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Some Transmission Problems of Waves and Viscoelastic Wave Equations With Delay and an Evolutionary Problem

Presented by

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Abstract

This thesis is devoted to the study of stability and decay rates of solutions for some wave transmission problems and viscoelastic wave equations with delay, in some cases the delay is a time function. And the existence, uniqueness of weak solution to a nonlinear history-dependent boundary value problem, and the same goal for an evolution of a viscoelastic plate in frictional contact with foundation.

The first part of this thesis is composed of three chapters. In Chapter 2, we consider a transmission system with a delay. We show the well-posedness as well as the exponential stability of the solution depending on the weight of linear damping and the weight of the delay term. In Chapter 3, we proved the well-possedness of a system with delay and memory. In Chapter 4, we prove a decay of a transmission problem with memory and delay, but in this case the delay is considered as a time-varying function.

The second part of this thesis is devoted to the study of mathematical models of contact. More precisely, in chapter 5, we introduce a mathematical model that describes the evolution of a viscoelastic plate in frictional contact with foundation, we derive the variational inequality for the displacement field, then we establishe the existence of a unique weak solution to the model.

Contents

Introduction

Several authors have studied transmission problems in the wave and viscoelastic wave equations and in thermoelasticity. Different stabilization results have been established. The authores in [28] studied a transmission problem in thermoelasticity where they proved an asymptotic behavior of one dimensional bodies composed of two different types of materials, one of them is of a thermoelastic type, the other has no thermal effect. They showed that the localized dissipation due to thermal effect is strong enough to produce exponential decay to zero of the total energy of the system, provided that the kernel $g(t)$ is positive and decays exponentially to zero. That is, it satisfies

$$g'(t) \leq -cg(t),$$

for all $t \geq 0$.
On of the results in this direction was also established by Dautray and Lions in [34]. In their work, the authors discussed the linear transmission problem for hyperbolic equations and proved existence and regularity of solution using classical methods such as Fourier series, and the variational formulations.

In general the stability of dissipative wave equation is an active area of research and many interesting publications have been appeared in the last three decades. For instance, we mention the work of Zuazua [107] where a uniform decay rate of the solution was obtained for a large class of nonlinear wave equation with frictional damping acting in the whole domain.

Zuazua and Freistas in [109] considered the system

$$\begin{cases} u_{tt} + 2\epsilon a(x)u_t = u_{xx}, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \varphi(x); u_t(x,0) = \psi(x), \end{cases} \tag{0.0.1}$$

where ϵ is a positive parameter, $a \in L^\infty(0,1)$, and initial condition (φ, ψ) are taken to lie in the energy space $X = H_0^1(0,1) \times L^2(0,1)$ endowed with the usual inner product defined by

$$\langle (f,g),(u,v)\rangle_X = \int_0^1 (f'\overline{u'} + g\overline{v})dx. \tag{0.0.2}$$

The authors gave sufficient conditions for the solution of (0.0.1) to be globally asymptotically stable in the space X for small values of ϵ. That is, $(u(.,t), u_t(.,t))$ converges to zero strongly in the topology of the space X, as t goes to ∞.
Similar results were obtained, for example, in [65] where stability is proved for the following linear problem

$$\begin{cases} u_{tt} - \Delta u + h(x)u_t + k(x)u = 0, & \text{in } (0,+\infty)\times\Omega, \\ u(t,x) = 0, & \text{in } (0,+\infty)\times\partial\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \Omega, \end{cases} \tag{0.0.3}$$

where $h : \Omega \longrightarrow \mathbb{R}$ may have negative values, but in a set which is small if compared to the set where it is positive. In this way the authors generalized the previous results in the 1-dimensional case by Freista-Zuazua [109].

Once the stability of the wave equations is obtained, the natural question that arises is about the rate of decay of the solution (the speed of the convergence of the solution to the steady state). This was the purpose of some investigations as in [72], where the authors were concerned with the existence and uniform decay rates of solutions of the wave equation with a source term and a nonlinear boundary damping. They considered the system

$$\begin{cases} u_{tt} - \Delta u = |u|^\rho u, & \text{in } \Omega\times(0,+\infty), \\ u = 0, & \text{on } \Gamma_0\times(0,+\infty), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta(u_t) = 0, & \text{on } \Gamma_1\times(0,+\infty), \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & \text{on } \Omega, \end{cases} \tag{0.0.4}$$

where Ω is a bounded star-sharped domain of $\mathbb{R}^n$, $n \geq 1$, with $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \neq \emptyset$; Γ_0 and Γ_1 are closed an disjoint and ν represents the unit outward normal to Γ. The authors, in this work obtained an explicit decay estimate of the energy depending on the damping term $\beta(u_t)$ and on the size of the initial data. To obtain thier results some growth assumption of the function β near the origin were assumed. The linear wave equation subject to nonlinear boundary feedback has been widely studied. For instance in (0.0.4) when $\beta(s) = s^p$, for some $p \geq 1$ and in the absence of the source term $|u|^\rho u$ Zuazua in [108] proved that the energy decays exponentially if $p = 1$ and polynomially if $p > 1$. In the later case he proved that the energy

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx,$$

decays with the following rate

$$E(t) \leq \frac{C}{(1+t)^{\frac{2}{p-1}}}, \qquad \forall t \geq 0,$$

for some positive constants C. When no growth assumption at the origin is imposed on the function β, Laseicka and Tataru in [62] studied the nonlinear wave equation subject to a nonlinear feedback acting on Γ_1. Considering their work, they were the first to prove that the energy decays to zero as fast as the solution of some associeted ODE and without assuming that the feedback has a polynomial growth in zero. More precisely, they showed that the energy of the solution $y(t)$ associeted to thier problem defined, for a nonlinear functions f_i, by

$$\mathcal{E}(t) = \frac{1}{2}(|\nabla y(t)|^2_{L_{2(\Omega)}} + |y(t)|^2_{L_{2(\Omega)}} + \int_{\Gamma_1} F_1(y)d\Gamma_1 + \int_\Omega F_0(y)d\Omega,$$

where

$$F_i(s) = \int_0^s f_i(t)dt, \quad i = 0, 1,$$

satisfies

$$\mathcal{E}(t) \leq S\left(\frac{t}{T_0} - 1\right)\mathcal{E}(0), \quad \forall t \geq T_0 > 0,$$

where $S(t)$ is the solution of the following ODE:

$$S'(t) + q(S(t)) = 0,$$

and q is a strictly increasing function which depends on the feedback β. See also [71] for a related work. Messaoudi and Mustafa in [81] considered the system

$$\begin{cases} u_{tt} - \Delta u + \alpha(t)g(u_t) = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \tag{0.0.5}$$

where α, g are specific functions. The authors investigated (0.0.5), in which the damping considered is modeled by a time dependent coefficient $\alpha(t)$, and they established an explicit and general decay result, depending on g and α.

It happens frequently in applications where the domain is occupied by several materials, whose elastic properties are different, joined together over the whole surface as in the work [8] where the authors considered the wave propagation over a domain consisting of two different types of materials in a transmission (or diffraction) problem. From the mathematical point of view, transmission problem for wave propagation consists of hyperbolic equation for which the corresponding elliptic operator has discontinuous coefficients.
Existence and, regularity, as well as the exact controllability for the transmission problem for the pure wave equation was also investigated in [66].

Subsequently Datko et al in [32], treated the following one dimentional problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, & 0 < x < 1, t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -ku_t(1,t-\tau), & t > 0, \end{cases} \tag{0.0.6}$$

which models the vibration of a string clamped at one end and the other end is free. Here $u(x,t)$ is the displacement of the string and a and k are positive constants and $\tau > 0$ is the time delay. Here the string is controlled by a boundary control force with a delay at the free end. They showed that, if the positive constants a and k satisfy

$$k\frac{e^{2a}+1}{e^{2a}-1} < 1, \tag{0.0.7}$$

then the delayed feedback system (0.0.6) is stable for all sufficiently small delays. On the other hand if

$$k\frac{e^{2a}+1}{e^{2a}-1} > 1, \tag{0.0.8}$$

then there exist a dense open set D in $(0,+\infty)$ such that for each $\tau \in D$, system (0.0.6) admits exponentially unstable solutions.
The transmission problem for viscoelastic wave equation was investigated by Mũnoz Rivera and Oquendo [82] where they proved the exponential decay of the solution using regularity result of Volterra's integral equations and regularizing properties of the viscosity.

Marzocchi, Rivera and Naso in [75] considered the system

$$\begin{cases} u_{tt} - au_{xx} + m\theta_x + f(u) & = h_1, \quad \text{in} \quad \Omega\times]0,+\infty[, \\ \theta_t - k\theta_{xx} + mu_{xt} & = h_2, \quad \text{in} \quad \Omega\times]0,+\infty[, \\ v_{tt} - bv_{xx} & = h_3, \quad \text{in} \quad]L_1, L_2[\times]0,\infty[, \end{cases} \tag{0.0.9}$$

where a, b, k and m are positive constants, $h_i : \Omega \to \mathbb{R}(i = 1, 2)$ and $h_3 : \mathbb{R} \to \mathbb{R}$ are nonlinear functions with some specified properties. The system is subjected to the following boundary conditions

$$\begin{aligned} & u(0,t) = u(L_3,t) = \theta(0,t) = \theta(L_3,t), \\ & u(L_i,t) = v(L_i,t), au_x(L_i,t) - m\theta(L_i,t) = bv_x(L_i,t), \quad (i = 1,2), \\ & \theta_x(L_i,t) = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1,2), \end{aligned} \tag{0.0.10}$$

and initial conditions

$$\begin{aligned} & u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ & \theta(x,0) = \theta_0(x), \qquad\qquad\qquad\qquad\quad x \in \Omega, \\ & v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in]L_1, L_2[, \end{aligned} \tag{0.0.11}$$

and f satisfied

$$sf(s) \geqslant 0 \qquad \forall s \in \mathbb{R}.$$

The authors have proved that in the linear homogenous case ($f = 0, h_i = 0, i = 1, 2$), the solution of the above system tends to zero with an exponential rate, as time goes to infinity. In the nonlinear case, this property is replaced by the existence of an absorbing set in the space of solutions provided that μ is sufficiently small, where f is supposed Lipschitz function and μ is its Lipschitz constant.

