subalgebras whose sum is all of $\mathfrak{g}$ (say, for $\mathfrak{g} = gl(n, \mathbb{K})$ the seaweed subalgebras are those subalgebras that preserve a pair of opposite partial flags in the vector space $\mathbb{K}^n$

In [12], Panyushev proposed the following conjecture.

Conjecture 2.3.8 *Let* $\mathfrak{g}$ *be a reductive Lie algebra over the algebraically closed field* $\mathbb{K}$ *Then*

1. *For any nonreductive seaweed subalgebra* $\mathfrak{s}$ *in* $\mathfrak{g}$*,* $\chi(\mathfrak{s}) < \operatorname{rank} \mathfrak{g}$
2. *For any proper parabolic subalgebra* $\mathfrak{p}$ *in* $\mathfrak{g}$*,* $\chi(\mathfrak{p}) < \operatorname{rank} \mathfrak{g}$

The second part of the conjecture is obviously just the corollary of the first part, since any parabolic subalgebra is also a seaweed subalgebra. On the other hand, it was shown in [11] that for classical Lie algebras the second part of the conjecture actually implies the first.

In the following we give some results of Raïs [41] : We use the following notations : $\mathfrak{g}$ is a Lie algebra, $\mathfrak{a}$ is an ideal, f a linear form on $\mathfrak{g}$ and $f_0 = f_{|\mathfrak{a}}$ is its restriction to $\mathfrak{a}$. We have

$$\dim \mathfrak{a}_{f_0} + \dim \mathfrak{g}_f \leq \dim(\mathfrak{g}/\mathfrak{a}) + 2\,\operatorname{codim}(\mathfrak{g}.f_0).$$

and we deduce the inequality of Panyushev

Theorem 2.3.9 *([49]) Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{a}$ *an ideal of* $\mathfrak{g}$ *Then :*

$$\chi(\mathfrak{g}) + \chi(\mathfrak{a}) \leq \dim(\mathfrak{g}/\mathfrak{a}) + 2\chi(\mathfrak{g}, \mathfrak{a}). \tag{2.3}$$

(so-called inequality of Panyushev) in fact, equality 2.3 is written

$$\dim \mathfrak{g} - \chi(\mathfrak{g}) = 2(\dim \mathfrak{a} - \chi(\mathfrak{g}, \mathfrak{a})).$$

Proposition 2.3.10 *We note Ω open Zariski in $\mathfrak{g}^*$ constituted by the linear forms such as f codim$(\mathfrak{g}.f_0) = \chi(\mathfrak{g}, \mathfrak{a})$ (i.e such that $f_0 = \ell_{|\mathfrak{a}}$ is a regular element for naurelle representation of $\mathfrak{g}$ in $\mathfrak{a}$). the equality : $\chi(\mathfrak{a}) + \chi(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{a}) + 2 + \chi(\mathfrak{g}, \mathfrak{a})$ is realised if and only if : For all f in Ω, we have : $\mathfrak{a}_{f_0} + \mathfrak{g}_f = \mathfrak{a}_f$.*

Here we examine in particular the case where $\mathfrak{a}$ is an abelian ideal which is a direct factor, i.e we assume that there exists a subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{a}$. When this is so, we $\mathfrak{a}_f = \mathfrak{a} \oplus \mathfrak{q}(f_0)$, where $\mathfrak{q}(f_0)$ is the annihilator of f_0 in $\mathfrak{q}$, i.e the set of x in $\mathfrak{q}$ such as : $< f, [x, \mathfrak{a}] >= 0$, for every linear form f.

Proposition 2.3.11 *The following assertions are equivalent :*

1. $\chi(\mathfrak{a}) + \chi(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{a}) + 2 + \chi(\mathfrak{g}, \mathfrak{a})$.

2. *For all f in Ω, the "generic stabilizer" $\mathfrak{q}(f_0)$ is abelian.*

This Proposition gives some examples where there is equality :

1. Let $\mathfrak{q}$ be a simple Lie algebra and $\rho : \mathfrak{q} \to g\ell(\mathbb{V})$ be a linear representation of finit dimension. When this dimension is strictly upper than that of $\mathfrak{q}$, it is known that the generic stabilizer is reduced to $\{0\}$. So there is equality in the inequality for Panyushev $\mathfrak{g} = \mathbb{V} \ltimes_\rho \mathfrak{q}$ with $\mathfrak{a} = \mathbb{V} \times \{0\}$.

2. Let $\mathfrak{q}$ be a Lie algebra and : $\rho : \mathfrak{q} \to g\ell(\mathfrak{q})$ be its adjoint representation. denotes by $\mathfrak{g} = \mathfrak{q} \ltimes_\rho \mathfrak{q}$ the Lie algebra of semidirect product associated with this representation. We know that ([42]) : $\chi(\mathfrak{g}) = 2\chi(\mathfrak{q})$. So we have, with $\mathfrak{a} = \mathfrak{q} \times \{0\}$:

$$\chi(\mathfrak{g}) + \dim \mathfrak{a} = 2\chi(\mathfrak{q}) + \dim \mathfrak{q},$$

$$\text{and } \dim(\mathfrak{g}/\mathfrak{a}) + 2\chi(\mathfrak{g}, \mathfrak{a}) = \dim \mathfrak{q} + 2\chi(\mathfrak{q}).$$

Therefore we have the equality, which may be looked as priori by noting that the generic stabilizer of the coadjoint representation of a Lie algebra is abelian. Finally we note another characterization in the case of equality, always when a is Abelian,which can have a geometric interpretation.

Proposition 2.3.12 *The following assertions are equivalent :*

1. $\chi(\mathfrak{g}) + \dim \mathfrak{a} = \dim(\mathfrak{g}/\mathfrak{a}) + 2\chi(\mathfrak{g}, \mathfrak{a})$
2. *For all f in Ω, we have* $\dim(\mathfrak{g}, f) = 2\dim(\mathfrak{g}, f_0)$.

Indeed, equality 1 can be written

$$\dim \mathfrak{g} - \chi(\mathfrak{g}) = 2(\dim \mathfrak{a} - \chi(\mathfrak{g}, \mathfrak{a})).$$

There are interesting examples where $\mathfrak{a}$ is not Abelian and where there is equality in Panyushev enquality.

Example 2.3.13 *We consider the Lie algebra $\mathfrak{g}$ of the group Mautner; $\mathfrak{g}$ has (P, Q, E, X) as a base with :*

$$[P; Q] = E, \ [X, P] = Q, \ [X, Q] = -P$$

The index of $\mathfrak{g}$ is obviously 2. *Consider the ideal $\mathfrak{a}$ admitting (P, Q, E) as a base; it is the Heisenberg algebra of dimension 3; therefore $\chi(\mathfrak{a}) = 1$ and thus :*

$$\chi(\mathfrak{g}) + \chi(\mathfrak{a}) = 3$$

A direct calculation shows that $\chi(\mathfrak{g}, \mathfrak{a}) = 1$ So :

$$\dim(\mathfrak{g}/\mathfrak{a}) + 2\chi(\mathfrak{g}, \mathfrak{a}) = 3.$$

Chapter 3

Index of Graded Filiform and Quasi Filiform Lie Algebras

3.1 Introduction

In this chapter we focus on the computation of the index for nilpotent Lie algebras, mainly the class of filiform and quasi-filiform Lie algebras. The filiform Lie algebras were introduced by M. Vergne (see [50]), she classified them up to dimension 6 and also characterized the graded filiform Lie algebras. The classification of filiform Lie algebras of larger dimensions is known. In particular L_n plays an important role in the study of filiform and nilpotent Lie algebras. It is known that any n-dimensional filiform Lie algebra may be obtained by deformation of the one of the filiform Lie algebras L_n. The classification of naturally graded quasi-filiform Lie algebras is known. They have the characteristic sequence $(n-2, 1, 1)$ where n is the dimension of the algebra. In this chapter we focus on filiform Lie algebras and quasi-filiform Lie algebras. We compute the index and provide the regular vectors of n-dimensional filiform Lie algebras for $n < 8$ and quasi-filiform Lie algebras.This chapter is organized as follows In the first Section, we review the nilpotent and filiform Lie algebras theories. Section 2, is dedicated to the two graded filiform Lie algebras L_n and Q_n. In Section 3, we consider the classification up to dimension 8 and compute for each filiform Lie algebra its index and the set of all regular vectors. In section 4 we compute the index of graded quasi-filiform Lie algebras, and provide corresponding regular vectors. In the last section we compute the index of Lie algebras whose nilradical is Q_{2n}.

3.2 Nilpotent and Filiform Lie algebras

In this Section, we review the theory of nilpotent and filiform Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. We set $\mathcal{C}^0\mathfrak{g} = \mathfrak{g}$ and $\mathcal{C}^k\mathfrak{g} = [\mathcal{C}^{k-1}\mathfrak{g}, \mathfrak{g}]$, for $k > 0$. A Lie algebra $\mathfrak{g}$ is said to be nilpotent if there exists an integer p such that $\mathcal{C}^p\mathfrak{g} = 0$. The smallest p such that $\mathcal{C}^p\mathfrak{g} = 0$ is called the nilindex of $\mathfrak{g}$. Then, a nilpotent Lie algebra has a natural filtration given by the central descending sequence:

$$\mathfrak{g} = \mathcal{C}^0\mathfrak{g} \supseteq \mathcal{C}^1\mathfrak{g} \supseteq \cdots \mathcal{C}^{p-1}\mathfrak{g} \supseteq \mathcal{C}^p\mathfrak{g} = 0.$$

We have the following characterization of nilpotent Lie algebras (Engel's Theorem).

Theorem 3.2.1 *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if the operator adx is nilpotent for all x in $\mathfrak{g}$.*

In the study of nilpotent Lie algebras, filiform Lie algebras, which were introduced by M. Vergne [50], play an important role. An n-dimensional nilpotent Lie algebra is called *filiform* if its nilindex p equals $n-1$. The filiform Lie algebras are the nilpotent algebras with the largest nilindex. If $\mathfrak{g}$ is an n-dimensional filiform Lie algebra, then we have $dim\mathcal{C}^i\mathfrak{g} = n - i \quad \text{for} \quad 2 \leq i \leq n.$

Another characterization of filiform Lie algebras uses characteristic sequences $c(\mathfrak{g}) = sup\{c(x) : x \in \mathfrak{g} \setminus [\mathfrak{g}, \mathfrak{g}]\}$, where $c(x)$ is the sequence, in decreasing order, of dimensions of characteristic subspaces of the nilpotent operator adx.

Definition 3.2.2 *An n-dimensional nilpotent Lie algebra is filiform if its characteristic sequence is of the form $c(\mathfrak{g}) = (n-1, 1)$.*

3.3 Index of Graded filiform Lie algebras

The classification of filiform Lie algebras was given by Vergne ([50]) up to dimension 6 and then extended to dimension 11 by several authors (see [23, 6, 24, 43, 20]).

In the general case there is two classes L_n and Q_n of filiform Lie algebras which plays an important role in the study of the algebraic varieties of filiform and more generally nilpotent Lie algebras.

Let $\{x_1, \cdots, x_n\}$ be a basis of the $\mathbb{K}$ vector space L_n, the Lie algebra structure of L_n is defined by the following non-trivial brackets : $[x_1, x_i] = x_{i+1} \quad i = 2, ..., n-1$.

Let $\{x_1, \cdots, x_{n=2k}\}$ be a basis of the $\mathbb{K}$ vector space Q_n, the Lie algebra structure of Q_n is defined by the following non-trivial brackets.

$$Q_n : [x_1, x_i] = x_{i+1} \quad i = 2, ..., n-1,$$
$$[x_i, x_{n-i+1}] = (-1)^{i+1} x_n \quad i = 2, ..., k \quad \text{ where } n = 2k.$$

The classification of n-dimensional graded filiform Lie algebras yields to two isomorphic classes L_n and Q_n when n is odd and to only the Lie algebra L_n when n is even.

It turns out that any filiform Lie algebra is isomorphic to a Lie algebra obtained as a deformation of a Lie algebra L_n.

3.3.1 Index of Filiform Lie algebras

We aim to compute the index of L_n and Q_n and regular vectors.

Index of L_n : Let $\{x_1, x_2, ..., x_n\}$ be a fixed basis of the vector space $\mathbb{V} = L_n$ and $\{x_1^*, ..., x_n^*\}$ be a basis of the dual space. Set $f = \sum_{i \geq 1} p_i x_i^* \in \mathbb{V}^*$.

Proposition 3.3.1 *For $n \geq 3$, the index of the n-dimensional filiform Lie algebra L_n is $\chi(L_n) = n - 2$. The regular vectors of L_n are of the form $f = p_1\ x_1^* + p_2\ x_2^* + p_s\ x_s^*$ where $s \in \{3, ..., n\}$ and $p_s \neq 0$.*

Proof. Since the corresponding matrix to the Lie algebra L_n is of the form

$$\begin{pmatrix} 0 & x_3 & \cdots & x_n & 0 \\ -x_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and its rank is 2, then $\chi(L_n) = n-2$. The second assertion is obtained by a direct calculation:

We set $x = \sum_{i=1}^n \ a_i x_i$, $y = \sum_{j=1}^n \ b_j x_j$, $f = \sum_{s=1}^n p_s x_s^*$ and $\mathfrak{g}^f = \{x \in \mathfrak{g} : f([x,y]) = 0 \ \ \forall y \in \mathfrak{g}\}$.