Recently, the stability of solutions in the wave and viscoelastic wave equations with delay became an active area of research. Nicaise and Pignoti in [84] investigated the effect of the time delay in boundary or internal stabilization of the wave equation in domains of $\mathbb{R}^n$. They gave some stability results under the condition that the weight of the damping term without delay is greater than the wight of the damping with delay. On the other hand, they showed that if this later condition is not satisfied, then there exist some delays for which the system is unstable. In a certain sense, their sufficient condition is also necessary in order to have a general stability result. More precisely, they considered the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & \text{in} \quad \Omega \times (0,+\infty), \\ u(x,t) = 0, & \text{on} \quad \Gamma_D \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau), & \text{on} \quad \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0(x), \quad \text{and} \quad u_t(x,0) = u_1(x), & \text{in} \quad \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in} \quad \Gamma_N \times (0,\tau), \end{cases} \tag{0.0.12}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set with a boundary Γ of classe C^2 and $\Gamma = \Gamma_D \cup \Gamma_N$ with $\Gamma = \overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$ and $\Gamma_D \neq \emptyset$. Moreover, $\tau > 0$ is the time delay, μ_1 and μ_2 are positive constants. Under the assumption $\mu_2 < \mu_1$ they obtained a stability result in a general space dimension by using a suitable energy-type estimates. In the other cases, if $\mu_2 \geq \mu_1$ they showed that there exists a sequence of arbitrary small delays so that instabilities could occur.

The asymptotic behavior for a coupled system of wave equations was studied by Rapposo and Bastos [8] by the same method used in [66]. The authors, considered, for k_1, k_2 and α being positive constants and $0 < L_0 < L$, the system

$$\begin{cases} u_{tt}(x,t) - k_1 u_{xx}(x,t) + \alpha u_t(x,t) = 0, & x \in (0,L_0), t > 0, \\ v_{tt}(x,t) - k_2 v_{xx}(x,t) = 0, & x \in (L_0,L), t > 0, \end{cases} \tag{0.0.13}$$

satisfying the boundary conditions

$$u(0,t) = v(L,t) = 0, \quad t > 0, \tag{0.0.14}$$

the transmission conditions

$$u(L_0,t) = v(L_0,t), \quad k_1 u_x(L_0,t) = k_2 v_x(L_0,t), \quad t > 0, \tag{0.0.15}$$

and te initial conditions

$$\begin{aligned} &u(x,0) = u^0(x), \quad u_t(x,0) = u^1, \quad x \in (0,L_0), \\ &v(x,0) = v^0(x), \quad v_t(x,0) = v^1, \quad x \in (L_0,L)). \end{aligned} \tag{0.0.16}$$

They investigated the asymptotic properties of the above system. The main result of their work was a theorem in which they showed that the solution of the transmission problem decays exponentially to zero as time goes to infinity, no matter how large is the difference $L - L_0$. The approach used consists of choosing appropriate multipliers to build a Lyapunov functional for the system. The energy associated to the equations (0.0.13)-(0.0.16) is

$$E(t) = E_1(t) + E_2(t)$$

with

$$E_1(t) = \frac{1}{2}\int_0^{L_0} [|u_t|^2 + k_1|u_x|^2]dx,$$

and

$$E_2(t) = \frac{1}{2}\int_{L_0}^{L} [|v_t|^2 + k_2|v_x|^2]dx.$$

They proved that there exist two positive constants C and c such that:

$$E(t) = E_1(t) + E_2(t) \leq CE(0)e^{-ct}.$$

Following their first work [84], Nicaise and Pignoti [85] studied the wave equation with Dirichlet boundary conditions on one part of the boundary of a domain Ω and dissipative boundary conditions with delay on the other part of the boundary. Namely, they considered the problem

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0, & \text{on} \quad \Gamma_0 \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu}(t) + \displaystyle\int_{\tau_1}^{\tau_2} \mu(s) u_t(t-s)ds + \mu_0 u_t = 0, & \text{on} \quad \Gamma_1 \times (0, = \infty), \\ u(x, 0) = u_0(x), \quad \text{and} \quad u_t(x, 0) = u_1(x), & \text{in} \quad \Omega, \\ u_t(x, -t) = f_0(x, -t), & \text{on} \quad \Gamma_1 \times (0, \tau_2), \end{cases} \tag{0.0.17}$$

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\dfrac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, τ_1 and τ_2 are two positive constants with $0 \leq \tau_1 < \tau_2$, μ_0 is a positive constant, $\mu : [\tau_1, \tau_2] \to \mathbb{R}$ is an L^∞ function, $\mu \geq 0$ almost everywhere, and the initial data (u_0, u_1, f_0) belongs to a suitable space. The above problem can be regarded as a problem with a memory acting only on the time interval (τ_1, τ_2).
If $\mu \equiv 0$, that is in absence of delay, the energy of the problem (0.0.17) is proved to decay exponentially to zero. See also some related works such as [33, 106]. Under the assumption

$$\mu_0 > \int_{\tau_1}^{\tau_2} \mu(s)ds, \tag{0.0.18}$$

the authors in [85] proved an exponential stability for problem (0.0.17).

Nig and Yan [110] studied the stabilization of the wave equation with variable coefficients and a delay in the dissipative boundary feedback. By virtue of the Riemannian geometry methods the energy-perturbed approach and the multiplier techniques, they established the uniform stability of the energy.

Cavalcanti et al. [26] consdered the viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + h(u_t) = f(u), \tag{0.0.19}$$

in $\Omega \times (0, \infty)$, subjected to initial conditions and boundary conditions of Dirichlet type. In the case where $f = 0$ and $h(u_t) = a(x)u_t$, That is for the problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + a(x)u_t = 0, \tag{0.0.20}$$

in $\Omega \times (0, \infty)$, where $a : \Omega \longrightarrow \mathbb{R}^+$ is a function, which may be null on a part of the domain Ω and by assuming $a(x) \geqslant a_0$ on $\omega \subset \Omega$ and

$$-\zeta_1 g(t) \leqslant g'(t) \leqslant -\zeta_2 g(t), \quad \forall t \geqslant 0, \tag{0.0.21}$$

the authors showed an exponential decay result under some geometric restrictions on the subset $w \subset \Omega$. The result in [26] was improved by Berrimi and Messaoudi [16], under weaker conditions on both a and g.

Cavalcanti et al.[23] investigated the following problem

$$|u_t|^\rho u_{tt} - \triangle u - \triangle u_{tt} + \int_0^t g(t-s)\triangle u(x,s)ds - \gamma\triangle u_t = 0, \quad t > 0, \tag{0.0.22}$$

in $\Omega \times (0, \infty)$. For $\gamma \geqslant 0$, they showed a global existence result. Furthermore, they obtained an exponential decay result for $\gamma > 0$ provided that the function g decays exponentially.

Fabrizio and Poidoro [37] treated the problem (0.0.20) with $a(x) = a_0$ and showed that the solution decays exponentially only if the relaxation kernel g does. That is to say the presence of the memory term may prevent the exponential decay. Such decay always hold for $g = 0$, due to the linear frictional damping term. Messaoudi in [78] considered a problem related to (0.0.19) and proved general decay result. In fact, his result allows a large class of relaxation functions and improves earlier result in which only the exponential and polynomial rates were established.

Cavalcanti and Oquendo [24] showed, using the piecewise multiple method, some stability results for a more general problem than the one considered in [26] . More precisely, they investigated the following problem

$$u_{tt} - k_0\Delta u + \int_0^t \operatorname{div}[a(x)g(t-s)\nabla u(x,s)]ds + b(x)h(u_t) + f(u) = 0, \tag{0.0.23}$$

and proved that, under the same conditions on the function g and for $a(x)+b(x) \geqslant \rho > 0$, an exponential stability result if g decays exponentially and h is linear, and a polynomial stability results for g decaying polynomially, that is

$$g'(s) \leq -c[g(s)]^p, \quad p > 1,$$

and h nonlinear. Recently, Messaoudi and Tatar [77] studied (0.0.22) with ($\gamma = 0$) and showed that the viscoelastic damping term is strong enough to stabilize the system. Cavalclcanti et al. in the paper [21] investigated a problem similar to (0.0.19) with a nonlinear feedback acting on the boundary of the domain Ω and showed uniform decay rates of the energy without imposing any restrictive growth assumption on the damping term.

Viscoelastic wave equation with delay was first considered by Kirane and Said-Houari in [58] they studied the existence and asymptotic stability of viscoelastic wave equation with delay, they considered the following linear viscoelastic wave equation with a linear damping and a delay term:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds & \\ +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \geqslant 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1, & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \in \Omega, t \in (0,\tau), \end{cases} \tag{0.0.24}$$

where $u = u(x,t), t \geq 0, x \in \Omega, \triangle$ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of $\mathbb{R}^N$, $(N \geq 1)$, μ_1, μ_2 are positive constants, $\tau > 0$ represents the time delay and u_0, u_1, f_0 are given functions belonging to suitable spaces. The purpose of their work was to study the existence and the asymptotic stability of problem (3.1.1) by relaxing the assumption in [84]. Introducing the delay term $\mu_2 u_t(x,t-\tau)$, made the problem different from those considered in the literature. First the authors used the Faedo-Galerkin approximations together with some energy estimates, and under some restriction on the parameters μ_1 and μ_2, they showed that the problem (3.1.1) is well-posed. Second under the hypothesis $\mu_1 < \mu_2$ between the weight of the delay term in the feedback and the weight of the term without delay, they proved a general decay result of the total energy of the problem (3.1.1).