Then $f([x,y]) = 0$ implies $\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^n a_i b_j p_s x_s^* \left([x_i, x_j]\right) = 0$. It is equivalent to

$$\sum_{s=1}^{n} \sum_{j=2}^{n-1} a_1 b_j p_s x_s^* \left([x_1, x_j]\right) - \sum_{i=2}^{n-1} a_i b_1 p_s x_s^* \left([x_1, x_j]\right) = 0.$$

Then we obtain $\sum_{s=1}^n \sum_{i=2}^{n-1} \left(a_1 b_i - a_i b_1\right) p_s x_s^* \left(x_{i+1}\right) = 0$. The equation $\sum_{i=2}^{n-1} \left(a_1 b_i - a_i b_1\right) p_{i+1} = 0$ should hold for all b_i. It leads to the following system

$$\begin{cases} a_1 p_{i+1} = 0, \quad 2 \le i \le n-1, \\ \sum_{i=2}^{n-1} a_i p_{i+1} = 0. \end{cases}$$

Therefore, one of the p_i satisfies $p_i \neq 0$ where i$\in \{3, ..., n\}$. ■

Index of Q_n :

Proposition 3.3.2 *For $n = 2k$ and $k \geq 2$, the index of the n-dimensional filiform Lie algebra Q_n is $\chi\left(Q_n\right) = 2$. The regular vectors of Q_n are of the form $f = \sum_{i=1}^n \ p_i x_i^*$ with $p_n \neq 0$.*

Proof. Since the corresponding matrix to the Lie algebra Q_n is of the form

$$\begin{pmatrix} 0 & x_3 & x_4 & \cdots & x_{n-1} & x_n & 0 \\ -x_3 & 0 & 0 & \cdots & 0 & -x_n & 0 \\ -x_4 & 0 & 0 & \cdots & x_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{n-1} & 0 & -x_n & \cdots & 0 & 0 & 0 \\ -x_n & x_n & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and its rank is n-2, then $\chi(Q_n) = 2$. The second assertion is obtained by the following calculation.

Let $\{x_1, x_2, ..., x_n\}$ be a fixed of basis of Q_n, $x = \sum_{i=1}^{n} a_i x_i$, $y = \sum_{j=1}^{n} b_j x_j$, $f = \sum_{s=1}^{n} p_s x_s^*$ and $\mathfrak{g}^f = \{x \in \mathfrak{g} : f([x, y]) = 0 \ \ \forall y \in \mathfrak{g}\}$.

The equation $f([x, y]) = 0$ may be written as $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{n} a_i b_j p_s x_s^* ([x_i, x_j]) = 0$. It is equivalent to $\sum_{i=2}^{n-1} (a_1 b_i - a_i b_1) p_{i+1} + \sum_{i=2}^{n-1} (-1)^{i+1} (a_1 b_{n-i+1} - a_{n-i+1} b_i) p_n = 0$. Then

$$\begin{cases} \sum_{i=2}^{n-1} (a_1 p_{i+1}) b_i = 0, \\ -b_1 \sum_{i=2}^{n-1} a_i p_{i+1} = 0, \\ \sum_{i=2}^{n-1} (-1)^{i+1} (a_1 p_n) b_{n-i+1} = 0, \\ -\sum_{i=2}^{n-1} (-1)^{i+1} (a_{n-i+1} p_n) b_i = 0. \end{cases}$$

Canceling the first and the last columns and the corresponding lines, leads to the following minor

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -x_n \\ 0 & 0 & \cdots & x_n & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -x_n & \cdots & 0 & 0 \\ x_n & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Hence, we obtain $f = \sum_{i=1}^{n} p_i x_i^*$, with $p_n \neq 0$. $\blacksquare$

Using Proposition 2.2.9, we obtain the following result.

Corollary 3.3.3 *The index of a filiform Lie algebra is less or equal to $n-2$.*

Proof. Any filiform Lie algebra $\mathcal{N}$ is obtained as a deformation of the Lie algebra L_n, since $\chi(L_n) = n-2$ and using Proposition 2.2.9, one has $\chi(\mathcal{N}) \leq n-2$. ■

3.4 Index of Filiform Lie algebras of dimension ≤ 8

In this section, we compute the index of n-dimensional Filiform Lie algebras with $n < 8$. Let $\mathfrak{g}$ be an n-dimensional Lie algebra. We set $\{x_1, x_2, ..., x_n\}$ be a fixed basis of $\mathbb{V} = \mathfrak{g}$, $\{x_1^*, x_2^*, ..., x_n^*\}$ and $f = \sum_{i\geq 1} p_i x_i^*$.

3.4.1 Filiform Lie algebras of dimension less than 6

Any n-dimensional Lie algebras with $n < 5$ is isomorphic to one of the following Lie algebras.

Dimension 1 and 2 We have only the abelian Lie algebras.

Dimension 3

$\mathcal{F}_3^1 : [x_1, x_2] = x_3$.

Dimension 4

$\mathcal{F}_4^1 : [x_1, x_2] = x_3$, $[x_1, x_3] = x_4$.

Dimension 5

$\mathcal{F}_5^1 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4$.

$\mathcal{F}_5^2 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4$ and $[x_2, x_3] = x_5$.

The computations of the index using Proposition 2.2.3 lead to the following result.

Proposition 3.4.1 *The index of n-dimensional filiform Lie algebras with $n \leq 5$ are*

$$\chi\left(\mathcal{F}_3^1\right) = 1, \quad \chi\left(\mathcal{F}_4^1\right) = 2, \quad \chi\left(\mathcal{F}_5^1\right) = 3, \quad \chi\left(\mathcal{F}_5^2\right) = 1.$$

The regular vectors of $\mathcal{F}_n^1$ for $n = 3, 4, 5$ are of the form $f = \sum_{i=1}^5 p_i x_i^*$ with one of $p_i \neq 0$ $i \in \{3, 4, 5\}$

The regular vectors of $\mathcal{F}_5^2$ are of the form $f = \sum_{i=1}^5 p_i x_i^*$ with $p_i \neq 0 . i \in \{3, 4, 5\}$

Proof. The filiform Lie algebras $\mathcal{F}_3^1$, $\mathcal{F}_4^1$ and $\mathcal{F}_5^1$ are of type L_n. For $\mathcal{F}_5^2$, the corresponding matrix is of rank 4, then the index is one. The regular vector are obtained by direct calculation. ■

3.4.2 Filiform Lie algebras of dimension 6

Any n-dimensional Lie algebras with $n = 6$ is isomorphic to one of the following Lie algebras.

$\mathcal{F}_6^1 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$

$\mathcal{F}_6^2 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_2, x_3] = x_6$

$\mathcal{F}_6^3 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_2, x_5] = x_6$, and $[x_3, x_4] = -x_6$

$\mathcal{F}_6^4 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_2, x_3] = x_5$, and $[x_2, x_4] = x_6$

$\mathcal{F}_6^5 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_2, x_3] = x_5 - x_6, [x_2, x_4] = x_6, [x_2, x_5] = x_6, [x_3, x_4] = -x_6$.

Proposition 3.4.2 *The index of 6-dimensional filiform Lie algebras are*

$$\chi\left(\mathcal{F}_6^i\right) = 2 \text{ for } i = 2, 4, 3, 5.$$

$$\chi\left(\mathcal{F}_6^1\right) = 4$$

The regular vectors of $\mathcal{F}_6^1$ are of the form $f = \sum_{i=1}^{6} p_i x_i^*$ with one of $p_i \neq 0. i = \{3, ..., 6\}$ (class of L_n algebra).

The regular vectors of $\mathcal{F}_6^2$ are of the form $f = p_1 x_1^* + p_2 x_2^* + p(x_3^* + x_4^* + x_5^*) + p_5 x_5^*$.

The regular vectors of $\mathcal{F}_6^4$ are of the form $f = \sum_{i=1}^{5} p_i x_i^*$ with $p_6 = 0$.

The regular vectors of $\mathcal{F}_6^i$ for $i = 3, 5$ are of the form $f = \sum_{i=1}^{6} p_i x_i^*$ with one of $p_i \neq 0$ $i \in \{3, ...6\}$.

3.4.3 Filiform Lie algebras of dimension 7

Any n-dimensional Lie algebras with $n = 7$ is isomorphic to one of the following Lie algebras.

$\mathcal{F}_7^1 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $\quad [x_1, x_6] = \alpha x_7$, $\quad [x_2, x_3] = (1 + \alpha) x_5$, $\quad [x_2, x_4] = (1 + \alpha) x_6$, $\;[x_3, x_4] = x_7$.

$\mathcal{F}_7^2 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_5$, $[x_2, x_4] = x_6$, $[x_2, x_5] = x_7$.

$\mathcal{F}_7^3 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_5 + x_6$, $[x_2, x_4] = x_6$, $[x_2, x_5] = x_7$.

$\mathcal{F}_7^4 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_6$, $[x_2, x_4] = x_7$, $[x_2, x_5] = x_7$, $[x_3, x_4] = -x_7$.

$\mathcal{F}_7^5 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_6 + x_7$, $[x_2, x_4] = x_7$.

$\mathcal{F}_7^6 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_6$, $[x_2, x_4] = x_7$.

$\mathcal{F}_7^7 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_7$

$\mathcal{F}_7^8 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$ (class of L_n algebra).

Proposition 3.4.3 *The index of 7-dimensional filiform Lie algebras are*

$$\chi\left(\mathcal{F}_7^i\right) = 3 \textit{ for } i = 2, 3, 5, 6, 7 \quad \chi\left(\mathcal{F}_7^4\right) = 1,$$

$$\chi\left(\mathcal{F}_7^1\right) = \begin{cases} 1 \textit{ if } \alpha \neq \{0, -1\}, \\ 3 \textit{ if } \alpha = 0. \end{cases}$$

$$\chi\left(\mathcal{F}_7^8\right) = 5.$$

The regular vectors of $\mathcal{F}_7^i$ are given by the following table

item	*regular vectors*
$i = 1$	$f = \sum_{i=1}^{5} p_i x_i^* + p(x_6^* + x_7^*)$, *if* $\alpha = 0$ $f = \sum_{i=1}^{6} p_i x_i^*$ *with* $p_i \neq 0$, *if* $\alpha \neq 0$
$i = 2$	$f = p_1 x_1^* + p_2 x_2^* + p(x_4^* + x_5^* + x_6^*)$ *with* $p \neq 0$
$i = 3$	$f = \sum_{i=1}^{4} p_i x_i^*$
$i = 4$	$f = \sum_{i=1}^{7} p_i x_i^*$ *with* $p_4 = 0, p_3 = 0$
$i = 5$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p_4 x_4^* + p(x_5^* + x_6^*)$
$i = 6$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p(x_4^* + x_5^*)$
$i = 7$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p_4 x_4^* + p(x_6^* + x_7^*)$
$i = 8$	$f = \sum_{i=1}^{7} p_i x_i^*$ *with one of* $p_i \neq 0$ $i \in \{3, ...7\}$

(3.1)

3.5 Index of Graded quasi-filiform Lie algebras

The classification of naturally graded quasi-filiform Lie algebras is known and given in [21]. They have the characteristic sequence $(n-2,1,1)$ where n is the dimension of the algebra.