Gerbi and Said-Houari in [42] treated the following linear damped wave equation with dynamic boundary conditions and a delay term boundary term:

$$\begin{cases} u_{tt} - \triangle u - \alpha \triangle u_t = 0, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \Gamma_0, t > 0, \\ u_{tt} = -\left(\dfrac{\partial u}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) \right), & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Gamma_1, t \in (0,\tau), \end{cases} \tag{0.0.25}$$

where $u = u(x,t), t \geq 0, x \in \Omega, \triangle$ denotes the Laplacian operator with respect to t and the variable x, Ω is a regular and bounded domain of $\mathbb{R}^N, (N \geq 1), \partial\Gamma = \Gamma_0 \cup \Gamma_1, mes(\Gamma_0) > 0, \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and $\dfrac{\partial u(x,t)}{\partial \nu}$ denotes the unit outer normal derivative, α, μ_1 and μ_2 are positive constants. Moreover, $\tau > 0$ represents the time delay and u_0, u_1, f_0 are given functions belonging to suitable spaces. This type of problems arise (for example) in modeling of longitudinal vibrations in a homogenous bar in which there are viscous effects. The term Δu_t indicates that the stress is proportional not only to the strain, but also to the strain rate, see [20]. From the mathematical point of view, these problems do not neglet acceleration terms on the boundary. Such type of boundary conditions are usally called *dynamic boundary conditions*. They are not only important from the theoretical point of view but arise in several physical applications. In one space dimension the problem (0.0.25) can modelize the dynamic evolutionof a viscoelastic rod that is fixed at one end and a tip mass attached to its free and,(see [19, 5, 29] for more details). In the two dimension space, as in [96]and in references therein, these boundary conditions arise when we consider the transverse motion of a flexible membrane Ω whose boundary may be affected by the vibrations only in a region. Also some dynamic boundary conditions as in problem(0.0.25) appear when we assume that Ω is an exterior domain of $\mathbb{R}^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [10] for more details), this type of boundary conditions are known as acoustic boundary conditions. In [42] the authors gave positive answers to the following two questions:

- Is it possible for the damping term $-\Delta u_t$ to stabilize system (0.0.25) when the weight of the delay is greater than the weight of the boundary damping (i.e. when $\mu_2 \geqslant \mu_1$)?

• Does the particular structure of the problem prevents the instability result obtained for the problem studied in [31]?
In answering these questions, the authors built appropriate Lyapunov functional which led to stability results.

Alabau, Nicaise, and Pignoti in [2] extented the result in [58] and considered the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - \displaystyle\int_0^{\infty} \mu(s)\Delta u(x,t-s)ds + ku_t(x,t-\tau) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,t) = u_0(x,t), & \text{in } \Omega \times (-\infty,0], \end{cases} \tag{0.0.26}$$

where the initial datum u_0 belongs to a suitable space, the constant $\tau > 0$ is the delay, k is a real number and the memory kernel $\mu : [0,+\infty) \longrightarrow [0,+\infty)$ is a locally absolutly continuous function satisfying

i) $\mu(0) = \mu_0 > 0$,

ii) $\displaystyle\int_0^{+\infty} \mu(t)dt = \tilde{\mu} < 1$,

iii) $\mu'(t) \leqslant -\alpha\mu(t)$, for some $\alpha > 0$,

they recalled that the above problem is exponentially stable for $k = 0$. Observing that for $\tau = 0$ and $k > 0$ the model (0.0.26) presents both viscoelastic and standard dissipative damping. Therefore, in the case, under the above assumptions on the kernel μ, the model is exponentially stable, and exponential stability also occurs for $k < 0$, under a suitable smallness assumption on $|k|$. They noted that the term $ku_t(t)$ with $k < 0$ is a so-called anti-damping, namely a damping with an opposite sign with respect to the standard dissipative one, and therefore it induces instability. Indeed, in absence of viscoelastic damping, i.e. for $\mu \equiv 0$, the solution of the above problem, with $\tau = 0$ and $k < 0$, grows exponentially to infinity.

The stabilization problem for model (0.0.26) has been studied by Guesmia in [48] by using a different approach based on the construction of a suitable Lyapunov functional.

In this thesis we study the stability of some transmission problems of wave and viscoelastic wave equation with delay. The modeling of physical phenomena requires

a set of techniques enabling a mathematical representation of the system studied. In the same sense, it can be said that theoretical modeling requires a precise knowledge of the phenomena intervening in the system and an ability to represent them by mathematical equations. And consequently it conditions the methods which will be used subsequently, to analyze its properties. The problem of stability is to find conditions that relate to systems so that they are stable globally, exponentially or polynomially, so that modeling makes sense. From a practical point of view, and more particularly in the field of engineering science, it is found that phenomena with delay occur naturally in physical processes. The transmission times of the information are given as the transfer times of the materials or even the measurement times. Then, in order to get closer to the real process, better modeling consists in designing the delay systems. The aim of this work is to study the stability of a transmission problem where a delay term occurs in the presence of a damping term without a viscoelastic term and the study of a problem similar to the first one with a Viscoelastic term, and a similar problem to the second by considering the delay as a time function.
The second part of this thesis is devoted to the study of mathematical models of contact. More precisely, we introduce a mathematical model that describes the evolution of a viscoelastic plate in frictional contact with foundation. we derive the variational inequality for the displacement field, then we establishe the existence of a unique weak solution to the model. At the end of this introduction, this thesis is organized as follows:

Chapter 1. In this chapter we recall some prelimenaries and basic results on functional analysis.

Chapter 2. In this chapter, we considered system

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in (L_1, L_2) \times (0,+\infty), \end{cases} \tag{0.0.27}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0, L_1[\cup]L_2, L_3[$, a, b, μ_1 and μ_2 are positive constants and $\tau > 0$ is the delay, $\mu_1 u_t(x,t)$ is the damping term and $\mu_2 u_t(x,t-\tau)$ is the delay term. By using the Faedo-Galerkin approximation and the semigroup approach, we proved the well-posedness of our problem. In addition, we established an exponential decay of the energy defined by

$$E(t) = E_1(t) + E_2(t) + \frac{\zeta}{2}\int_\Omega \int_0^1 y^2(x,\rho,t)d\rho dx,$$

where

$$E_1(t) = \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{a}{2}\int_\Omega u_x^2(x,t)dx,$$

and

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)dx,$$

such that $\tau\mu_2 \leq \zeta \leq \tau(2\mu_1 - \mu_2)$. The decay proved provided that the weight of the delay is less than the weight of the damping (i.e.$\mu_2 \leq \mu_1$). To achieve the decay estimate, we introduced an appropriate Lyapunov functional which leads to the desired result.

Chapter 3. This chapter is devoted to the study of the well-posedness and stability of the solutions for a transmission problem. In this problem we considered system

$$\begin{cases} u_{tt} - au_{xx} + \displaystyle\int_0^{+\infty} g(s)u_{xx}(x,t-s)ds + \mu_1 u_t(x,t) + |\mu_2|\, u_t(x,t-\tau) = 0, & (x,t) \in \Omega\times]0,+\infty[, \\ v_{tt} - bv_{xx} = 0, & (x,t) \in]L_1, L_2[\times]0,+\infty[, \end{cases} \tag{0.0.28}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0, L_1[\times]L_2, L_3[$, $\mu_1 \in \mathbb{R}_*^+$, and $\mu_2 \in \mathbb{R}$. The function g represents the memory, $\mu_1 u_t(x,t)$ is a damping term, and $|\mu_2|u_t(x,t-\tau)$ is the delay term. The constant μ_2 is a real number not necessary positive. By using the semigroup theory we proved the well-posedness of the problem provided that the weight of the delay is less than the weight of the damping (i.e. $|\mu_2| \leq \mu_1$). Also, a result of stability of solutions was obtained.

Chapter 4. The goal of this chapter is to investigate the decay of a transmission problem with memory and time-varying delay. We considered system

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^t g(t-s)u_{xx}(x,s)ds + \mu_1 u_t(x,t) & \\ +|\mu_2|u_t(x,t-\tau(t)) = 0, & (x,t) \in \Omega\times(0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2)\times(0,+\infty), \end{cases}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0, L_1[\cup]L_2, L_3[$, a, b, μ_1, are positive constants, the constant μ_2 is a real number not necessary positive, $\tau(t) > 0$ is the delay function. The memory and the damping, and the time-varying delay are in left-side of the first equation. We proved an exponential decay of the energy by introducing a Lyapunov functional under the assumption $|\mu_2| \leq \mu_1$.
Here, the energy is defined by

$$E(t) = E_1(t) + E_2(t) + \frac{\zeta}{2} \int_\Omega \int_{t-\tau(t)}^t u_t^2(x, s) ds dx,$$

with

$$E_1(t) = \frac{1}{2} \int_\Omega u_t^2(x, t)\, dx + \frac{\beta(t)}{2} \int_\Omega u_x^2(x, t) dx + \frac{1}{2} \int_\Omega (g \Box u_x) dx,$$

and

$$E_2(t) = \frac{1}{2} \int_{L_1}^{L_2} v_t^2(x, t)\, dx + \frac{b}{2} \int_{L_1}^{L_2} v_x^2(x, t)\, dx$$

where ζ is a positive constant defined by

$$\frac{|\mu_2|}{\sqrt{1-d}} \leq \zeta \leq 2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}, \qquad 0 < d < 1,$$

and the function β is defined in Lemma 4.2.1 .

Chapter 5. In this chapter we consider a nonlinear initial boundary value problem in a two-dimensional rectangle. We derive variational formulation of the problem which is in the form of an evolutionary variational inequality in a product Hilbert space. Then, we establish the existence of a unique weak solution to the problem and prove the continuous dependence of the solution with respect to some parameters. Finally, we consider a second variational formulation of the problem, the so-called dual variational formulation, which is in a form of a history-dependent inequality associated to a time-dependent convex set. We study the link between the two variational formulations and establish existence, uniqueness and equivalence results.

Chapter 1

Preliminaries of Functional Analysis

1.1 Introduction

This chapter presents preliminary material from functional analysis which will be used in subsequent chapter in this thesis. Part of the results are stated without proofs, since they are standard and can be found in many references [17, 34, 50, 68, 52, 54, 91]. Nevertheless, we pay a particular attention to the result which are repeatly used in next chapters of the thesis, they include the Projection Lemma, the Riez representation theorem and among others. All the linear spaces considered in this thesis are assumed to be real linear spaces.

1.2 Definitions and Elementary Properties.

1.2.1 Definitions

Definition 1.2.1. : Let X be a vector space. A *scalar product* (u, v) is a bilinear form on $X \times X$ with values in $\mathbb{R}$ (i.e, a map from $X \times X$ to $\mathbb{R}$ that is linear in both variables) such that

$$\begin{aligned}
&(u, v) = (v, u) \qquad \forall u, v \in X \qquad (symetry),\\
&(u, u) \geq 0 \qquad \forall u \in X \qquad (positive),\\
&(u, u) \neq 0 \qquad \forall u \neq 0 \qquad (definite).
\end{aligned}$$

Let us recall that a scalar product satisfies the Cauchy-Schwartz inequality

$$|(u, v)| \leq (u, u)^{\frac{1}{2}} (v, v)^{\frac{1}{2}}, \quad \forall u, v \in X.$$

We shall often denote by ($|.|$ (instead of $\|.\|$) norms arising from scalar products.

Definition 1.2.2. : A *Hilbert space* is a vector space X equiped with a scalar product such that X is complete for the norm $|.|$.

In what follows, X will always denote a Hilbert space.

1.2.2 Useful results

We introduce in what follows some useful results which are valid in Hilbert spaces. This concerns the projection operator, some properties related to orthogonality and the Ries representation theorem, Lax-Milgram Theorem, togather with its consequences.

Theorem 1.2.3. *(The Banach Fixed Point Theorem). Let K be a nonempty closed subset of a Banach space $(X, \|.\|_X)$. Assume that $\Lambda : K \longrightarrow K$ is a contraction, i.e. there exists a constant $\alpha \in [0, 1)$ such that*

$$\|\Lambda u - \Lambda v\|_X \leq \alpha \|u - v\|_X \qquad \forall u, v \in K. \tag{1.2.1}$$

Then there exists a unique $u \in K$ such that $\Lambda u = u$.

Projection theorem

The projection operators represent an important class of nonlinear operators defined in Hilbert spaces, to introduce them we need the following existence and uniqueness result.

Theorem 1.2.4. *(The projection Lemma) Let K be a nonempty closed convex subset of a Hilbert space X. Then, for each $f \in X$ there exists a unique element $u \in K$ such that*

$$\|u - f\|_X = \min_{v \in K} \|v - f\|_X. \tag{1.2.2}$$

Definition 1.2.5. Let K be a nonempty closed convex subset of a Hilbert space X. Then, for each $f \in X$ the element u which satisfies (1.2.2) is called the projection of f on K and is usually denoted $\mathcal{P}_K f$. Moreover, the operator $\mathcal{P}_K :\longrightarrow K$ is called the projection operator on K.

It follows from Definition (1.2.5) that

$$f = \mathcal{P}_K f \Leftrightarrow f \in K. \tag{1.2.3}$$

We conclude from (1.2.5) that the element $f \in X$ is a fixed point of the projection operator $\mathcal{P}_K$ iif $f \in K$.

Next, we present the following characterization of the projection.

Proposition 1.2.6. *Let K be a nonempty closed convex subset of a Hilbert space X and let $f \in X$. Then $u = \mathcal{P}_K f$ if only if*

$$u \in K, \qquad (u, v-u)_X \geq (f, v-u)_X \qquad \forall v \in K. \tag{1.2.4}$$

Not that, besides the caracterization of the projection in termes of inequalities, Proposition 2.5 provides, implicitly, the existence of a unique solution to the inequality (1.2.2). Moreover, using this proposition it is easy to prove the following results.

Proposition 1.2.7. *Let K be a nonempty closed convex subset of a Hilbert space X. Then the projection operator $\mathcal{P}_K$ satisfies the following inequalities:*

$$(\mathcal{P}_K u - \mathcal{P}_K v, v-u)_X \geq 0 \quad \forall u, v \in K, \tag{1.2.5}$$

$$\|\mathcal{P}_K u - \mathcal{P}_K v\|_X \leq \|v-u\|_X \quad \forall u, v \in K. \tag{1.2.6}$$

Proposition 1.2.8. *Let K be a nonempty closed convex subset of a Hilbert space X and let $G_K : X \longrightarrow X$ be the operator defined by*

$$G_K u = u - \mathcal{P}_K u \quad \forall u \in X. \tag{1.2.7}$$

Then, the following properties holds:

$$(G_K u - G_K v, v-u)_X \geq 0, \quad \forall u, v \in K, \tag{1.2.8}$$

$$\|G_K u - G_K v\|_X \leq 2\|v-u\|_X, \quad \forall u, v \in K, \tag{1.2.9}$$

$$(G_K u, u-v)_X \leq 0, \quad \forall u \in X, v \in K, \tag{1.2.10}$$

$$G_K u = 0_X \quad \textit{iff} \quad u \in K. \tag{1.2.11}$$

Duality and weak convergence

It is easy, in a Hilbert space, to write down continuous linear functionals. Pick any $f \in X$; then the map $u \longmapsto (f, u)$ is a continuous linear function on X. It is a remarkable fact that all continuous linear functionals on X are obtained in this fashion.

Theorem 1.2.9. *(The Riesz representation Theorem) Let $(X, (.,.)_X)$ be a Hilbert space and let $l \in X'$. Then there exists a unique $u \in X$ such that*

$$l(v) = (u, v)_X \quad \forall v \in X. \tag{1.2.12}$$

Moreover,

$$\|l\|_{X'} = \|u\|_X. \tag{1.2.13}$$

The Riesz representation theorem also allows to identify a Hilbert space with its dual and, with its bidual which, roughly speaking, shows that each Hilbert space is reflexive. Based on this result we have the following important proprety which represents a particular case of the well-known Eberlein-Smulyan theorem.

Theorem 1.2.10. *If X is a Hilbert space, then any bounded sequence in X has a weakly convergent subsequence.*

It follows that if X is a Hilbert space and the sequence $\{u_n\} \subset X$ is bounded, that is, $\sup_n \|u_n\|_X < \infty$, then there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and an element $u \in X$ such that $u_{n_k} \rightharpoonup u$ in X, where $\rightharpoonup$ means weakly convergence. Furthermore, if the limit u is independent of the subsequence, then the sequence $\{u_n\}$ converges weakly to u, as stated in the following result.

Theorem 1.2.11. *Let X be a Hilbert space and let $\{u_n\}$ be a bounded sequence of elements in X such that each weakly convergent subsequence of $\{u_n\}$ converges weakly to the same limit $u \in X$. Then $u_n \rightharpoonup u$ in X.*

1.3 The Theorem of Stampacchia and Lax-Milgram

Definition 1.3.1. A bilinear form $a : X \times X \longrightarrow \mathbb{R}$ is said to be
(i) *continuous* if there is a constant C such that

$$|a(u, v)| \leq C|u||v|, \quad \forall u, v \in X,$$

(ii) *coercieve* if there is a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha|v| \quad \forall v \in X.$$

Theorem 1.3.2. ***(Stampacchia)***. *Assume that $a(u,v)$ is a continuous coercieve bilinear form on X. Let $K \subset X$ be a nonempty closed and convex subset. Then, given any $\varphi \in X^\star$, there exists a unique element $u \in K$ such that*

$$a(u, v-u) \geq \langle \varphi, v-u \rangle \quad v \in K. \tag{1.3.1}$$

Moreover, if a is symmetric, then u is characterized by the property

$$\boxed{u \in K \quad \text{and} \quad \frac{1}{2}a(u,u) - \langle \varphi, u \rangle = \min_{v \in K}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v \rangle\right\}.} \tag{1.3.2}$$

Theorem 1.3.3. ***(Banach fixed-point theorem - The contraction mapping principal)*** *Let K be a nonempty complete metric space and let $S : K \longrightarrow K$ be a strict contraction, i.e.,*

$$d(Sv_1, Sv_2) \leq kd((v_1, v_2) \qquad \forall v_1, v_2 \in K \quad \textit{with} \quad k < 0. \tag{1.3.3}$$

Then K has a unique fixed point, $u = Su$.

Theorem 1.3.4. ***(Lax-Milgram)*** *Assume that $a(u,v)$ is a continuous coercive bilinear form on X. Then, given any $\varphi \in X^\star$, there exist a unique element $u \in X$ such that*

$$a(u,v) = \langle \varphi, v \rangle \qquad \forall v \in X. \tag{1.3.4}$$

Moreover, if a is symmetric, then u is characterized by the proprety

$$\boxed{u \in X \quad \textit{and} \quad \frac{1}{2}a(u,u) - \langle \varphi, u \rangle = \min_{v \in X}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v \rangle\right\}.} \tag{1.3.5}$$

1.4 Element of Nonlinear Analysis

In this study of variational inequalities presented in Chapter 5 of this thesis we need several results on nonlinear operators and convex functions that we introduce in this section.

1.4.1 Monotone operators

The projection operator on a convex subset K of a Hilbert space is, in general, a nonlinear operator on X . Its properties (2.3) and (2.4) can be extincted as follows.

Definition 1.4.1. Let X be a space with inner product $(.,.)_X$ and norm $\|.\|$ and let $A : X \longrightarrow X$ be an operator . The operator A is said to be monotone if

$$(Au - Av, u - v)_X \geq 0 \quad \forall u, v \in X.$$

The operator A is strictly monotone if

$$(Au - Av, u - v)_X > 0 \quad \forall u, v \in X, u \neq v.$$

and strongly monotone if there exists a constant $m > 0$ such that

$$(Au - Av, u - v)_X \geq m\|u - v\|_X^2, \quad \forall u, v \in X. \tag{1.4.1}$$

The operator A is nonexpensive if

$$\|Au - Av\|_X \leq \|u - v\|_X \quad \forall u, v \in X.$$

and Lipschitz continuous if there exists $M > 0$ such that

$$\|Au - Av\|_X \leq M\|u - v\|_X \quad \forall u, v \in X. \tag{1.4.2}$$

Finally, the operator A is hemicontinuous if the real valued function

$$\theta \longmapsto (A(u + \theta v), w)_X \quad \text{is continous on} \quad \mathbb{R}, \quad \forall u, v \in X.$$

and A is continuous if

$$u_n \longrightarrow u \quad in \quad X \Rightarrow Au_n \longrightarrow Au \quad \text{in} \quad X.$$

It follows from the definition above each strongly monotone operator is strictly monotone and a nonexpensive operator is Lipschitz continuous, with Lipschitz constant $M = 1$. Also, it is easy to check that the Lipschitz continuous operator is continuous and a continuous operator is hemicontinuous. Moreover, it follows from proposition (1.2.7) that the projection operators are monotone and nonexpensive.

In many applications it is not necessary to define nonlinear operators on the entire space X. Indeed, in the study of variational inequalities presented in Chapter 5 we shall consider strongly monotone Lipschitz continuous operators defined on a subset $K \subset X$. For this reason we complete the definition (1.4.1) with the following one.