Definition 3.5.1 *[21] An n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is said to be quasi-filiform if $C^{n-3}\mathfrak{g} \neq 0$ and $C^{n-2}\mathfrak{g} = 0$, where $C^0\mathfrak{g} = \mathfrak{g}$, $C^i\mathfrak{g} = \left[\mathfrak{g}, C^{i-1}\mathfrak{g}\right]$.*

In the following we describe the classification of naturally quasi-graded filiform Lie algebras

let $\mathcal{B} = \{x_0 x_2, ..., x_{n-1}\}$ be a basis of $\mathfrak{g}$:

3.5.1 Naturally graded Quasi-filiform Lie algebras

We consider the following classes of n-dimensional Lie algebras which are naturally graded quasi-filiform Lie algebras,

we set

Split : $L_{n-1} \oplus \mathbb{C}$ $(n \geq 4)$:

$[x_0, x_i] = x_{i+1}. \qquad 1 \leq i \leq n-3.$

$Q_{n-1} \oplus \mathbb{C}$ $(n \geq 7,\ n \text{ odd})$,

$[x_0, x_i] = x_{i+1}. \qquad 1 \leq i \leq n-3,$

$[x_i, x_{n-2-i}] = (-1)^{i-1} x_{n-2}. \qquad 1 \leq i \leq \frac{n-3}{2}.$

Principal : $\mathcal{L}_{(n,r)}$ $\left(n \geq 5,\ r \text{ odd},\ 3 \leq r \leq 2\left[\frac{n-1}{2}\right] - 1\right)$:

$[x_0, x_i] = x_{i+1}, \qquad 1 \leq i \leq n-3,$

$[x_i, x_{r-i}] = (-1)^{i-1} x_{n-1}, \qquad 1 \leq i \leq \frac{r-1}{2},$

$Q_{(n,r)}$ $(n \geq 7,\ \ n \text{ odd},\ r \text{ odd}, 3 \leq r \leq n-4)$:

$[x_0, x_i] = x_{i+1}, \qquad 1 \leq i \leq n-3,$

$[x_i, x_{r-i}] = (-1)^{i-1} x_{n-1}, \qquad 1 \leq i \leq \frac{r-1}{2},$

$[x_i, x_{n-2-i}] = (-1)^{i-1} x_{n-2}, \qquad 1 \leq i \leq \frac{n-3}{2}.$

Terminal : $\mathcal{T}_{(n,n-3)}$ $(n \text{ even}, n \geq 6)$:

$[x_0, x_i] = x_{i+1}, \qquad 1 \leq i \leq n-3,$

$[x_{n-1}, x_1] = \frac{n-4}{2} x_{n-2},$

$[x_i, x_{n-3-i}] = (-1)^{i-1} (x_{n-3} + x_{n-1}), \qquad 1 \le i \le \frac{n-4}{2},$

$[x_i, x_{n-2-i}] = (-1)^{i-1} \frac{n-2-2i}{2} x_{n-2}, \qquad 1 \le i \le \frac{n-4}{2},$

$\mathcal{T}_{(n,n-4)}$ $(n \text{ odd}, n \ge 7)$,

$[x_0, x_i] = x_{i+1}, \qquad 1 \le i \le n-3,$

$[x_{n-1}, x_1] = \frac{n-5}{2} x_{n-4+i}, \qquad 1 \le i \le 2,$

$[x_i, x_{n-4-i}] = (-1)^{i-1} (x_{n-4} + x_{n-1}), \qquad 1 \le i \le \frac{n-5}{2},$

$[x_i, x_{n-3-i}] = (-1)^{i-1} \frac{n-3-2i}{2} x_{n-2}, \qquad 1 \le i \le \frac{n-5}{2},$

$[x_i, x_{n-2-i}] = (-1)^{i-1} (i-1) \frac{n-3-i}{2} x_{n-2}, \qquad 1 \le i \le \frac{n-3}{2}.$

Moreover, we have the following 7-dimensional and 9-dimensional Lie algebra [17].

$$\varepsilon_{(7,3)} : \begin{cases} [x_0, x_i] = x_{i+1}, \quad 1 \le i \le 4, \\ [x_6, x_i] = x_{3+i}, \quad 1 \le i \le 2, \\ [x_1, x_2] = x_3 + x_6, \\ [x_1, x_i] = x_{i+1}, \quad 3 \le i \le 4. \end{cases}$$

$$\varepsilon^1_{(9,5)} : \begin{cases} [x_0, x_i] = x_{i+1}, \quad 1 \le i \le 6, \\ [x_8, x_i] = 2x_{5+i}, \quad 1 \le i \le 2, \\ [x_1, x_4] = x_5 + x_8, \quad [x_1, x_5] = 2x_6, \\ [x_1, x_6] = 3x_7, \quad [x_2, x_3] = -x_5 - x_8, \\ [x_2, x_4] = -x_6, \quad [x_2, x_5] = -x_7. \end{cases}$$

$$\varepsilon^2_{(9,5)} : \begin{cases} [x_0, x_i] = x_{i+1}, \quad 1 \le i \le 6, \\ [x_8, x_i] = 2x_{5+i}, \quad 1 \le i \le 2, \\ [x_1, x_4] = x_5 + x_8, \quad [x_1, x_5] = 2x_6, \\ [x_1, x_6] = x_7, \quad [x_2, x_3] = -x_5 - x_8, \\ [x_2, x_4] = -x_6, [x_2, x_5] = x_7, [x_3, x_4] = -2x_7. \end{cases}$$

$$\varepsilon^3_{(9,5)}:\begin{cases}[x_0,x_i]=x_{i+1}, \quad 1\le i\le 6,\\ [x_0,x_8]=x_6, \quad 1\le i\le 2,\\ [x_1,x_4]=x_8, \quad [x_3,x_4]=-3x_7,\\ [x_2,x_4]=-x_6, \quad [x_1,x_5]=2x_6,\\ [x_2,x_3]=-x_8, \quad [x_2,x_5]=2x_7.\end{cases}$$

are graded quasi-filiform Lie algebras.

Theorem 3.5.2 *[21] Every n-dimensional naturally graded quasi-filiform Lie algebra is isomorphic to one of the following Lie algebras :*

- *If n is even to $L_{n-1}\oplus\mathbb{C}$, $\mathcal{T}_{(n,n-3)}$, or $\mathcal{L}_{(n,r)}$ with r odd and $3\le r\le n-3$.*
- *If n is odd to $L_{n-1}\oplus\mathbb{C}$, $Q_{n-1}\oplus\mathbb{C}$, $\mathcal{L}_{(n,n-2)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{L}_{(n,r)}$, or $Q_{(n,r)}$ with r odd, and $3\le r\le n-4$. In the case of $n=7$ and $n=9$, we add $\varepsilon_{(7,3)}$, $\varepsilon^1_{(9,5)}$, $\varepsilon^2_{(9,5)}$, $\varepsilon^3_{(9,5)}$.*

Index of graded quasi-filiform Lie algebras :

In the following we compute the index of graded quasi–filiform Lie algebras. Let $\mathfrak{g}$ be a n-dimensional graded quasi-filiform Lie algebra

Theorem 3.5.3 *Index of graded quasi-filiform Lie algebras are*

case where n ***is even***

1. $\chi(L_{n-1}\oplus\mathbb{C})=n-2.$
2. $\chi(\mathcal{T}_{(n,n-3)})=2.$
3. $\chi(\mathcal{L}_{(n,r)})=n-r-1, \quad 3\le r\le n-3.$

case where n ***is odd :***

1. $\chi(L_{n-1}\oplus\mathbb{C})=n-2.$
2. $\chi(Q_{n-1}\oplus\mathbb{C})=3.$

3. $\chi(\mathcal{L}_{(n,n-2)}) = 3.$

4. $\chi(\mathcal{T}_{(n,n-4)}) = 3.$

5. $\chi(\mathcal{L}_{(n,r)}) = n - r - 1,\ \hat{E}\ 3 \leq r \leq n-3.$

6. $\chi(Q_{(n,r)}) = 3.$

7. $\chi\left(\varepsilon_{(7,3)}\right) = 3.$

8. $\chi\left(\varepsilon^{1}_{(9,5)}\right) = 3.$

9. $\chi\left(\varepsilon^{i}_{(9,5)}\right) = 2, \quad i = 2, 3.$

Proof. case where n is even

The corresponding matrix to the graded quasi-filiform Lie algebra $L_{n-1} \oplus \mathbb{C}$ is of the form

$$\begin{pmatrix} 0 & x_2 & \cdots & x_{n-1} & 0 & 0 \\ -x_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Its rank is 2, then $\chi(L_{n-1} \oplus \mathbb{C}) = n - 2.$

The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{T}_{(n,n-3)}$ is of the form

$$\begin{pmatrix}
0 & x_2 & x_3 & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & \cdots & x_{n-3}+x_{n-1} & \left(\frac{n-4}{2}\right)x_{n-2} & 0 & \left(\frac{n-4}{2}\right)x_{n-2} \\
-x_3 & 0 & 0 & \cdots & -\left(\frac{n-6}{2}\right)x_{n-2} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-3} & -x_{n-3}-x_{n-1} & \left(\frac{n-6}{2}\right)x_{n-2} & \cdots & 0 & 0 & 0 & 0 \\
-x_{n-2} & -\left(\frac{n-4}{2}\right)x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\left(\frac{n-4}{2}\right)x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

Its rank is $n-2$, then $\chi(\mathcal{T}_{(n,n-3)}) = 2$.

The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{L}_{(n,r)}$ is of the form

$$\begin{pmatrix}
0 & x_2 & x_3 & \cdots & x_r & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & \cdots & -x_{n-1} & \cdots & 0 & 0 & 0 & 0 \\
-x_3 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_r & x_{n-1} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-3} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
-x_{n-2} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

For $3 \le r \le n-3$, its rank is $r+1$. Then $\chi(\mathcal{L}_{(n,r)}) = n-r-1$.

case where n is odd :

The corresponding matrix to the graded quasi-filiform Lie algebra $L_{n-1} \oplus \mathbb{C}$ is of the form

$$\begin{pmatrix} 0 & x_2 & \cdots & x_{n-1} & 0 & 0 \\ -x_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Its rank is 2, then $\chi(L_{n-1} \oplus \mathbb{C}) = n - 2$.

The corresponding matrix to the graded quasi-filiform Lie algebra $Q_{n-1} \oplus \mathbb{C}$ is of the form

$$\begin{pmatrix} 0 & x_2 & x_3 & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\ -x_2 & 0 & 0 & \cdots & 0 & -x_{n-2} & 0 & 0 \\ -x_3 & 0 & 0 & \cdots & -x_{n-2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{n-3} & 0 & -x_{n-2} & \cdots & 0 & 0 & 0 & 0 \\ -x_{n-2} & x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

Its rank is $n - 3$, then $\chi(Q_{n-1} \oplus \mathbb{C}) = 3$.

The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{L}_{(n,n-2)}$ is of the form

$$\begin{pmatrix}
0 & x_2 & x_3 & x_4 & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & 0 & \cdots & 0 & -x_{n-1} & 0 & 0 \\
-x_3 & 0 & 0 & 0 & \cdots & x_{n-1} & 0 & 0 & 0 \\
-x_4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 \\
-x_{n-3} & 0 & -x_{n-1} & 0 & \cdots & 0 & 0 & 0 & 0 \\
-x_{n-2} & x_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

Its rank is $n-3$, then $\chi(\mathcal{L}_{(n,n-2)}) = 3$.

The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{L}_{(n,r)}$ is of the form

$$\begin{pmatrix}
0 & x_2 & x_3 & \cdots & x_r & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & \cdots & -x_{n-1} & \cdots & 0 & 0 & 0 & 0 \\
-x_3 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_r & x_{n-1} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-3} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
-x_{n-2} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

Its rank is $r+1$, then $\chi(\mathcal{L}_{(n,r)}) = n-r-1$, $3 \leq r \leq n-4$.

The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{T}_{(n,n-4)}$ is of the form

$$
\begin{pmatrix}
0 & x_2 & x_3 & x_4 & \cdots & x_{n-4} & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & 0 & \cdots & x_{n-4}+x_{n-1} & \frac{n-5}{2}x_{n-3} & 0 & 0 & -\left(\frac{n-5}{2}\right)x_{n-3} \\
-x_3 & 0 & 0 & 0 & \cdots & -\left(\frac{n-7}{2}\right)x_{n-3} & \frac{n-5}{2}x_{n-2} & 0 & 0 & -\left(\frac{n-5}{2}\right)x_{n-2} \\
-x_4 & 0 & 0 & 0 & \vdots & -2\left(\frac{n-6}{2}\right)x_{n-2} & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-4} & -x_{n-4}-x_{n-1} & \left(\frac{n-7}{2}\right)x_{n-3} & 2\left(\frac{n-6}{2}\right)x_{n-2} & \cdots & 0 & 0 & 0 & 0 & 0 \\
-x_{n-3} & -\left(\frac{n-5}{2}\right)x_{n-3} & -\left(\frac{n-5}{2}\right)x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
-x_{n-2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & \left(\frac{n-5}{2}\right)x_{n-3} & \left(\frac{n-5}{2}\right)x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Its rank is $n-3$, then $\chi\left(\mathcal{T}_{(n,n-4)}\right)=3$.