Definition 1.4.2. Let X be a space with inner product $(.,.)_X$ and norm $\|.\|_X$ and let $K \subset X$. An operator $A : K \longrightarrow X$ is said to be strongly monotone if there exists a constant $m > 0$ such that

$$(Au - Av, u - v)_X \geq m\|u - v\|_X^2 \quad \forall u, v \in K.$$

The operator A is Lipschitz continuous if there exist $M > 0$ such that

$$\|Au - Av\|_X \leq M\|u - v\|_X \quad \forall u, v \in K.$$

The following result involving monotone operators will be used in Chapter 5 of this thesis, in the analysis of elliptic variational inequalities.

Proposition 1.4.3. *Let space $(X, (.,.)_X)$ be an inner product and let $A : X \longrightarrow X$ be a monotone hemicontinuous operator. Assume that u_n is a sequence of elements in X which converges weakly to the element $u \in X$, i.e.*

$$u_n \longrightarrow u \quad \textit{in } X \quad \textit{as} \quad n \longrightarrow \infty.$$

Moreover, assume that

$$\limsup_{n \longrightarrow \infty} (Au_n, u_n - u)_X \leq 0.$$

Then, for all $v \in X$ the following inequality holds:

$$\liminf_{n \longrightarrow \infty} (Au_n, u_n - u)_X \geq (Au, u - v)_X.$$

We proceed with the following existence and uniqueness result in the study of nonlinear equations involving monotone operators.

Theorem 1.4.4. *Let X be a Hilbert space and let $A : X \longrightarrow X$ be a strongly monotone Lipschitz continuous operator. Then, for each $f \in X$ there exists a unique element $u \in X$ such that $Au = f$.*

Proof. Since A is strongly monotone and Lipschitz continuous it follows from Definition (1.4.1) that there exist two constants $m > 0$ and $M > 0$ such that (1.4.1) and (1.4.2) hold. Moreover, we have

$$M \geq m. \tag{1.4.3}$$

Let $f \in X$ and let $\rho > 0$ be given. We consider the operator $S_\rho : X \longrightarrow X$ defined by

$$S_\rho u = u - \rho(Au - f) \qquad \forall u \in X.$$

It follows from this definition that

$$\|S_\rho u - S_\rho v\|_X = \|(u - v) - \rho(Au - Av)\|_X \qquad \forall u, v \in X.$$

and, using (1.4.1), (1.4.2) yields

$$\begin{aligned} \|S_\rho u - S_\rho v\|_X^2 &= \|(u - v) - \rho(Au - Av)\|_X^2 \\ &= \|u - v\|_X^2 - 2\rho(Au - Av, u - v) + \rho^2\|Au - Av\|_X^2 \\ &(1 - 2\rho m + \rho^2 M^2)\|u - v\|_X^2 \forall u, v \in X. \end{aligned}$$

Next, using (1.4.3) it is easy to see that if $0 < \rho < \dfrac{2m}{M^2}$ then

$$0 \leq 1 - 2\rho m + \rho^2 M^2 < 1.$$

Therefore, with this choice of ρ, it follows that

$$\|S_\rho u - S_\rho v\|_X \leq k(\rho)\|u - v\|_X \qquad \forall u, v \in X. \tag{1.4.4}$$

where $k(\rho) = (1 - 2\rho m + \rho^2 M^2)^{\frac{1}{2}} \in [0, 1)$. Inequality (1.4.4) shows that S_ρ is a contraction on the space X and, using Theorem (1.3.3), we obtain that there exists $u \in X$ such that

$$S_\rho u = u - \rho(Au - f)_X \qquad \forall u, v \in X. \tag{1.4.5}$$

Equality (1.4.5) yields $Au = f$, which proves the existence part of the theorem.
Next, consider two elements $u \in V$ and $v \in V$ such that $Au = f$ and $Av = f$. It follows that

$$(Au, u - v)_X = (f, u - v)_X, \qquad ,(Av, u - v)_X = (f, u - v)_X.$$

we substract these equalities to obtain

$$(Au - Av, u - v)_X = 0,$$

then we use assumption (1.4.1) to find that $u = v$, which proves the uniqueness part of the theorem. □

Theorem (1.4.4) shows that if $A : X \longrightarrow X$ is strongly monotone Lipschitz continuous operator defined on a Hilbert space X, then A is inversible. Then properties of its inverse, denoted A^{-1}, are given by the following result.

Proposition 1.4.5. *Let X be a Hilbert space and let $A : X \longrightarrow X$ is strongly monotone Lipschitz continuous operator. Then, $A^{-1} : X \longrightarrow X$ is strongly monotone Lipschitz continuous operator.*

1.4.2 Convex lower semicontinuous functions

Convex lower semicontinuous functions represent a crucial ingredient in the study of variational inequalities. To introduce them, we start with the following definition.

Definition 1.4.6. Let X be a linear space and let K be a nonempty convex subset of X. A function $\varphi : K \longrightarrow \mathbb{R}$ is said to be convex if

$$\varphi((1-t)u + tv) \leq (1-t)\varphi(u) + t\varphi(v), \tag{1.4.6}$$

for all $u, v \in K$ and $t \in [0, 1]$. The function φ is strongly convex if the inequality in (1.4.6) is strict for $u \neq v$ and $t \in (0, t)$.

We note that if $\varphi, \psi : K \longrightarrow \mathbb{R}$ are convex and $\lambda \geq 0$, then the functions $\varphi + \psi$ and $\lambda\varphi$ are convex.

Definition 1.4.7. Let $(X, \|.\|)_X$ be a normed space and let K be a nonempty convex subset of X. A function $\varphi : K \longrightarrow \mathbb{R}$ is said to be lower semicontinuous (l,s,c)at $u \in K$ if

$$\liminf_{n \longrightarrow \infty} \varphi(u_n) \geq \varphi(u), \tag{1.4.7}$$

for each sequence $u_n \subset K$ converging to u in X. The function φ is l.s.c. if it is l.s.c. at every point $u \in K$. When inequality (1.4.7) holds for each sequence $u_n \subset K$ that converges weakly to u, the function φ is said to be weakly lower semicontinuous at u. The function φ is weakly l.s.c. if it is weakly l.s.c. at every point $u \in K$.

We note that if $\varphi, \psi : K \longrightarrow \mathbb{R}$ are l.s.c. functions and $\lambda \geq 0$, then the functions $\varphi + \psi$ and $\lambda\varphi$ are also lower semicontinuous. Moreover, if $\varphi : K \longrightarrow \mathbb{R}$ is a continuous function then it is also lower semicontinuous. The converse is not true and a lower semicontinuous function can be discontinuous. Since strong convergence in X implies weak convergence, it follows that a weakly lower semicontinous function is lower semicontinuous. Moreover, the following results hold.

Proposition 1.4.8. *Let $(X, \|.\|_X)$ be a Banach space, K a nonempty closed convex subset of X and $\varphi : K \longrightarrow \mathbb{R}$ a convex function. Then φ is lower semicontinuous if and only if it is weakly semicontinuous.*

Proposition 1.4.9. *Let $(X, \|.\|_X)$ be a normed space, K a nonempty closed convex subset of X and $\varphi : K \longrightarrow \mathbb{R}$ a convex function. Then φ is bounded from below by an affine function, i.e. there exist $l \in X'$ and $\alpha \in \mathbb{R}$ such that $\varphi(v) \geq l(v) + \alpha$ for all $v \in K$.*

Example.1 The norme function $v \longmapsto \|v\|_X$ is weakly lower semicontinuous. The second example of lower semicontinuous function is provided by the following result.

Proposition 1.4.10. *Let $(X, \|.\|_X)$ be a normed space, and let $a : X \times X \longrightarrow \mathbb{R}$ be a bilinear symmetric continuous and positive form. Then the function $v \longmapsto a(v, v)$ is strictly convex and lower semicontinuous.*

In particular, it follows from Proposition (1.4.10) that, if $(X, (.,.)_X)$ is an inner product space then the function $v \longmapsto \|v\|_X^2$ is strictly convex and lower semicontinuous. We now recall the definition of Gâteaux differentiable functions.

Definition 1.4.11. Let $(X, (.,.)_X)$ be an inner product spacce, $\varphi : X \longrightarrow \mathbb{R}$ and $u \in X$. Then φ is Gâteaux differentiale at u if there exists an element $\nabla\varphi(u) \in X$ such that

$$\lim_{t \longrightarrow 0} \longrightarrow \frac{\varphi(u + tv) - \varphi(u)}{t} = (\nabla\varphi(u), v)_X \quad \forall v \in X. \tag{1.4.8}$$

The element $\nabla\varphi(u)$ which satiffies (1.4.8) is unique and is called the gradient of φ at u. The function $\varphi : X \longrightarrow \mathbb{R}$ is said to be Gâteaux differentiable if it is Gâteaux differentiable at every point of X. In this case the operator $\nabla\varphi : X \longrightarrow X$ which maps every element $u \in X$ into the elment $\nabla\varphi(u)$ is called the gradient operator of φ. The convexity of Gâteaux differentiable functions can be characterized as follows.

Proposition 1.4.12. *Let $(X, (.,.)_X)$ be an inner product space and let $\varphi : X \longrightarrow \mathbb{R}$ be a Gâteaux differentiable function. Then the following statements are equivalent:*
(i) φ is a convex function,
(ii) φ satisfies the inequality

$$\varphi(v) - \varphi(u) \geq (\nabla\varphi(u), v - u) \quad \forall u, v \in X, \tag{1.4.9}$$

(iii) The gradient of φ is a monotone operator, that is

$$(\nabla\varphi(u) - \nabla\varphi(v), u - v)_X \geq 0 \quad \forall u, v \in X. \tag{1.4.10}$$

From the previous proposition we easily deduce the following result.

Corollary 1.4.12.1. *let $(X, (.,.)_X)$ be an inner product space and let $\varphi : X \longrightarrow \mathbb{R}$ be a convex Gâteaux differentiable function. Then, φ is lower semicontinuous.*

1.5 Elliptic Variational Inequalities

In this section we provide an existence and uniqueness result in Theorem (1.4.4). Thus, given an Hilbert space X, an operator $A : X \longrightarrow X$, a subset $K \subset X$ and an element $f \in X$, we consider the problem of finding an element u such that

$$u \in K, \quad (Au, v - u)_X \geq (f, v - u)_X \quad \forall v \in K. \tag{1.5.1}$$

An inequelity of the form (1.5.1) is called an *elliptic variational inequality of the first kind*. The first result we present in the study of the variational inequality (1.5.1) is the folowing.

Theorem 1.5.1. *Let X be a Hilbert space and let $K \subset X$ be a nonempty, closed convex subset. Assume that $A : K \longrightarrow X$ is strongly monotone Lipschitz continuous operator. Then, for each $f \in X$ the variational inequality (1.5.1) has a unique solution.*

Proof. [102], Let $f \in X$ and $\rho > 0$ be given. We consider the operator $S_\rho : K \longrightarrow K$ defined by

$$S_\rho u = \mathcal{P}_K(u - \rho(Au - f)) \qquad \forall u \in K.$$

Where $\mathcal{P}_K$ denotes the projection operator on K. Using (1.2.7) it follows that

$$\|S_\rho u - S_\rho v\|_X \leq \|(u - v) - \rho(Au - Av)\|_X \qquad \forall u, v \in K.$$

Then, using arguments similar to those used in the proof of Theorem (1.4.4) we obtain

$$\|S_\rho u - S_\rho v\|_X \leq k(\rho)\|u - v\|_X \qquad \forall u, v \in K,$$

where $k(\rho) = (1 - 2\rho m + \rho^2 M^2)^{\frac{1}{2}}$ and m, M are the constants in (1.4.1) and (1.4.2), respectively. Also, with a convenient choice of ρ we may assume that $k(\rho) \in [0, 1)$. It follows now from Theorem (1.2.3) that there exists $u \in K$ such that

$$S_\rho u = \mathcal{P}(u - \rho((Au - f)) = u \tag{1.5.2}$$

We combine now (1.5.2) and Proposition (1.2.6) to see that u satisfies (1.5.1), which proves the existence part of the theorem.
Next, we consider two solutions u and v to (1.5.1). It follows that $u \in K, v \in K$ and, moreover

$$(Au, v - u)_X \geq (f, v - u)_X, \qquad (Av, u - v)_X \geq (f, u - v)_X.$$

We add these inequalities to see that

$$(Au - Av, u - v)_X \leq 0,$$

then we use assumption (1.4.1) to ontain $u = v$, which proves the uniqueness part.
Assume now the case when $K = X$. Then, taking $v = u \mp w$it is easy to see that the variational inequality (1.5.1) is equivalent to the variational equation

$$(Au, w)_X = (f, w)_X \qquad \forall w \in X$$

which, in turn, is equivalent to the nonlinear equation $Au = f$. We conclude from above that Theorem (1.5.1) represents an extension of Theorem (1.4.4). $\square$

1.6 History-dependent Variational Inequalities

In this section we extend the existence and uniqueness result in Theorem (1.5.1) to a special class of time-dependent variational inequalities. To this end we need to introduce some background of spaces of functions defined on a time interval with values in an abstract Hilbert space.

1.6.1 Spaces of vector-valued functions

Let $T > 0$ and let X be a Hilbert space. We denote by $C([0, T]; X)$ the space of continuous functions defined on $[0, T]$ with values on X. It is well known that $C([0, T]; X)$

is a Banach space with the norm

$$\|v\|_{C([0,T];X)} = \max_{t\in[0,T]} \|v(t)\|_X.$$

We also recall that a function $v : [0,T] \longrightarrow X$ is said to be *differentiable at* $t_0 \in [0,T]$ if there exists an element in X, denoted $\dot{v}(t_0)$ and called the *derivative* of v at t_0, such that

$$\lim_{h\longrightarrow 0} \|\frac{1}{h}(v(t_0+h) - v(t_0)) - \dot{v}(t_0)\|_X,$$

where the limit is taken with respect to h with $t_0 + h \in [0,T]$. The derivative at $t_0 = 0$ is defined as a right-sided limit, and that at $t_0 = T$ as a left-sided limit. The function v is said to be *differentiable* on $[0,T]$ if it is defferentiable at every $t_0 \in [0,T]$. in this case the function $\dot{v} : [0,T] \longrightarrow X$ is called the *derivative* of v. The function v is said to be *continuously differentiable* on $[0,T]$ if it is differentiable and its derivative is continuous. We denote by $C^1([0,T];X)$ the space of continuously differentiable functions on $[0,T]$ with values in x and we recall that this is a Banach space with the norm

$$\|v\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|v(t)\|_X + \max_{t\in[0,T]} \|\dot{v}(t)\|_X.$$

Using the properties of the integral it is easy to see that if $f \in C([0,T];X)$ then the function $g : [0,T] \longrightarrow X$ given by

$$g(t) = \int_0^t f(s)ds \quad \forall t \in [0,T],$$

belongs to $C^1([0,T];X)$ and , moreover, $\dot{g} = f$. Moreover, we recall that for a function $v \in C^1([0,T];X)$ the following equality holds:

$$v(t) = \int_0^t \dot{v}(s)ds \quad \forall t \in [0,T]. \tag{1.6.1}$$

Finally, for a subset $K \subset X$ we still use the notation $C([0,T];K)$ and $C^1([0,T];K)$ for the set of continuous and continuously differentiable functions defined on $[0,T]$ with values in K, respectively.
We present now a fixed point result which is useful to prove the solvability of nonlinear equations and variational inequalities with history-dependent operators.

Proposition 1.6.1. *Let $\Lambda : C([0,T];X) \longrightarrow C([0,T];X)$ be an operator which satisfies the following property; there exist $k \in [0,1)$ and $c \geq 0$ such that*

$$\begin{aligned} \|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X \leq \|\eta_1(t) - \eta_2(t)\|_X + c\int_0^t \|\eta_1(s) - \eta_2(s)\|_X ds \\ \forall \eta_1, \eta_2 \in C([0,T];X), \quad t \in [0,T]. \end{aligned} \tag{1.6.2}$$

Then, there exists a unique element $\eta^\star \in C([0,T];X)$ such that $\Lambda\eta^\star = \eta^\star$.

A proof of Proposition (1.6.1) can be found in [102], for instance.
We use below the notation $C([0,T])$ for the space of real valued continuous functions defined on the compact interval $[0,T] \subset \mathbb{R}$. The following inequality is useful to obtain uniqueness result in the study of nonlinear equations and variational inequalities with history-dependent operators.

Lemma 1.6.2. *(The Gronwall Inequality) Let $f, g \in C([0,T])$ a positive functions, and assume that there exists $c > 0$ such that*

$$f(t) \leq g(t) + c\int_0^t f(s)ds \quad \forall t \in [0,T]. \tag{1.6.3}$$

Then

$$f(t) \leq g(t) + c\int_0^t g(s)e^{c(t-s)}ds \quad \forall t \in [0,T]. \tag{1.6.4}$$

Moreover, if g is nondecreasing, then

$$f(t) \leq g(t)e^{ct} \quad \forall t \in [0,T]. \tag{1.6.5}$$

A proof of Lemma (1.6.2) can be found [102], for instance.

1.6.2 History-dependent quasivariational inequalities.

In the rest of this section we follow to introduce the concept of history-dependent quasivariational inequalities for which we provide an existence and uniqueness see [102].
Let X be a real Hilbert space with inner product $(.,.)_X$ and associated norm $\|.\|_X$; Let K be subset of X and consider the operator $A : X \longrightarrow X$, and $\mathcal{S} : C([0,T];X) \longrightarrow$

$C([0,T];X)$ and let $f : [0,T] \longrightarrow X$. We are interested in the problem of finding a function $u \in C([0,T];X)$ such that, for all $t \in [0,T]$, the inequality below holds

$$\begin{aligned} u(t) \in K, \quad & (Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X \\ & \geq (f(t), v - u(t))_X \quad \forall v \in K. \end{aligned} \tag{1.6.6}$$

To avoid any confusion we not that here and below the notation $Au(t)$ and $\mathcal{S}u(t)$ are short hand notation for $A(u(t))$ and $(\mathcal{S}u)(t)$, for all $t \in [0,T]$. In the study of (1.6.6) we assume that

$$K \text{ is a nonempty closed convex subset of } X, \tag{1.6.7}$$

and A is strongly monotone and Lipschitz continuous operator, i.e.

$$\left\{ \begin{array}{l} \text{(a) There exists } m > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \geq m \, \|u_1 - u_2\|_X^2 \\ \qquad\quad \forall\, u_1,\, u_2 \in X. \\ \text{(b) There exists } M > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \leq M \, \|u_1 - u_2\|_X \quad \forall\, u_1,\, u_2 \in X. \end{array} \right. \tag{1.6.8}$$

Moreover, we assume that the operator $\mathcal{S}$ satisfies the following condition:

$$\left\{ \begin{array}{l} \text{There exists } L_S > 0 \text{ such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \leq L_S \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \qquad\quad \forall\, u_1,\, u_2 \in C([0,T];X),\ \forall\, t \in [0,T]. \end{array} \right. \tag{1.6.9}$$

Finally, we suppose that

$$f \in C([0,T];X). \tag{1.6.10}$$

Note that condition (1.6.9) is satisfied for the operator $\mathcal{S} : C([0,T];X) \longrightarrow C([0,T];X)$ given by

$$\mathcal{S}v(t) = R\left(\int_0^t v(s)ds + v_0\right) \quad \forall v \in C([0,T];X), \quad \forall t \in [0,T], \tag{1.6.11}$$

where $R : X \longrightarrow X$ is a Lipschitz continuous operator and $v_0 \in X$. It is also satisfied for *Volterra's operator* $\mathcal{S} : C([0,T];X) \longrightarrow C([0,T];X)$ given by

$$\mathcal{S}v(t) = \int_0^t R(t-s)v(s)ds \quad \forall v \in C([0,T];X), \quad \forall t \in [0,T], \tag{1.6.12}$$

where $R \in C([0,T];\mathcal{L}(X))$. Indeed, in the case of the operator (1.6.11), inequality (1.6.9) holds with L_S being the Lipschitz constant of the operator R, and in the case of the opertator (1.6.12) it holds with

$$L_S = \|R\|_{C([0,T];\mathcal{L}(X))} = \max_{t\in[0,T]} \|R(t)\|_{\mathcal{L}(X)}. \tag{1.6.13}$$

Clearly, in the case of the operators (1.6.11), and (1.6.12) the current value $\mathcal{S}v(t)$ at the moment t depends on the values of v at the moments $0 \leq t$ and, therefore, we refer the operators of the form (1.6.11) or (1.6.12) as *history-dependent operators.* We extend this definition to all the operators $\mathcal{S} : C([0,T];X) \longrightarrow C([0,T];X)$ which satisfies condition (1.6.9) and for this reason, we say that the quasivariational inequalities of the form (1.6.6) are *history-dependent quasivariational inequalities.*

The main result of this section is the following.