The corresponding matrix to the graded quasi-filiform Lie algebra $Q_{(n,r)}$ is of the form

$$
\begin{pmatrix}
0 & x_2 & x_3 & \cdots & x_r & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & \cdots & -x_{n-1} & \cdots & 0 & -x_{n-2} & 0 & 0 \\
-x_3 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_r & x_{n-1} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-3} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
-x_{n-2} & x_{n-2} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}
$$

For $3 \leq r \leq n-4$ its rank is $n-3$. Then $\chi(Q_{(n,r)})=3$. ■

Remark 3.5.4 *There are no Frobenius quasi-filiform Lie algebra.*

Regular vectors

Proposition 3.5.5 *The regular vectors of the families $\mathcal{T}_{(n,n-3)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{L}_{(n,r)}$ and $Q_{(n,r)}$ are given by the following functionals f where x_i^* are the element of the dual basis and p_i are parameters.*

1. $\mathcal{T}_{(n,n-3)}$:
$$f=\sum_{i=0}^{n-1} p_i\, x_i^* \text{ with } p_{n-2}\neq 0.$$
2. $\mathcal{T}_{(n,n-4)}$:
$$f=\sum_{i=0}^{n-1} p_i x_i^* \text{ with } p_{n-2}\neq 0.$$
3. $\mathcal{L}_{(n,r)}$: *n odd or even and $r<n-2$:*
$$f=\sum_{i=0}^{n-1} p_i x_i^* \text{ with } p_{n-1}\neq 0 \text{ and one of } p_i\neq 0 \text{ where } i\in\{r+1,...,n-2\}.$$
4. $Q_{(n,r)}$:
$$f=\sum_{i=0}^{n-1} p_i\, x_i^* \text{ with } p_{n-2}\neq 0.$$
5. $\mathcal{L}_{(n,n-2)}$:
$$f=\sum_{i=0}^{n-1} p_i x_i^* \text{ with } p_{n-1}\neq 0.$$

Proof. $\mathcal{T}_{(n,n-3)}$:

The associate system of the graded quasi-filiform Lie algebra $\mathcal{T}_{(n,n-3)}$ is of the form:
$$\begin{cases}\sum\limits_{i=1}^{n-3} a_i p_{i+1}=0,\\ a_0p_2-a_{n-4}(p_{n-3}+p_{n-1})-\frac{n-4}{2}a_{n-3}p_{n-2}+\frac{n-4}{2}a_{n-1}p_{n-2}=0,\\ a_0p_{i+1}+(-1)^i\,a_{n-3-i}(p_{n-3}+p_{n-1})-(-1)^i\,\frac{n-2-2i}{2}a_{n-2-i}p_{n-2}=0,\quad i=2,...n-4,\\ a_0p_{n-2}+\frac{n-4}{2}a_1p_{n-2}=0,\\ -\frac{n-4}{2}a_1p_{n-2}=0.\end{cases}$$
It turns out that $p_{n-2}\neq 0$ gives a solution of this system such that $\dim\mathfrak{g}^f=\chi_{\mathfrak{g}}$, then the regular vectors are given by : $f=\sum_{i=0}^{n-1} p_i\, x_i^*$ with $p_{n-2}\neq 0$.

$\mathcal{T}_{(n,n-4)}$:

The associate system is of the form:

$$\begin{cases} \sum\limits_{i=1}^{n-3} a_i p_{i+1} = 0, \\ a_0 p_2 - a_{n-5}(p_{n-4} + p_{n-1}) - \frac{n-5}{2} a_{n-4} p_{n-3} + \frac{n-5}{2} a_{n-1} p_{n-3} = 0, \\ a_0 p_3 + a_{n-6}(p_{n-4} + p_{n-1}) + \frac{n-7}{2} a_{n-5} p_{n-3} - \frac{n-5}{2} a_{n-4} p_{n-2} + \frac{n-5}{2} a_{n-1} p_{n-2} = 0, \\ a_0 p_{i+1} + (-1)^i a_{n-4-i}(p_{n-4} + p_{n-1}) + (-1)^i \frac{n-3-2i}{2} a_{n-3-i} p_{n-3} - (-1)^i \frac{n-3-i}{2} a_{n-2-i} p_{n-2} = 0 \\ \qquad\qquad i = 3, ...n-5, \\ a_0 p_{n-3} = 0, \\ -\frac{n-5}{2} a_1 p_{n-3} + \frac{n-5}{2} a_2 p_{n-2} = 0, \\ a_0 p_{n-3} + \frac{n-5}{2} a_1 p_{n-3} + \frac{n-5}{2} a_2 p_{n-2} = 0. \end{cases}$$

It follows that $p_{n-2} \neq 0$ gives a solution of this system such that $\dim \mathfrak{g}^f = \chi_{\mathfrak{g}}$, then the regular vectors are given by : $f = \sum_{i=0}^{n-1} p_i \, x_i^*$ with $p_{n-2} \neq 0$.

$\mathcal{L}_{(n,r)}$ n odd or even and $r < n-2$.

We cancel the columns $(r+1)$ until $(n-1)$ and the corresponding lines. We obtain the following minor

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -p_{n-1} \\ 0 & 0 & \cdots & p_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -p_{n-1} & \cdots & 0 & 0 \\ p_{n-1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

It is of non-zero determinant and this leads to $f = \sum_{i=0}^{n} p_i x_i^*$, with $p_{n-1} \neq 0$ and one of the p_i satisfies $p_i \neq 0$ where $i \in \{r+1, ..., n-2\}$.

The same reasoning and calculations are used for $Q_{(n,r)}$ and $\mathcal{L}_{(n,n-2)}$. ■

Remark 3.5.6 *Since $L_{n-1} \oplus \mathbb{C}$ and $Q_{n-1} \oplus \mathbb{C}$ are the central extension of L_n and Q_n, then the regular vectors could be given using Proposition 2.2.6.*

3.6 Index of Lie algebras whose nilradical is L_n or Q_{2n}

Snobel and Winternitz determined the Lie algebras whose nilradical is isomorphic to the filiform Lie algebra L_n. In their work this algebra is denoted by $\mathfrak{n}_{n,1}$ and it is defined with respect to the basis$\{x_1, ..., x_n\}$ by

$$[x_i, x_n] = x_{i-1}, \quad i = 2, ..., n-1.$$

Theorem 3.6.1 *[17] Let τ be a solvable Lie algebra over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and having as nilradical $\mathfrak{n}_{n,1}$. Then it is isomorphic to one of the following Lie algebras.*

1. *If dim $\tau = n+1$, set $\mathcal{B} = \{x_1, ..., x_n, f\}$ be a basis of τ.*

   $\tau_{n+1,1}$ *defined as*

   $$[f, x_i] = (n-2+\beta)\, x_i, \quad i = 1, ..., n-1,$$

   $$[f, x_n] = x_n.$$

   $\tau_{n+1,2}$ *defined as*

   $$[f, x_i] = x_i, \quad i = 1, ..., n-1.$$

   $\tau_{n+1,3}$ *defined as*

   $$[f, x_i] = (n-i)\, x_i, \quad i = 1, ..., n-1,$$

   $$[f, x_n] = x_n + x_{n-1}.$$

2. *If dim $\tau = n+2$, set $\mathcal{B} = \{x_1, ..., x_n, f_1, f_2\}$ be a basis of τ.*

   $\tau_{n+2,1}$ *defined as*

   $$[f_1, x_i] = (n-1-i)\, x_i, i = 1, ..., n-1,$$

   $$[f_2, x_i] = x_i, \;\; i = 1, ..., n-1,$$

   $$[f_1, x_n] = x_n, \;\; i = 1, ..., n-1.$$

3.6.1 Index of Lie algebras $\mathfrak{n}_{n,1}$ whose nilradical is L_n

Proposition 3.6.2 *Index of Lie algebras $\mathfrak{n}_{n,1}$ whose nilradical is L_n are*

$$\textit{If dim } \tau = n+1, \ \textit{then } \chi(\tau_{n+1,i}) = n-1, i = 1,2,3.$$

$$\textit{If dim } \tau = n+2, \ \textit{then } \chi(\tau_{n+2,1}) = n-2.$$

Proof. Set $\dim \tau = n+1$. The corresponding matrix to the algebra $\tau_{n+1,1}$ is of the form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -(n-2+\beta)\,x_1 \\ 0 & 0 & \cdots & 0 & 0 & -(n-2+\beta)\,x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -(n-2+\beta)\,x_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & -x_n \\ (n-2+\beta)\,x_1 & (n-2+\beta)\,x_2 & \cdots & (n-2+\beta)\,x_{n-1} & x_n & 0 \end{pmatrix}$$

Its rank is 2, then $\chi(\tau_{n+1,1}) = n-1$.

The corresponding matrix of the Lie algebra $\tau_{n+1,2}$ is of the form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -x_1 \\ 0 & 0 & \cdots & 0 & 0 & -x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -x_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ x_1 & x_2 & \cdots & x_{n-1} & 0 & 0 \end{pmatrix}$$

Its rank is 2, then $\chi(\tau_{n+1,2}) = n-1$.

The corresponding matrix to the Lie algebra $\tau_{n+1,3}$ is of the form:

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & -(n-1)\,x_1 \\
0 & 0 & \cdots & 0 & 0 & 0 & -(n-2)\,x_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & -(n-(n-2))\,x_{n-2} \\
0 & 0 & \cdots & 0 & 0 & 0 & -x_{n-1} \\
0 & 0 & \cdots & 0 & 0 & 0 & -x_n - x_{n-1} \\
(n-1)\,e_1 & (n-2)\,x_2 & \cdots & (n-(n-2))\,x_{n-2} & x_{n-1} & x_n + x_{n-1} & 0
\end{pmatrix}$$

Its rank is 2, then $\chi(\tau_{n+1,3}) = n-1$.

If $\dim \tau = n+2$, the corresponding matrix to the Lie algebra $\tau_{n+2,1}$ is of the form:

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & (n-2)\,x_1 & x_1 \\
0 & 0 & \cdots & 0 & 0 & (n-3)\,x_2 & x_2 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & (n-n)\,x_{n-1} & x_{n-1} \\
0 & 0 & \cdots & 0 & 0 & x_n & 0 \\
-(n-2)\,x_1 & -(n-3)\,x_2 & \cdots & -(n-n)\,x_{n-1} & -x_n & 0 & 0 \\
-x_1 & -x_2 & \cdots & -x_{n-1} & 0 & 0 & 0
\end{pmatrix}$$

Its rank is 4, then $\chi(\tau_{n+2,1}) = n-2$. ■

Regular vectors

If $\dim \tau = n+1$

1. $\tau_{n+1,1}$

$$f = \sum_{i=1}^{n} p_i\, x_i^* \text{ with } p_1, ..., p_n \neq 0.$$

2. $\tau_{n+1,2}$

$$f = \sum_{i=1}^{n} p_i\, x_i^* \text{ with } p_1, ..., p_{n-1} \neq 0.$$

3. $\tau_{n+1,3}$

$$f=\sum_{i=1}^{n}p_i\,x_i^{*}\text{ with }p_1,...,p_n\neq 0.$$

If dim $\tau=n+2$

4. $\tau_{n+2,1}$

$$f=\sum_{i=1}^{n}p_i\,x_i^{*}\text{ with }p_1,...,p_n\neq 0.$$

Proof. Straightforward calculations following Remark 2.2.7. ■

3.6.2 Lie algebras whose nilradical is Q_{2n}

Proposition 3.6.3 *[17] Any real solvable Lie algebra of dimension $2n+1$ whose nilradical Q_{2n} is isomorphic to one of the following Lie algebras :*

let $\mathcal{B}=\{x_1,...,x_{2n},y\}$ be a basis of τ

1. $\tau_{2n+1}(\lambda_2)$

$$[x_1,x_k]=x_{k+1},\quad 2\leq k\leq 2n-2,$$

$$[x_k,x_{2n+1-k}]=(-1)^k\,x_{2n},\quad 2\leq k\leq n,$$

$$[y,x_1]=x_1,$$

$$[y,x_k]=(k-2+\lambda_2)\,x_k,\quad 2\leq k\leq 2n-2,$$

$$[y,x_{2n}]=(2n-3+2\lambda_2)\,x_{2n}.$$

2. $\tau_{2n+1}(2-n,\varepsilon)$

$$[x_1,x_k]=x_{k+1},\quad 2\leq k\leq 2n-2,$$

$$[x_k,x_{2n+1-k}]=(-1)^k\,x_{2n}\quad 2\leq k\leq n,$$

$$[y,x_1]=x_1+\varepsilon x_{2n},\quad \varepsilon=-1,0,1,$$

$$[y,x_k]=(k-n)\,x_k,\quad 2\leq k\leq 2n-1,$$

$$[y,x_{2n}]=x_{2n}.$$

3. $\tau_{2n+1}\left(\lambda_2^5, \cdots, \lambda_2^{2n-1}\right)$

$$[x_1, x_k] = x_{k+1}, \quad 2 \leq k \leq 2n-2,$$

$$[x_k, x_{2n+1-k}] = (-1)^k x_{2n}, \quad 2 \leq k \leq n,$$

$$[y, x_{2+t}] = x_{2+t} + \sum_{k=2}^{\left[\frac{2n-3-t}{2}\right]} \lambda_2^{2k+1} x_{2k+1+t}, \quad 0 \leq t \leq 2n-6,$$

$$[y, x_{2n-k}] = x_{2n-k}, \quad k = 1, 2, 3,$$

$$[y, x_{2n}] = 2x_{2n}.$$

Index of Lie algebras whose nilradical is Q_{2n}

Proposition 3.6.4 *Index of n-dimensional Lie algebras whose nilradical is Q_{2n} are*

$$\chi\left(\tau_{2n+1}(\lambda_2)\right) = 1,$$

$$\chi\left(\tau_{2n+1}(2-n, \varepsilon)\right) = 1,$$

$$\chi\left(\tau_{2n+1}\left(\lambda_2^5, ..., \lambda_2^{2n-1}\right)\right) = 1.$$

Proof. The corresponding matrix of the Lie algebra $\tau_{2n+1}(\lambda_2)$ is of the form:

$$\begin{pmatrix}
0 & x_3 & x_4 & \cdots & 0 & 0 & -x_1 \\
-x_3 & 0 & 0 & \cdots & x_{2n} & 0 & -\lambda_2 x_2 \\
-x_4 & 0 & 0 & \cdots & 0 & 0 & -(n-(n-1)+\lambda_2)\, x_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -x_{2n} & 0 & \cdots & 0 & 0 & -(n-1+\lambda_2)\, x_{2n-1} \\
0 & 0 & 0 & \cdots & 0 & 0 & -(2n-3+2\lambda_2)\, x_{2n} \\
x_1 & \lambda_2 x_2 & (n-(n-1)+\lambda_2)\, x_3 & \cdots & (n-1+\lambda_2)\, x_{2n-1} & (2n-3+2\lambda_2)\, x_{2n} & 0
\end{pmatrix}$$

Its rank is $2n$, then $\chi(\tau_{2n+1}(\lambda_2)) = 1$.