Theorem 1.6.3. *Let X be an Hilbert space and assume that (1.6.7)-(1.6.10) hold. Then, the variational inequality (1.6.6) has a unique solution $u \in C([0,T],K)$.*

Theorem (1.6.3) represents a particular case of a more general result obtained in [102, 101]. Nevertheless, for the convenience of the reader we decide to provide below a complete proof of this theorem. It is based on a fixed point argument and will be established in several steps. We assume in what follows that (1.6.7)-(1.6.10) hold. In the first step let $\eta \in C([0,T],X)$ be given and denote by $y_{\eta(t)} \in C([0,T],X)$ the function

$$y_{\eta(t)} = \mathcal{S}\eta(t) \quad \forall t \in [0,T]. \tag{1.6.14}$$

Consider now the problem of finding a function $u_\eta : [0,T] \longrightarrow X$ such that for all $t \in [0,T]$, the inequality below holds:

$$\begin{aligned} u_\eta(t) \in K \quad & (Au_\eta(t), v - u_\eta(t))_X + (y_\eta(t), v - u_\eta(t))_X \\ & \geq (f(t), v - u_\eta(t) \qquad \forall v \in K. \end{aligned} \tag{1.6.15}$$

We have the following existence and uniqueness result.

Lemma 1.6.4. *There exists a unique solution $u_\eta \in C([0,T],K)$ to problem (1.6.15).*

Proof. Using assumptions (1.6.8) and (1.6.9) it follows from Theorem (1.5.1) that there exists a unique element $u_{\eta(t)}$ that solves (1.6.15), for each $t \in [0,T]$. Let us show that

$u_\eta : [0,T] \longrightarrow K$ is continuous and, to this end, consider $t_1, t_2 \in [0,T]$. For the sake of simplicity we denote $u_\eta(t_i) = u_i, \eta(t_i) = \eta_i, y_\eta(t_i) = y_i, f(t_i) = f_i$ for $i = 1, 2$. Using (1.6.15) we obtain

$$\begin{aligned} u_1 \in K, \qquad &(Au_1, v - u_1)_X + (y_1, v)_X - (y_1, u_1)_X \\ &\qquad \geq (f_1, v - u_1)_X \qquad \forall v \in K, \end{aligned} \tag{1.6.16}$$

$$\begin{aligned} u_2 \in K, \qquad &(Au_2, v - u_2)_X + (y_2, v)_X - (y_2, u_2)_X \\ &\qquad \geq (f_2, v - u_2)_X \qquad \forall v \in K, \end{aligned} \tag{1.6.17}$$

We take $v = v_2$ in (1.6.16) and $v = v_1$ in (1.6.17), then we add the resulting inequalities to deduce that

$$\begin{aligned} (Au_1 - Au_2, u_1 - u_2)_X &\leq (y_1, u_2)_X - (y_1, u_1)_X + (y_2, u_1)_X \\ &\quad -(y_2, u_2)_X + (f_1 - f_2, u_1 - u_2)_X. \end{aligned}$$

Next, we use assumption (1.6.8)(a) and Cauchy-Schwarz inequality to obtain

$$m\|u_1 - u_2\|_X \leq \|y_1 - y_2\|_X + \|f_1 - f_2\|_X. \tag{1.6.18}$$

We deduce from (1.6.18) that $t \longmapsto u_\eta(t) : [0,T] \longrightarrow K$ is a continuous function, which concludes the proof. □

In the second step we use Lemma (1.6.4) to consider the operator

$$\Lambda : C([0,T], X) \longrightarrow C([0,T], K) \subset C([0,T], X), \tag{1.6.19}$$

defined by equality

$$\Lambda(\eta) = u_\eta \qquad \forall \eta \in C([0,T], X). \tag{1.6.20}$$

We have the following fixed point result.

Lemma 1.6.5. *The operator Λ has a unique fixed point $\eta^\star \in C([0,T], K)$.*

Proof. Let $\eta_1, \eta_2 \in C([0,T], X)$ and let y_i be the function defined by (1.6.14) for $\eta = \eta_i$, i.e. $y_i = y_{\eta_i}$, for $i = 1, 2$. We also denoted by u_i the solution of the variational inequality (1.6.15) for $\eta = \eta_i$, i.e. $u_i = u_{\eta_i}, i = 1, 2$. Let $t \in [0,T]$. From definition (1.6.20) we have

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X = \|u_1(t) - u_2(t)\|_X. \tag{1.6.21}$$

Moreover, an argument similar to that in the proof of (1.6.18) shows that

$$m\|u_1(t) - u_2(t)\|_X \leq \|y_1(t) - y_2(t)\|_X. \tag{1.6.22}$$

Next, we use (1.6.14) and the property (1.6.9) of the operator $\mathcal{S}$ to see that

$$\|y_1(t) - y_2(t)\|_X = \|\mathcal{S}\eta_1(t) - \mathcal{S}\eta_2(t)\|_X \leq L_S \int_0^t \|\eta_1(s) - \eta_2(s)\|_X ds. \tag{1.6.23}$$

and, using this inequality in (1.6.22) yields

$$\|u_1(t) - u_2(t)\|_X \leq \frac{L_S}{m} \int_0^t \|\eta_1(s) - \eta_2(s)\|_X ds. \tag{1.6.24}$$

We combine now (1.6.21) and (1.6.24) to see that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X \leq \frac{L_S}{m} \int_0^t \|\eta_1(s) - \eta_2(s)\|_X ds. \tag{1.6.25}$$

Finally, we use (1.6.25) and Proposition (1.6.1) to obtain that the operator Λ has a unique fixed point $\eta^\star \in C([0,T],X)$.
Since Λ has values on $C([0,T],K)$, we deduce that $\eta^\star \in C([0,T],K)$, which concludes the proof. □

We have all the ingredients to prove Theorem (1.6.3).

Proof. ***Existence.*** Let $\eta^\star \in C([0,T],K)$ be the fixed point of the operator Λ, i.e. $\Lambda\eta^\star = \eta^\star$. It follows from (1.6.14) and (1.6.20) that, for all $t \in [0,T]$, the following equalities hold:

$$y_{\eta^\star} = \mathcal{S}\eta^\star(t), \qquad u_{\eta^\star} = \eta^\star(t). \tag{1.6.26}$$

We write now the inequality (1.6.15) for $\eta = \eta^\star$ and then use the equalities (1.6.26) to conclude that the function $\eta^\star \in C([0,T],K)$ is solution to the quasivariational inequality (1.6.6).

Uniqueness. The uniqueness part is a consequence of the uniquenes of the fixed point of the operator Λ can be proved as follows. Denote by $\eta^\star \in C([0,T],K)$ the solution of the quasivariational inequality (1.6.6) obtained above, and let $\eta \in C([0,T],K)$ be a solution of this inequality. Also, consider the function $y_\eta \in C([0,T],X)$ defined by (1.6.14). Then, it follows from (1.6.6) that η is solution to the variational inequality (1.6.15) and, since by Lemma (1.6.4) this inequality has a unique solution, denoted u_η, we conclude that

$$\eta = u_\eta \tag{1.6.27}$$

Equality (1.6.27) shows that $\Lambda\eta = \eta$ where Λ is the operator defined by (1.6.20) it follows that $\eta = \eta^\star$, which concludes the first proof of the uniqueness part.
A direct proof of the uniqueness part can be obtained by using the Gronwell argument and is as follows. Assume that u_1, u_2 are two solutions of the variational inequality (1.6.6) with regularity $C([0,T],K)$ and let $t \in [0,T]$. We use (1.6.6) to see that

$$\begin{aligned}
&(Au_1(t) - Au_2(t), u_1(t) - u_2(t))_X,\\
&\quad\le (\mathcal{S}u_1(t), u_2(t))_X - (\mathcal{S}u_1(t), u_1(t))_X,\\
&\qquad +(\mathcal{S}u_2(t), u_1(t))_X - (\mathcal{S}u_1(t), u_2(t))_X,\\
&\qquad\quad = (\mathcal{S}u_2(t) - \mathcal{S}u_1(t), u_1(t) - u_2(t))_X,
\end{aligned}$$

and then, using assumptions (1.6.8) yields

$$m\|u_1(t) - u_2(t)\|_X^2 \le \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \|u_1(t) - u_2(t)\|_X.$$

We employ this inequality and assumption (1.6.9) on the operator $\mathcal{S}$ to find that

$$\|u_1(t) - u_2(t)\|_X \le \frac{L_S}{m} \int_0^t \|u_1(s) - u_2(s)\|_X ds.$$

We use the Lemma (1.6.2) to see that $u_1(t) = u_2(t)$ for all $t \in [0,T]$, which concludes the second proof of the uniqueness part of the theorem. □

We end this section with the following consequence of Theorem (1.6.3).