The corresponding matrix of the algebra $\tau_{2n+1}(2-n, \varepsilon)$ is of the form

$$\begin{pmatrix}
0 & x_3 & x_4 & \cdots & x_{2n-1} & 0 & 0 & -x_1 - \varepsilon x_{2n} \\
-x_3 & 0 & 0 & \cdots & 0 & x_{2n} & 0 & -(n-2)\, x_2 \\
-x_4 & 0 & 0 & \cdots & -x_{2n} & 0 & 0 & -(n-2)\, x_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_{2n-1} & 0 & x_{2n} & \cdots & 0 & 0 & 0 & -(n-(2n-1))\, x_{2n-1} \\
0 & -x_{2n} & 0 & \cdots & 0 & 0 & 0 & -x_{2n} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
x_1 + \varepsilon x_{2n} & (n-2)\, x_2 & (n-2)\, x_3 & \cdots & (n-(2n-1))\, x_{2n-1} & x_{2n} & 0 & 0
\end{pmatrix}$$

Its rank is $2n$, then $\chi(\tau_{2n+1}(2-n, \varepsilon)) = 1$.

Since the corresponding matrix of the algebra $\tau_{2n+1}\left(\lambda_2^5, ..., \lambda_2^{2n-1}\right)$ is of rank $2n$ then the index is 1. ■

Remark 3.6.5 *The procedure described in Remark 2.2.7 could be used to compute the regular vectors of Lie algebras whose nilradical is* Q_{2n}.

Chapter 4

Index of Hom-Lie Algebras and semidirect products

4.1 Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [26]. A Hom-Lie algebra is a triple $(\mathfrak{g}, [-,-], \alpha)$, where α is a linear self-map, in which the skewsymmetric bracket satisfies an $\alpha-$twisted variant of the Jacobi identity, called the Hom-Jacobi identity. When α is the identity map, the Hom-Jacobi identity reduces to the usual Jacobi identity, and $\mathfrak{g}$ is a Lie algebra. In [34] Makhlouf and Silvestrov introduced the notion of Hom associative algebra $(\mathcal{A}, \mu, \alpha)$ in which α is a linear self-map of the vector space $\mathcal{A}$ and the bilinearity operation μ satisfies an α-twisted version of associativity. Associative algebras are a special cases of Hom-associative algebras in which α is the identity map. The dual notion of Hom-coalgebra was considered in [33].

We start by considering representations of Hom-Lie algebras. The representation theory of an algebraic object reveals some of its profound structures hidden underneath. In [44] Y. Sheng defined representations of Hom-Lie algebras and corresponding Hom-cochain complexes. In particular, he obtain the adjoint representation and the trivial representation of Hom-Lie algebras. A complete theory of the index exist for Lie algebra however we have not this theory in the case of Hom Lie algebras. The first main purpose of this chapter is to introduce the index of semidirect products of Hom-Lie algebras. In the second Section we summarize the definitions and basics of Hom-Lie algebras from [26, 34, 53].

In Section 3, we study the index of Hom-Lie algebras. We introduce the notion of the index of Hom-Lie algebras in the case of coadjoint and an arbitrary representation. Moreover, we compare the index of Lie algebra with the index of Hom-Lie algebra obtained by twisting. In the last Section we give the index of semidirect products of Hom-Lie algebras, but before we explore the coadjoint representations of semidirect products of Hom-Lie algebras.

4.2 Preliminary

We work in this chapter over an algebraically closed fields $\mathbb{K}$ of characteristic 0

4.2.1 Hom-Lie Algebras

The notion of Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov in [26] motivated initially by examples of deformed Lie algebras coming from twisted discretizations.

Definition 4.2.1 *[26] A Hom-Lie algebra is a triple* $(\mathfrak{g}, [-,-], \alpha)$ *consisting of a vector space* $\mathfrak{g}$*, a skew-symmetric bilinear map* $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *and a linear map :* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *satisfying the following Hom-Jacobi identity:*

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \tag{4.1}$$

Let $(\mathfrak{g}_1, [-,-]_1, \alpha_1)$, $(\mathfrak{g}_2, [-,-]_2, \alpha_2)$ be two Hom-Lie algebras. A linear map $\beta : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Hom-Lie algebra morphism if it satisfies

$$\begin{cases} \beta\,[x, y]_1 = [\beta(x), \beta(y)]_2\,,\ \forall x, y \in \mathfrak{g}_1 \\ \beta \circ \alpha_1 = \alpha_2 \circ \beta. \end{cases}$$

The map β is said to be a weak Hom-Lie algebras morphism if it satisfies only the first condition.

Remark 4.2.2 *We recover classical Lie algebra when* $\alpha = id_{\mathfrak{g}}$ *and the identity* (4.1) *is the Jacobi identity*

Definition 4.2.3 *[26] A Hom-Lie algebra is called a multiplicative Hom-Lie algebra if is an algebraic morphism, i.e. for any* $x, y \in \mathfrak{g}$ *we have* $\alpha([x, y]) = [\alpha(x), \alpha(y)]$.

Definition 4.2.4 *[26] A sub-vector space* $\mathfrak{H} \subset \mathfrak{g}$ *is a Hom-Lie sub-algebra of* $(\mathfrak{g}, [-,-], \alpha)$ *if* $\alpha(\mathfrak{H}) \subset \mathfrak{H}$ *and* $\mathfrak{H}$ *is closed under the braket operation i.e.*

$$[h, h\prime] \in \mathfrak{H}, \forall h, h\prime \in \mathfrak{H}.$$

Consider the direct sum of two Hom-Lie algebras, we have :

Proposition 4.2.5 *[26] Given two Hom-Lie algebras* $(\mathfrak{g}, [-,-], \alpha)$ *and* $(\mathfrak{H}, [-,-], \beta)$*, there is a Hom-Lie algebra* $(\mathfrak{g} \oplus \mathfrak{H}, [-,-], \alpha + \beta)$*, where the skew-symmetric bilinear map*

$$[-,-] : \mathfrak{g} \oplus \mathfrak{H} \times \mathfrak{g} \oplus \mathfrak{H} \to \mathfrak{g} \oplus \mathfrak{H}$$

is given by

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]), \forall x_1, x_2 \in \mathfrak{g}, y_1, y_2 \in \mathfrak{H},$$

and the linear map

$$(\alpha + \beta) : \mathfrak{g} \oplus \mathfrak{H} \to \mathfrak{g} \oplus \mathfrak{H}$$

is given by

$$(\alpha + \beta)(x, y) = (\alpha(x), \beta(y)), \forall x \in \mathfrak{g}, y \in \mathfrak{H}.$$

A morphism of Hom-Lie algebras : $(\mathfrak{g}, [-,-], \alpha)$ *and* $(\mathfrak{H}, [-,-], \beta)$ *is a linear map* $\phi : \mathfrak{g} \to \mathfrak{H}$*, such that*

$$\phi[x, y] = [\phi(x), \phi(y)], \forall x, y \in \mathfrak{g} \tag{4.2}$$

$$\phi \circ \alpha = \beta \circ \phi \tag{4.3}$$

Denote by $\mathfrak{g}_\phi \subset \mathfrak{g} \oplus \mathfrak{H}$ *the graph of a linear map* $\phi : \mathfrak{g} \to \mathfrak{H}$.

Theorem 4.2.6 *Let* $\mathfrak{g} = (\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra and* $\beta : \mathfrak{g} \to \mathfrak{g}$ *be a weak Hom-Lie algebra morphism, then* $(\mathfrak{g}, \beta[-,-], \beta_\alpha)$ *is a Hom-Lie algebra.*

Corollary 4.2.7 *[53] Let* $(\mathfrak{g}, [-,-])$ *be a Lie algebra and* $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ *be a Lie algebra endomorphism. Then* $\mathfrak{g}_\alpha = (\mathfrak{g}, [-,-]_\alpha, \alpha)$ *is a Hom-Lie algebra, where* $[-,-]_\alpha = \alpha \circ [-,-]$*. Moreover, suppose that* $\mathfrak{g}'$ *is another Lie algebra and that* $\alpha' : \mathfrak{g}' \rightarrow \mathfrak{g}'$ *is a Lie algebra endomorphism. If* $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ *is a Lie algebras endomorphism that satisfies*

$$f \circ \alpha = \alpha' \circ f$$

then

$$f : (\mathfrak{g}, [-,-]_\alpha, \alpha) \rightarrow (\mathfrak{g}', [-,-]_{\alpha'}, \alpha')$$

is a morphism of multiplicative Hom-Lie algebras.

The following example is obtained by using $\sigma-$derivation and it not of the above type.

Example 4.2.8 *(Jackson* Sl_2*). The Jackson* qSl_2 *is a* $q-$*deformation of the classical* Sl_2*. This family of Hom-Lie algebras was constructed in [45] using a quasi-deformation scheme based on discretizing by means of Jackson* $q-$*derivations a representation of* $Sl_2(\mathbb{K})$ *by one-dimensional vector fields (first order ordinary differential operators) and using the twisted commutator bracket defined in [26]. It carries a Hom-Lie algebra structure but not a Lie algebra structure. It is defined with respect to a basis* $\{x_1, x_2, x_3\}$ *by the brackets and a linear map* α *such that*

$$\begin{aligned} &[x_1, x_2] = -2qx_2 && \alpha(x_1) = qx_1 \\ &[x_1, x_3] = 2x_3 && \alpha(x_2) = q_2x_2 \\ &[x_2, x_3] = -\tfrac{1}{2}(1+q)x_1, && \alpha(x_3) = qx_3 \end{aligned}$$

where q *is a parameter in* $\mathbb{K}$ *if* $q = 1$ *we recover the classical* Sl_2*.*

Proposition 4.2.9 *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a multiplicative Hom-Lie algebra with* α *bijective, then* $(\mathfrak{g}, \alpha^{-1}[-,-])$ *is a Lie algebra.*

Proof. We set in the Theorem 4.2.6, $\beta = \alpha^{-1}$ it shows that a multiplicative Hom-Lie algebras with bijective twisting map correspond to a Lie algebras. The Lie algebra $(\mathfrak{g}, \alpha^{-1}[-,-])$ is called the induced Lie algebra. ■

4.3 Representations of Hom-Lie algebras

Definition 4.3.1 *[4] Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra. A representation of* $\mathfrak{g}$ *is a triple* $(\mathbb{V}, \rho, \beta)$*, where* $\mathbb{V}$ *is a* $\mathbb{K}-$*vector space,* $\beta \in End(\mathbb{V})$ *and*

$$\rho : \mathfrak{g} \to gl(\mathbb{V})$$

is a linear map satisfying

$$\rho([x,y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x), \forall x, y \in \mathfrak{g}.$$

In Particular a representation of a multiplicative Hom-Lie algebra $(\mathfrak{g}, [-,-]_{\alpha}, \alpha)$ on a vector space $\mathbb{V}$is a representation of the Hom-Lie algebra satisfiying in addition :

$$\alpha\left(\rho(x)(v)\right) = \rho(\alpha(x))(\beta(v)), \qquad \forall v \in \mathbb{V}, \qquad \forall x \in \mathfrak{g}.$$

In the following, we explore the dual representations and coadjoint representations of Hom-Lie algebras.

Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra and $(\mathbb{V}, \rho, \beta)$ be a representation of $\mathfrak{g}$. Let $\mathbb{V}^*$ be the dual vector space of $\mathbb{V}$. We define a linear map $\rho^* : \mathfrak{g} \to End(\mathbb{V}^*)$ by $\rho^*(x) = -{}^t\rho(x)$.

Proposition 4.3.2 *[4] Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra and* $ad : \mathfrak{g} \to gl(\mathfrak{g})$ *be an operator defined for* $x \in \mathfrak{g}$ *by* $ad(x)(y) = [x,y]$*. Then* $(\mathfrak{g}, ad, \alpha)$ *is a representation of* $\mathfrak{g}$*.*

Indeed the operator *ad* which is equivalent to Hom-Jacobi condition. We call the representation defined in the previous proposition adjoint representation of the Hom-Lie algebra.

Proposition 4.3.3 *[4] Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra and* $(\mathfrak{g}, ad, \alpha)$ *be the adjoint representation of* $\mathfrak{g}$*, where* $ad : \mathfrak{g} \to gl(\mathfrak{g})$*. We set* $ad^* : \mathfrak{g} \to gl(\mathfrak{g}^*)$ *and* $ad^*(x)(f) = -f \circ ad(x)$*. Then* $(\mathfrak{g}^*, ad^*, \alpha^*)$ *is a representation of* $\mathfrak{g}$ *if and only if*

$$\alpha([[x,y],z]) = [x, [\alpha(y), z]] - [y, [\alpha(x), z]] \qquad \forall x, y, z \in \mathfrak{g} \tag{4.4}$$

Proposition 4.3.4 *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra and* $(\mathbb{V}, \rho, \beta)$ *be a representation of* $\mathfrak{g}$*. The triple* $(\mathbb{V}^*, \rho^*, \beta^*)$*, where* $\rho^* : \mathfrak{g} \to gl(\mathbb{V}^*)$ *is given by* $\rho^*(x) = -{}^t\rho^*(x)$*, defines a representation of the Hom-Lie algebra* $(\mathfrak{g}, [-,-], \alpha)$ *if and only if*

$$\beta \circ \rho([x,y]) = \rho(x)\rho(\alpha(y)) - \rho(y)\rho(\alpha(x)).$$

4.4 Index of Hom-Lie algebras

4.4.1 For a coadjoint representation

Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra. We assume that coadjoint representation exists, that is proposition 4.3.3 satisfies. Let f be a bilinear form on $\mathfrak{g}$ then set

$$\begin{aligned} \mathfrak{g}_f &= \left\{ x \in \mathfrak{g} : ad^* (x) (f) = 0 \right\}, \\ &= \left\{ x \in \mathfrak{g} : f ([x, y]) = 0, \forall y \in \mathfrak{g} \right\}. \end{aligned}$$

Then we have the following definition

Definition 4.4.1 *The index of* $\mathfrak{g}$*, is defined by*

$$\chi(\mathfrak{g}) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}_f.$$

Example 4.4.2 *(Jackson* Sl_2*). The index is given by*

$\chi(\mathfrak{g} = qSl_2) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}_f = 1.$

Indeed the associated matrix is of the form $\begin{pmatrix} 0 & -2qx_2 & 2x_3 \\ 2qx_2 & 0 & -\frac{1}{2}(1+q)x_1 \\ -2x_3 & \frac{1}{2}(1+q)x_1 & 0 \end{pmatrix}.$

It is of rank 2 hence the index equal 1.

4.4.2 For an arbitrary representation

Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra, and $(\mathbb{V}, \rho, \beta)$ be a representation of $\mathfrak{g}$ where

$$\rho : \mathfrak{g} \to \mathfrak{g}l(\mathbb{V}), \qquad x \mapsto \rho(x) = \rho(x)(v) = x \cdot v.$$

We set

$$\begin{aligned} \mathfrak{g}_v &= \left\{ x \in \mathfrak{g} : x \cdot v = 0, \quad v \in \mathbb{V} \right\}, \\ \text{and } \mathfrak{g} \cdot v &= \left\{ x \cdot v \quad ; \quad x \in \mathfrak{g}, \quad v \in \mathbb{V} \right\}. \end{aligned}$$

The set $\mathfrak{g}_v$ is the stabiliser of v. We say that $v \in \mathbb{V}$ is regular if $\mathfrak{g}_v$ has a minimum dimension.

i.e.

$$\dim \mathfrak{g}_v = \min \{\dim \mathfrak{g}_w, w \in \mathbb{V}\}.$$

Since $\dim \mathfrak{g}_v + \dim \mathfrak{g} \cdot v = \dim \mathfrak{g}$, $v \in \mathbb{V}$ is regular if

$$\dim \mathfrak{g} \cdot v = \max_{w \in \mathbb{V}} \{\dim \mathfrak{g} \cdot w\}.$$

So if we consider the dual representation of $\mathbb{V}$ on $\mathbb{V}^*$ when it exists, we have the following Lemma

Lemma 4.4.3 $\max_{f \in \mathbb{V}^*} \dim \mathfrak{g} \cdot f = \dim \mathfrak{g} - \min \{\dim \mathfrak{g}_f, f \in \mathbb{V}^*\}.$

Therefor we define the index of a Hom-Lie algbra with respect to a given representation

Definition 4.4.4 *The integer*

$$\begin{aligned}\chi(\mathfrak{g}, \rho) &= \dim \mathbb{V} - \max_{f \in \mathbb{V}^*} \{\dim \mathfrak{g}.f\}, \\ &= \dim \mathbb{V} - \dim \mathfrak{g} + \min \{\dim \mathfrak{g}_f,\ f \in \mathbb{V}^*\}.\end{aligned}$$

is called the index of the representation $(\mathbb{V}, \rho, \beta)$ *of the Hom-Lie algebra* $(\mathfrak{g}, [-,-], \alpha)$.

Proposition 4.4.5 *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie the index of* $\mathfrak{g}$, $\chi(\mathfrak{g}, \rho)$ *can be writen as*

$$\begin{aligned}\chi(\mathfrak{g}, \rho) &= \dim \mathbb{V} - \max_{f \in \mathbb{V}^*} \{\dim \mathfrak{g}.f\}, \\ &= \min \{\dim \mathfrak{g}_f\ ;\ f \in \mathfrak{g}^*\}, \\ &= \dim \mathbb{V} - \operatorname{rank}_{\mathbb{k}(\mathbb{V})} (x_i \cdot v_j)_{ij}, \text{ (See Proposision 2.3 [37], p17)},\end{aligned}$$

where $\mathbb{K}(\mathbb{V})$ *is the quotient fields of the symmetric algebras* $S(\mathbb{V})$*.*

Proof. Consider the bilinear form $\mathcal{B}$ with values in $\mathbb{V}$

$$\mathcal{B} = \mathcal{B}(\mathfrak{g}, \mathbb{V}) : \mathfrak{g} \times \mathbb{V} \to \mathbb{V}, \qquad (x, v) \mapsto x \cdot v.$$

Evaluating this form for an arbitrary element $f \in \mathbb{V}^*$ gives a form with values in $\mathbb{K}$ it follows

$\mathcal{B}_f : \mathfrak{g} \times \mathbb{V} \to \mathbb{V} \xrightarrow{f} \mathbb{K}$ and $\mathcal{B}_f(x,v) = f(x \cdot v)$. The kernel (resp. image) of $\mathcal{B}_f$ is $\mathfrak{g}_f$ (resp. $\mathfrak{g} \cdot f$).We have

$$\ker(\mathcal{B}_f) = \mathfrak{g}_f = \{x \in \mathfrak{g} \ ; \ f(x \cdot v) = 0\} \text{ and}$$
$$\mathrm{Im}(\mathcal{B}_f) = \mathfrak{g} \cdot f = \{f(x \cdot v) \ ; \ x \in \mathfrak{g}, v \in \mathbb{V}\}.$$

Hence

$$\chi(\mathfrak{g}, \rho) = \dim \mathbb{V} - \max_{f \in \mathbb{V}^*}(\operatorname{rank} \mathcal{B}_f).$$

Let $n = \dim \mathfrak{g}$ and $m = \dim \mathbb{V}$. Having chosen bases for $\mathfrak{g}$ and $\mathbb{V}$,we may regard $\mathcal{B}$ as $n \times m-$matrix with integer in $\mathbb{V}$, taking $\{x_1, ..., x_n\}$ a basis for $\mathfrak{g}$ and $\{v_1, ..., v_m\}$ a basis for $\mathbb{V}$ then

$$\mathcal{B} = (x_i \cdot v_j)_{ij}, \qquad i = 1, ..., n, \ j = 1, ..., m \text{ and}$$
$$\mathcal{B}_f = (f(x_i \cdot v_j))_{ij} \quad i = 1, ..., n, \ j = 1, ..., m$$

Therefore

$$\begin{aligned}\chi(\mathfrak{g}, \rho) &= \dim \mathbb{V} - \max_{f \in \mathbb{V}^*}\left(\operatorname{rank}(f(x_i \cdot v_j))_{ij}\right), \ i = 1, ..., n, j = 1, ..., m, \\ &= \dim \mathbb{V} - \operatorname{rank}_{\Bbbk(\mathbb{V})}(x_i \cdot v_j)_{ij}, \qquad i = 1, ..., n, j = 1, ..., m.\end{aligned}$$

Hence

$$\chi(\mathfrak{g}, \rho) = \dim \mathbb{V} - \operatorname{rank}(x_i \cdot v_j)_{ij}, \ i = 1, ..., n, j = 1, ..., m.$$

■

4.4.3 Index of twisted Hom-Lie algebras

Let $(\mathfrak{g}, [-,-])$ be a Lie algebra and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra morphism. According to Corollary 4.2.7, the twist $\mathfrak{g}_\alpha = (\mathfrak{g}, [-,-]_\alpha, \alpha)$, where $[x,y]_\alpha = \alpha[x,y]$ is a Hom-Lie algebra. We aim to compare the index of Lie algebra with the index of Hom-Lie algebra obtained by twisting.

Proposition 4.4.6 *Let $\mathfrak{g}_\alpha = (\mathfrak{g}, [-,-]_\alpha, \alpha)$ be a Hom-Lie algebra, and ad to be the adjoint representation. then*

$$\chi(\mathfrak{g}_\alpha) = n - \operatorname{rank}_{\mathbb{k}(\mathbb{V})} (\alpha([e_i, e_j]))_{ij}.$$

Proof. For all $x \in \mathfrak{g}$, ad_x is a $\mathbb{K}-$linear map $\mathfrak{g}$ operate on $\mathfrak{g}^*$as

$$\mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{g}^*, \qquad (x, f) \mapsto x \cdot f.$$

$$\forall y \in \mathfrak{g} : (x \cdot f)(y) = f([x, y]_\alpha),$$

$$\phi_f : \mathfrak{g}\times\mathfrak{g}\to\mathbb{K}, \qquad (x, y) \mapsto \phi_f(x, y) = f([x, y]_\alpha),$$

$$\mathfrak{g}_f = \{x \in \mathfrak{g},\ f([x, y]_\alpha) = 0, \forall y \in \mathfrak{g}\},$$

or

$$\operatorname{Ker} f = \{x \in \mathfrak{g},\ f([x, y]_\alpha) = 0, \forall y \in \mathfrak{g}\}. \tag{4.5}$$

Particular cases : if the algebra is multiplicative then

$$f([x, y]_\alpha) = f(\alpha[x, y]) = (f \circ \alpha)([x, y]).$$

We denote the kernel of the map $(f \circ \alpha)$ by $\mathfrak{g}_{(f\circ\alpha)}$,

$$\mathfrak{g}_{(f\circ\alpha)} = \operatorname{Ker}(f \circ \alpha) = \{x \in \mathfrak{g},\ (f \circ \alpha)([x, y]) = 0, \forall y \in \mathfrak{g}\}, \tag{4.6}$$

and

$$\operatorname{Im}(f \circ \alpha) = \{(f \circ \alpha)([x, y]), \forall x, y \in \mathfrak{g}\}.$$

Applying the rank theorem we have

$$\dim \mathfrak{g} = \dim \ker(f \circ \alpha) + \dim \operatorname{Im}(f \circ \alpha),$$

$$\dim \ker(f \circ \alpha) = \dim \mathfrak{g} - \dim \operatorname{Im}(f \circ \alpha),$$

$$= n - \dim \operatorname{Im}(f \circ \alpha).$$

Other we have

$$\min_{(f\circ\alpha)\in\mathfrak{g}^*} \{\dim \ker(f \circ \alpha)\} = n - \max_{(f\circ\alpha)\in\mathfrak{g}^*} \{\dim \operatorname{Im}(f \circ \alpha)\},$$

We know that

$$\chi(\mathfrak{g}_\alpha) = \min\{\dim\ker(f\circ\alpha), \quad (f\circ\alpha)\in\mathfrak{g}^*\},$$

then

$$\chi(\mathfrak{g}_\alpha) = n - \max\{\dim\operatorname{Im}(f\circ\alpha), \quad (f\circ\alpha)\in\mathfrak{g}^*\}.$$

Let $B = \{e_1,,,e_n\}$ a basis de $\mathfrak{g}$, for all x, y of $\mathfrak{g}$ we have

$$x = \sum_i x_i e_i, \qquad y = \sum_j y_j e_j,$$

then

$$\begin{aligned}
(f\circ\alpha)([x,y]) &= (f\circ\alpha)\left(\left[\sum_i x_i e_i, \sum_j y_j e_j\right]\right) \\
&= (x_1,...,x_n)\,(f\circ\alpha)([e_i,e_j])\begin{pmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{pmatrix} \\
&= X^t A Y; \qquad \text{and } A = ((f\circ\alpha)([e_i,e_j]))_{ij}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\chi(\mathfrak{g}_\alpha) &= n - \max\operatorname{rank}((f\circ\alpha)([e_i,e_j]))_{ij}, \\
&= n - \operatorname{rank}(f(\alpha([e_i,e_j])))_{ij}, \\
&= n - \operatorname{rank}\left(f\left([e_i,e_j]_\alpha\right)\right)_{ij}, \\
&= n - \operatorname{rank}_{\Bbbk(\mathbb{V})}\left([e_i,e_j]_\alpha\right)_{ij}, \\
&= n - \operatorname{rank}_{\Bbbk(\mathbb{V})}(\alpha([e_i,e_j]))_{ij}.
\end{aligned}$$

Then

$$\chi(\mathfrak{g}_\alpha) = n - \operatorname{rank}_{\Bbbk(\mathbb{V})}(\alpha([e_i,e_j]))_{ij}.$$

$\blacksquare$

Theorem 4.4.7 *Let* $(\mathfrak{g}, [-,-])$ *be a Lie algebra, and* $(\mathfrak{g}, [-,-]_\alpha, \alpha)$ *the Hom-Lie algebra then we have* $\chi(\mathfrak{g}_\alpha) \geq \chi(\mathfrak{g})$. *Morever if* f *is a regular vector of* $\mathfrak{g}$ *then it is a regular vector of* $\mathfrak{g}_\alpha$.

Proof. Since

$$\text{rank}((f \circ \alpha)) \leq \min(\text{rank}\, f, \text{rank}\, \alpha),$$

$$\text{rank}((f \circ \alpha)) \leq \text{rank}\, f,$$

and

$$\chi(\mathfrak{g}_\alpha) \geq \chi(\mathfrak{g}).$$

■

Remark 4.4.8 *(Case where* α *is bijective) Let* $(\mathfrak{g}_\alpha, [-,-]_\alpha, \alpha)$ *be a Hom-Lie algebra, if* α *is bijective then* $\chi(\mathfrak{g}_\alpha) = \chi(\mathfrak{g})$. *Indeed* $\text{rank}((f \circ \alpha)) = \text{rank}\, f$. *(*$\alpha$ *bijective,* $\text{Im}(\alpha) = \mathfrak{g}$, *so* $\text{Im}(f \circ \alpha) = f(\mathfrak{g}) = \text{Im}\, f$, *then* $\text{rank}((f \circ \alpha)) = \text{rank}\, f$ *).*

Example 4.4.9 *(Morphism of Lie algebra and index). Let* $(\mathfrak{g}, [-,-])$ *be a Lie algebra, and* $(\mathfrak{g}, [-,-]_\alpha, \alpha)$ *a Hom-Lie algebra,* $\{x_1, x_2, ..., x_n\}$ *a fixed bases of* $\mathfrak{g}$. *We search the morphisms corresponding to this algebra and to calculate the index in this cas. The twisting principle leads for the dimensional affine Lie algebras defined as* $\mathfrak{g}_2^1 : [x_1, x_2] = x_2$ *to two Hom-Lie algebras : the first is the abelian Hom-Lie algebras* $\mathfrak{g}_{2,\alpha,1}^1 : [x_1, x_2]_\alpha = 0$,*and it is given by the homomorphism* α *defined, with respect to the previous basis by the following matrix* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

The second are $\mathfrak{g}_{2,\alpha,2}^2$ *defined as* $\mathfrak{g}_{2,\alpha,2}^2 : [x_1, x_2]_\alpha = dx_2$,

the homomorphism α *is given by the following matrix* $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$.

Where

$$\alpha(x_1) = ax_1 + bx_2, \quad \alpha(x_2) = cx_1 + dx_2.$$

The $3-$*dimensional Lie algebras defined as* $\mathfrak{g}_3^1 : [x_1, x_2] = x_3$ *leads to the Hom-Lie algebras defined as :* $\mathfrak{g}_{3,\alpha,3}^1 : [x_1, x_2]_\alpha = (a_1 b_2 - b_1 b_2) x_3$, *and it is given by the homomorphism* α

defined, with respect to the previous basis by the following matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & a_1b_2 - b_1a_2 \end{pmatrix}$

The Lie algebra $\mathfrak{g}_3^2 : [x_1, x_2] = x_2, \quad [x_1, x_3] = \beta x_3, \quad \beta \neq 0$ *leads to four Hom-Lie algebras defined as :*

1. $\mathfrak{g}_{3,\alpha,1}^2 : [x_1, x_2]_\alpha = 0, \qquad [x_1, x_3]_\alpha = 0,$ *this is an abelian Hom-Lie albebra, the homomorphism* α *is given by the following matrix* $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2. $\mathfrak{g}_{3,\alpha,2}^2 : [x_1, x_2]_\alpha = b_2x_3, \qquad [x_1, x_3]_\alpha = 0,$ *the homomorphism* α *is given by the following matrix* $\begin{pmatrix} \frac{1}{\beta} & b_1 & c_1 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}$

3. $\mathfrak{g}_{3,\alpha,3}^2 : [x_1, x_2]_\alpha = 0, \qquad [x_1, x_3]_\alpha = \beta b_3 x_2,$ *the homomorphism* α *is given by the following matrix* $\begin{pmatrix} \beta & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & 0 \end{pmatrix}$

4. $\mathfrak{g}_{3,\alpha,4}^2 : [x_1, x_2]_\alpha = b_2x_2, \qquad [x_1, x_3]_\alpha = \beta c_3 x_3,$ *the homomorphism* α *is given by the following matrix* $\begin{pmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$

The Lie algebra $\mathfrak{g}_3^3 : [x_1, x_2] = x_2 + x_3, \quad [x_1, x_3] = x_3$ *leads to tow Hom-Lie algebras defined as.*

1. $\mathfrak{g}_{3,\alpha,1}^3 : [x_1, x_2]_\alpha = 0, \quad [x_1, x_3]_\alpha = 0,$ *this is an abelian Hom-Lie albebra, the homomorphism* α *is given by the following matrix* $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2. $\mathfrak{g}^3_{3,\alpha,2} : [x_1,x_2]_\alpha = b_2x_2 + (b_2+c_2)x_3, \quad [x_1,x_3]_\alpha = b_2x_2$, *the homomorphism* α *is given by the following matrix* $\begin{pmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & b_2 \end{pmatrix}$

The Lie algebra $\mathfrak{g}^3_4 : [x_1,x_2] = 2x_2, \quad [x_1,x_3] = -2x_3, \quad [x_1,x_3] = x_1$ *leads to tow Hom-Lie algebras defined as :*

1. $\mathfrak{g}^3_{4,\alpha,1} : [x_1,x_2]_\alpha = \frac{2}{b_3}x_2, \quad [x_1,x_3]_\alpha = -2b_3x_3, \quad [x_2,x_3]_\alpha = -x_1$, *the homomorphism* α *is given by the following matrix* $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \frac{1}{b_3} \\ 0 & b_3 & 0 \end{pmatrix}$

2. $\mathfrak{g}^3_{4,\alpha,2} : [x_1,x_2]_\alpha = \frac{2}{C_3}x_2, \quad [x_1,x_3]_\alpha = -2c_3x_3, \quad [x_2,x_3]_\alpha = x_1$, *the homomorphism* α *is given by the following matrix* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c_3} & 0 \\ 0 & 0 & c_3 \end{pmatrix}$

Where

$$\alpha(x_1) = a_1x_1 + b_1x_2 + c_1x_3,$$

$$\alpha(x_2) = a_2x_1 + b_2x_2 + c_2x_3,$$

$$\alpha(x_3) = a_3x_1 + b_3x_2 + c_3x_3.$$

Evaluation of the index

In dimension 2 we have

$\chi\left(\mathfrak{g}^1_{2,\alpha,1}\right) = 2,$

$\chi\left(\mathfrak{g}^1_{2,\alpha,2}\right) = 0.$

In dimendion 3 we have

1. $\chi\left(\mathfrak{g}^1_{3,\alpha,1}\right) = \begin{cases} 1 \ \textit{if}\ a_1b_2 - b_1a_2 \neq 0, \\ 3 \ \textit{else}. \end{cases}$

2. $\chi\left(\mathfrak{g}_{3,\alpha,1}^{2}\right)=0.$

$$\chi\left(\mathfrak{g}_{3,\alpha,2}^{2}\right)=\begin{cases}1 & \text{if } c_2\neq 0,\\ 3 & \text{else.}\end{cases}$$

$$\chi\left(\mathfrak{g}_{3,\alpha,3}^{2}\right)=\begin{cases}1 & \text{if } b_3\neq 0,\\ 3 & \text{else.}\end{cases}$$

$$\chi\left(\mathfrak{g}_{3,\alpha,4}^{2}\right)=\begin{cases}1 & \text{if } b_2, c_3\neq 0,\\ 3 & \text{else.}\end{cases}$$

3. $\chi\left(\mathfrak{g}_{3,\alpha,1}^{3}\right)=0.$

$$\chi\left(\mathfrak{g}_{3,\alpha,2}^{3}\right)=\begin{cases}1 & \text{if } b_2, c_2\neq 0,\\ 3 & \text{else.}\end{cases}$$

4. $\chi\left(\mathfrak{g}_{3,\alpha,1}^{4}\right)=1 \qquad \text{with } b_3\neq 0,$

$\chi\left(\mathfrak{g}_{3,\alpha,2}^{4}\right)=1 \qquad \text{with } c_3\neq 0.$

4.5 Index of Multiplicative Simple Hom-Lie algebras

Definition 4.5.1 *Let* $(\mathfrak{g},[-,-],\alpha)$ *be a Hom-Lie algebra. A subspace* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is called a Hom-Lie subalgebra of* $(\mathfrak{g},[-,-],\alpha)$ *if* $\alpha(\mathfrak{h})\subseteq\mathfrak{h}$ *and* $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$. *In particular, a Hom-Lie subalgebra* $\mathfrak{h}$ *is said to be an ideal of* $(\mathfrak{g},[-,-],\alpha)$ *if* $[\mathfrak{h},\mathfrak{g}]\subseteq\mathfrak{h}$. $\mathfrak{g}$ *is called an abelian Hom-Lie algebra if* $[x,y]=0$ *for any* $x,y\in\mathfrak{g}$.

Definition 4.5.2

$$C(\mathfrak{g})=\{x\in\mathfrak{g}\mid[x,y]=0,[\alpha(x),y]=0,\forall y\in\mathfrak{g}\}$$

is called the center of $(\mathfrak{g},[-,-],\alpha)$.

Proposition 4.5.3 *Let* $(\mathfrak{g},[-,-],\alpha)$ *be a multiplicative Hom-Lie algebra, then* $(\ker(\alpha),[-,-],\alpha)$ *is an ideal.*

Proof. Obviously $\alpha(x) = 0 \in \ker(\alpha)$ for any $x \in \ker(\alpha)$. Since $\alpha([x,y]) = [\alpha(x), \alpha(y)] = [0,y] = 0$ for any $x \in \ker(\alpha)$ and $y \in \mathfrak{g}$, we get $[x,y] \in \ker(\alpha)$. Therefore $(\ker(\alpha), [-,-], \alpha)$ is an edial of $(\mathfrak{g}, [-,-], \alpha)$. ■

Definition 4.5.4 *Let $(\mathfrak{g}, [-,-], \alpha)\,(\alpha \neq 0)$ be a Hom-Lie algebra. $(\mathfrak{g}, [-,-], \alpha)$ is called simple Hom-Lie algebra if $(\mathfrak{g}, [-,-], \alpha)$ has no propre ideals and is not abelian. $(\mathfrak{g}, [-,-], \alpha)$ is called semisimple Hom-Lie algebra if $\mathfrak{g}$ is a direct sum of certain ideals.*

Let $(\mathfrak{g}, [-,-], \alpha)$ be a multiplicative Hom-Lie algebra. By Proposition 4.5.3 α must be a monomorphism, thus α is an automorphism of $(\mathfrak{g}, [-,-], \alpha)$.

Definition 4.5.5 *Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra. the Lie algebra $(\mathfrak{g}, [-,-]')$ is called the induced Lie algebra of $(\mathfrak{g}, [-,-], \alpha)$ if $[x,y] = \alpha\left([x,y]'\right) = [\alpha(x), \alpha(y)]', \forall x, y \in \mathfrak{g}$.*

Proposition 4.5.6 *Let $(\mathfrak{g}, [-,-], \alpha)$ be a multiplicative simple Hom-Lie algebra, define $[x,y]' = \alpha^{-1}([x,y])\,\forall x, y \in \mathfrak{g}$, then $(\mathfrak{g}, [-,-]')$is a Lie algebra and α is olso a Lie algebra automorphism.*

Theorem 4.5.7 *The induced Lie algebra of the multiplicative simple Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$ is semisimple and can be decomposed into direct sum of isomorphic simple ideals, in addition α acts simply transitively on simple ideals of the induced Lie algebra.*

Theorem 4.5.8 *The index of a multiplicative simple Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$ is the same as of the induced Lie algebra of the multiplicative simple Hom-Lie algebra $\left(\mathfrak{g}, \alpha^{-1}[-,-]\right)$.*

Proof. By Remark 4.4.8. ■

Hence, we have the following Proposition :

Proposition 4.5.9 *The index of a multiplicative simple Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$ is upper than or equal to 0.*

Proof. Since a Simple Lie algebras is never Frobenius, then the index is upper than or equal 0. ■

4.6 Index of semidirect products of Hom-Lie Algebras

In this section we introduce the adjoint and coadjoint representation of semi-direct product of the Hom-Lie algebra $\mathfrak{g} \ltimes_\rho \mathbb{V}$.

Proposition 4.6.1 *[4] Let* $(\mathfrak{g}, [-,-]_\mathfrak{g}, \alpha)$ *be a Hom-Lie algebra and* $(\mathbb{V}, \rho, \beta)$ *be a representation of* $\mathfrak{g}$*. The direct summand* $\mathfrak{g} \oplus \mathbb{V}$ *with a bracket defined by*

$$[(x+u), (y+v)] = ([x,y]_\mathfrak{g}, \rho(x)(v) - \rho(y)(u)) \quad \forall x, y \in \mathfrak{g}, \quad \forall u, w \in \mathbb{V}.$$

And the twisted map $\gamma : \mathfrak{g} \oplus \mathbb{V} \to \mathfrak{g} \oplus \mathbb{V}$ *defined by*

$$\gamma(x+w) = \alpha(x) + \beta(u) \quad \forall x \in \mathfrak{g} \quad \forall u \in \mathbb{V}.$$

is a Hom-Lie algebra.

We call the direct sum $\mathfrak{g} \oplus \mathbb{V}$ semi-direct product of $\mathfrak{g}$ and $\mathbb{V}$, it is denoted by $\mathfrak{g} \ltimes_\rho \mathbb{V}$. We identify the dual space $(\mathfrak{g} \ltimes_\rho \mathbb{V})^*$ with $\mathfrak{g}^* \oplus \mathbb{V}^*$.

Since $\mathfrak{g} \ltimes_\rho \mathbb{V}$ is Hom-Lie algebra, the Hom-Jacobi condition on $x, y, z \in \mathfrak{g}$ and $u, v, w \in \mathbb{V}$ is

$$\begin{aligned}
&\circlearrowleft_{(x,u),(y,v),(z,w)} [\gamma(x+u), [y+v, z+w]], \\
&= \circlearrowleft_{(x,u),(y,v),(z,w)} [\alpha(x) + \beta(u), [y,z]_\mathfrak{g} + \rho(y)(w) - \rho(z)(v)]. \\
&= 0.
\end{aligned}$$

We can determine a representation of semi-direct product of the Hom-Lie algebra $\mathfrak{g} \ltimes_\rho \mathbb{V}$. We call this representation the adjoint representation of semi-direct product of the Hom-Lie algebra $\mathfrak{g} \ltimes_\rho \mathbb{V}$, and it satisfies the condition

$$ad([x+u, y+v]) \circ \gamma = ad(\gamma(x+u)) \circ ad(y+v) - ad(\gamma(y+v)) \circ ad(x+u) \tag{4.7}$$

In the following, we explore the coadjoint representations of semi-direct product of the Hom-Lie algebra $\mathfrak{g} \ltimes_\rho \mathbb{V}$.

4.6.1 The coadjoint representation

let $\mathfrak{q} = \mathfrak{g} \ltimes_\rho \mathbb{V}$ we consider $\mathfrak{q}^* = \mathfrak{g}^* \oplus \mathbb{V}^*$, the dual space of q. An element of q^* is denoted by $\eta = (g, f)$; $\forall (x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}$. We set $ad^* : \mathfrak{q} \to gl(\mathfrak{q}^*)$ defined by $ad^*(x+u)(\eta) = -\eta \circ ad(x+u)$ and $\gamma^* : \mathfrak{q}^* \to \mathfrak{q}^*$ an even homomorphism defined by $\gamma^*(\eta) = \eta \circ \gamma$.

We compute the right hand side of the identity 4.7.

$(ad^*(\gamma(x+u)) \circ ad^*(y+v) - ad^*(\gamma(y+v)) \circ ad^*(y+v))(\eta)(z+w)$

$= (ad^*(\gamma(x+u))(ad^*(y+v)(\eta)) - ad^*(\gamma(y+v))(ad^*(y+v)(\eta)))(z+w)$

$= -ad^*(y+v)(\eta)(ad(\gamma(x+u))(z+w)) + ad^*(x+u)ad(\eta)(\gamma(x+u))(z+w))$

$= \eta(ad(y+v)ad(\gamma(x+u))(z+w)) - \eta(ad(x+u)ad(\gamma(x+u))(z+w))$

$= \eta(ad(y+v)ad(\gamma(x+u)) - ad(x+u)ad(\gamma(x+u))(z+w)).$

On the other hand

$$\begin{aligned} ((ad^*([x+u, y+v])\gamma^*)(\eta))(z+w) &= (ad^*([x+u, y+v])(\eta \circ \gamma)(z+w) \\ &= -\eta \circ \gamma (ad([x+u, y+v])(z+w)). \end{aligned}$$

Thus 4.7 is satisfies. We call the representation ad^* the coadjoint representation.

We obtain the following corollary

Corollary 4.6.2 *Let $(\mathfrak{q}, [-,-], \gamma)$ be a Hom-Lie algebra and $(\mathfrak{q}, ad, \gamma)$ be the adjoint representation of $\mathfrak{q}$. The triple $(\mathfrak{q}, ad^*, \gamma^*)$ defines a representation of $(\mathfrak{q}, [-,-]\gamma)$ if and only if*

$$\gamma \circ ad([x+u, y+v]) = ad(x+u) \circ ad(\gamma(y+v)) - ad(y+v) \circ ad(\gamma(x+u))$$

We call the representation ad^* the coadjoint representation and it given by

$$\left(ad^*_{\mathfrak{q}}(x, v)\right)(g, f) = \left(ad^*_{\mathfrak{g}}(x)(g) - v * f, x \cdot f\right),$$

such that

$$\mathfrak{g} \times \mathbb{V}^* \to \mathbb{V}^*$$

$$(x, f) \mapsto x \cdot f$$

and

$$\mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$$

$$(v, f) \mapsto v * f, \qquad \forall x \in \mathfrak{g} : (v * f)\, x = f\,(xv)\,.$$

4.6.2 The stabilizer of an arbitrary point of $\mathfrak{q}^*$

Let $\mathcal{K}_g$ denote the Kirillov form on $\mathfrak{g}$, i.e. $\forall\,(x_1, x_2) \in \mathfrak{g}$: $\mathcal{K}_g\,(x_1, x_2) = g\,[x_1, x_2]$ then $\ker(\mathcal{K}_g) = \mathfrak{g}_g$, the stabiliser of g. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then $\mathcal{K}_g|\mathfrak{h}$ can also be regarded as the Kirillov form associated with $g|\mathfrak{h} \in \mathfrak{h}^*$.

Proposition 4.6.3 *For any $\eta = (g, f) \in q$, we have*

$$\begin{aligned} \mathfrak{q}_\eta &= \left\{(x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}, ad^*_{\mathfrak{q}}(x, v)(g, f) = 0\right\} \\ &= \left\{(x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}, \left(ad^*_{\mathfrak{g}}\,(x)\,(g) - v * f, x \cdot f\right) = 0\right\} \\ &= \left\{(x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}, ad^*_{\mathfrak{g}}\,(x)\,(g) = v * f \;\; et\; x \cdot f = 0\right\}. \end{aligned}$$

such that $ad^*_{\mathfrak{g}}\,(x)\,(g) = g\,[x, y] = v * f : y \in \mathfrak{g}$

and $x \cdot f = 0 \Rightarrow x \in \ker\left(\mathcal{K}_g|\mathfrak{g}_f\right),$

such that $\mathcal{K}_g(x, f) = x \cdot f$ then

$$\mathfrak{q}_\eta = \left\{(x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}, ad^*_{\mathfrak{g}}\,(x)\,(g) = v * f \text{ et } x \in \ker\left(\mathcal{K}_g|\mathfrak{g}_f\right)\right\}.$$

We not by $\mathfrak{g}_f$ is the kernel of $f\,([x, y])$ so $(\mathfrak{g}_f)^\perp = \mathfrak{g}{\cdot}f$ such that the space

$$\begin{aligned} \{v \in \mathbb{V}, v * f = 0\} &= \{v \in \mathbb{V}, f\,(xv) = 0, \forall x \in \mathfrak{g}\} \\ &= (\mathfrak{g}{\cdot}f)^\perp \\ &= \ker \mathcal{B}_f. \end{aligned}$$

it follows that $\mathfrak{q}_\eta$ is the direct sum of the space $(\mathfrak{g}{\cdot}f)^\perp$ and the space $\ker\left(\mathcal{K}_g|\mathfrak{g}_f\right)$, so $\mathfrak{q}_\eta = \ker\left(\mathcal{K}_g|\mathfrak{g}_f\right) \ltimes \ker \mathcal{B}_f$.

Lemma 4.6.4 *Let $(\mathfrak{q}, [-, -], \gamma)$ be a Hom-Lie algebra,*

$$\chi_{\mathfrak{q}} = \chi_{\mathfrak{g}} + \chi_{(\mathfrak{g}, \rho)}.$$

Remark 4.6.5 *Let Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie Algebra and* $ad : \mathfrak{g} \to g\ell(\mathfrak{g})$ *be the adjointe representation. We means by* $\mathfrak{g} \ltimes_{ad} \mathfrak{g}$ *the Hom-Lie algebra of semi-direct product associated to the adjoint representation. then*

$$\chi(\mathfrak{q}) = 2\chi(\mathfrak{g}).$$

Conclusion

The index is an important concept in the representation theory and invariant theory, it was introduced by Dixmier in [9]. The theory of index for Lie algebras has applications in invariant theory of invariants, deformations and quantum groups.

In this thesis we study the index for some classes of Lie algebras and we generalize the theory to Hom-Lie algebras. The first part deals with nilpotent Lie algebras. We study the classes of filiform and quasi-filiform Lie algebras. First, we review the nilpotent and filiform Lie algebras theories. Then we consider the classification up to dimension 8 and compute for each filiform Lie algebra its index and the set of all regular vectors. We also compute the index of graded quasi-filiform Lie algebras, and provide the corresponding regular vectors. At last we compute the index of Lie algebras whose nilradical is Q_{2n}. In the second part dealing with Hom-Lie algebras. A Hom-Lie algebra structure is given by a skewsymmetric bracket on a vector space $\mathfrak{g}$, the structure is twisted by a homomorphism α which modified The Jacobi identity. This theory has been extended by Larsson and Silvestrov to the quasi-Lie algebras in [27]. A complete theory of the index exist for Lie algebra, however, we have not this theory in the case of Hom Lie algebras. The first main purpose of this part is to introduce the index of semidirect products of Hom-Lie algebras. We study the index of Hom-Lie algebras. We introduce the notion of the index of Hom-Lie algebras in the case of coadjoint and an arbitrary representation. Moreover, we compare the index of Lie algebra with the index of Hom-Lie algebra obtained by twisting. Finally, we give the index of semidirect products of Hom-Lie algebras, but before we explore the coadjoint representations of semidirect products of Hom-Lie algebras.

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ملخص

الغرض من هذه الرسالة هو دراسة مؤشر لجبر لي وتعميمه جبر هوم لي ، ندرس الفئة المضادة إلى الجبر شبه بسيط وهي فئة جبر لي عديمة القوة.نهتم خصوصا بجبر لي فيليفورم وشبه فيليفورم .في الجزء الثاني من العمل أنشأنا نظرية المؤشر في حالة جبر هوم لي. ندرس مؤشر جبر هوم لي البسيط المضاعف ومؤشر الضرب شبه المباشر لجبر هوم لي. وعلاوة على ذلك نتبع بتشوه وتحريف تطور المؤشر .وتقدم أيضا الكثير من الأمثلة.

Résumé

L’objet de cette thèse est l’étude de l’indice pour les algèbres de Lie et leur généralisation, les algèbres Hom-Lie. On étudie la classe opposée aux algèbres semi-simples qui est la classe des algèbres de Lie nilpotentes, on s’intéresse spécialement aux algèbres de Lie filiformes et quasi-filiformes. Dans la deuxième partie du travail, on établit la théorie de l’indice dans le cas des algèbres Hom-Lie. On étudie l’indice des algèbres Hom-Lie multiplicatives simples ainsi que l’indice du produit semi-direct d’algèbres Hom-Lie. Par ailleurs, on suit par déformation et par twist de Yau l’évolution de l’indice. De nombreux exemple sont aussi proposés.

Abstract

The objective of this thesis is to study index for some classes of Lie algebras and to generalize the theory to Hom-Lie algebras. The first part deals with nilpotent Lie algebras. We study the classes of filiform and quasi-filiform Lie algebras. In the second part dealing with Hom-Lie algebras, we generalize the theory. We introduce and study the notion of index for Hom-Lie algebras. We study the index of multiplicative simple Hom-Lie algebras, provide some key constructions and discuss index of semi-direct products of Hom-Lie algebras. A lot of examples are given.