Corollary 1.6.5.1. *Let X be Hilbert space and assume that (1.6.8)-(1.6.10) hold. Then there exists a unique function $u \in C([0,T],X)$ such that*

$$(Au(t), v)_X + (\mathcal{S}u(t), v)_X = (f(t), v)_X \quad \forall v \in [0,T]. \tag{1.6.28}$$

Proof. We consider the problem of finding a function $u \in C([0,T],X)$ such that

$$\begin{aligned}
&(Au(t), w - u(t))_X + (\mathcal{S}u(t), w - u(t))_X\\
&\qquad \ge (f(t), w - u(t))_X \quad \forall w \in X, \forall t \in [0,T].
\end{aligned} \tag{1.6.29}$$

Assume that (1.6.29) holds and let $v \in X$ and $t \in [0,T]$. We successively take $w = u(t) \mp v$ in (1.6.29) to obtain

$$\begin{aligned}
&(Au(t), v)_X + (\mathcal{S}u(t), v)_X \ge (f(t), v)_X,\\
&(Au(t), v)_X + (\mathcal{S}u(t), v)_X \le (f(t), v)_X,
\end{aligned}$$

which show that (1.6.28) holds, too. Conversely, assume that (1.6.28) holds and $w \in X$ and $t \in [0, T]$. Then we take $v = w - u(t)$ in (1.6.28) to obtain (1.6.29). We conclude from above that the variational equality (1.6.28) is equivalent with variational inequality of Theorem (1.6.3) which guarantees the unique solvability of the variational inequality (1.6.29). □

1.7 Semi-groupes of linear operators

1.7.1 Definition

Definition 1.7.1. Let X be a Banach space and let $(S(t))_{t\geq 0}$ a family of linear continuous operators on X. $S(t)$ is a C_0-semigroup if
1) $S(0) = Id$
2) for all $t, s \geq 0, S(t+s) = S(t)S(s)$
3) for all $x \in X, t \longmapsto S(t)x$ is continue from $\mathbb{R}_+$ to X. We say that $(S(t))_{t\in\mathbb{R}}$ is a C_0-semigroup if these properties extend on all negative t and s. We talke about contraction semigroup is $S(t)$ is a contraction for all $t \geq 0$ and about compact semigroup if $S(t)$ is compact for all $t > 0$. We say that the semigroup is continuous uniformely if $S(t)$ goes to zero I in $\mathcal{L}(X)$ as $t \longrightarrow 0$.

The definition directly implies the characteristic properties of the semigroup.

Proposition 1.7.2. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup, there exists $M > 1$ and $\lambda \in \mathbb{R}$ such that*

$$\forall t \geq 0, \quad \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\lambda t}. \tag{1.7.1}$$

Proof. For all $x \in X, (S(t)x)_{t\in[0,1]}$ is bounded in X. From Banch- Stanhaus Theorem , the family $(S(t)x)_{t\in[0,1]}$ is bounded in $\mathcal{L}(X)$ by $M \geq 1$. For all $t \geq 0$, let $n = [t]$ we have

$$\|S(1)S(1)S(1).....S(1)S(t-n)\| \leq MM^t = Me^{t\ln M}. \tag{1.7.2}$$

□

Example : Exponential Matrix
Let A a linear operator, continuous on X; We can define

$$\forall t \in \mathbb{R}, \quad S(t) = e^{At} = \sum_{k\geq 0} \frac{1}{k!} A^k. \tag{1.7.3}$$

The exponential properties shows that $S(t)$ is a uniformly continuous group. In addition, $S(t)$ is differentialble and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax. \tag{1.7.4}$$

In an other word, $S(t)$ is the corespondent flo to the differential equation $u'(t) = Au(t)$.

1.7.2 Infinitesimal generator

Inspired by the example above, we try to write all semigroup as an exponential operator.

Definition 1.7.3. A linear operator A defined by

$$Av = \lim_{t \longrightarrow 0^+} \frac{S(t)v - v}{t}, \tag{1.7.5}$$

with its domain of definition

$$D(A) = \left\{ x \in X, \lim_{t \longrightarrow 0^+} \frac{S(t)v - v}{t} \textbf{ exists in} X \right\}, \tag{1.7.6}$$

is called the infinitesimal generator of the family of semigroups $(S(t))_{t \geq 0}$.

Proposition 1.7.4. *Let $S(t)$ be a C_0-semigroup generated by A, then*
1) $u_0 \in X, \int_0^t S(s)u_0 \in D(A)$ *and*

$$A\left(\int_0^t S(\tau)u_0 d\tau\right) = S(t)u_0 - u_0. \tag{1.7.7}$$

2) *If $u_0 \in D(A)$, then $S(t)u_0 \in D(A)$ for all $t \geq 0$, $S(t)u_0$ is C^1 class and*

$$\frac{d}{dt}S(t)u_0 = AS(t)u_0 = S(t)Au_0. \tag{1.7.8}$$

1.7.3 The Hille-Yosida and Lumer-Philips theorem

Theorem 1.7.5. ***(Hille Yosida)*** *A linear operator A is infnitesimal generator of semigroup $S(t)$ on X satisfying $\|S(t)\| \leq e^{wt}$ with $w \in \mathbb{R}$ if and only if*

1) A is closed with dense domain $D(A)$,
2) $\rho(A) \supset]w, +\infty[$, where $\rho(A)$ is the resolvant set of A and ,

$$\forall \lambda \in]w, +\infty[, \quad \|(A - \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda - w}.$$

Dissipative operators and the Lumer-Phillips theorem

Let X be a Banch space (real or complex)and $X^\star$ be its dual. From Hahn-Banch Theorem, for every $x \in X$ there exists $x^\star \in X^\star$ satisfying

$$\langle x^\star, x\rangle = \|x\|^2 = \|x^\star\|^2.$$

Therefore the duality set

$$\mathcal{J} = \left\{x^\star \in X^\star; \langle x^\star, x\rangle = \|x\|^2 = \|x^\star\|^2\right\}.$$

is nonempty for every $x \in X$.

Definition 1.7.6. We say that an operator $(A, D(A))$ is dissipative if for every $x \in D(A)$ there is $x^\star \in \mathcal{J}$ such that

$$\mathcal{R}\langle x^\star, Ax\rangle \leq 0. \tag{1.7.9}$$

If X is real space, then the real part in above definition can be dropped.

Theorem 1.7.7. *A linear operator A is dissipative if and only if for all $\lambda > 0$ and $x \in D(A)$,*

$$\|(\lambda I - A)x\| \geq \lambda\|x\|. \tag{1.7.10}$$

Theorem 1.7.8. *(Lumer-Phillips) Let A be a linear operator with dense domain $D(A)$ in X.*

1) *If A is dissipative and there is λ_0 such that $\lambda_0 I - A$ is surjective, then A is the infinitesimal generator of a C_0-semigroup of contractions in X.*

2) *If A is the generator of a C_0-semigroup of contractions on X, then $\lambda I - A$ is surjective for all $\lambda > 0$ and is dissipative. Moreover, for every $x \in D(A)$ and every $x^\star \in \mathcal{J}$ we have $\mathcal{R}\langle x^\star, Ax\rangle \leq 0$..*

1.8 The Hille-Yosida Theorem

1.8.1 Definition and Elementary properties

Definition 1.8.1. An unbounded linear operator $A : D(A) \subset X \longrightarrow X$ is to be said monotone or accretive or, $(-A)$ is dissipative, if it satisfies

$$(Av, v) \geq 0 \qquad \forall v \in D(A).$$

It is called maximal monotone if on addition, $R(I+A)=X$, i.e.,

$$\forall f \in X \quad \exists u \in D(A) \text{ such that } u + Au = f.$$

Proposition 1.8.2. *Le A be a maximal monotone operator. Then*
(a) $D(A)$ is dense in X,
(b) A is a closed operator,
(c) For every $\lambda > 0, (I+\lambda A)$ is bejective from $D(A)$ onto X,$(I+\lambda A)^{-1}$ is a bounded operator, and $\|(I+\lambda A)^{-1}\|_{\mathcal{L}(X)} \leq 1$.

Remark 1.8.3. If A is a maximal monotone then λA is also maximal monotone for every $\lambda > 0$. However, if A and B are maximal monotone operators, then $A+B$, defined on $D(A) \cap D(B)$, need not be maximal monotone.

Definition 1.8.4. Let A be a maximal monotone operator. For every $\lambda > 0$, set

$$\boxed{J_\lambda = (I+\lambda A)^{-1} \quad \textsf{and} \quad A_\lambda = \frac{1}{\lambda}(I - J_\lambda)} \tag{1.8.1}$$

J_λ is called the resolvent of A, and A_λ is the Yosida approximation (or regularization) of A. Keep in mind that $\|J_\lambda\|_{\mathcal{L}(X)} < 1$.

Proposition 1.8.5. *Let A be a maximal monotone operator. Then*

$$\begin{array}{clll}
(a_1) & A_\lambda v = A(J_\lambda v) & \forall v \in X \text{ and } \forall \lambda > 0, \\
(a_2) & A_\lambda v = J_\lambda(Av) & \forall v \in D(A) \text{ and } \forall \lambda > 0, \\
(b) & |A_\lambda v| = |Av| & \forall v \in D(A) \text{ and } \forall \lambda > 0, \\
(c) & \lim\limits_{\lambda \to 0} J_\lambda v = v & \forall v \in X, \\
(d) & \lim\limits_{\lambda \to 0} A_\lambda v = Av & \forall v \in D(A), \\
(e) & (A_\lambda v, v) \geq 0 & \forall v \in X \text{ and } \forall \lambda > 0, \\
(f) & |A_\lambda v| \geq 0 & \forall v \in X \text{ and } \forall \lambda > 0.
\end{array}$$

Remark 1.8.6. Proposition (1.8.5) implies that $(A_\lambda)_\lambda$ is a family of bounded operators that " approximate" the *unbounded* operator A as $\lambda \longrightarrow 0$. This approximation will be used very often; Of course, in general, $\|A_\lambda\|_{\mathcal{L}(X)}$ "blows up" as $\lambda \longrightarrow 0$.

The Hille-Yosida theorem in Banach spaces

The Hille-Yosida theorem extends to Banach spaces. The precise statement is the following. Let E be a Banach space and let $A : D(A) \subset E \longrightarrow E$ be an unbounded linear operator. One says that A is m-accretive if $\overline{D(A)} = E$ and for every $\lambda > 0$, $I + \lambda A$ is bijective from $D(A)$ onto E with $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(E)} \leq 1$.

Theorem 1.8.7. ***(Hille-Yosida)***. *Let A be m-accretive. Then given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0, +\infty); E) \cap C([0, +\infty); D(A)),$$

such that

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = 0 \quad \text{ on } [0, +\infty), \\ u(0) = 0 \end{cases} \tag{1.8.2}$$

Moreover,

$$\|u(t)\| \leq \|u_0\| \;\text{ and } \|\frac{du}{dt}(t)\| = \|Au(t)\| \leq \|Au_0\| \quad \forall t \geq 0.$$

For the proof see [17] The map $u_0 \longmapsto u(t)$ extended by continuity to all of E is denoted by $S_A(t)$. It is a continuous semigroup of contractions on E. Conversely, given any continuous semigroup of contractions $S(t)$, ther exists a unique m-accretive operator A such that $S(t) = S_A(t) \quad \forall t \geq 0$. For the proof, see P. Lax, A. Pazy,

Part I

Transmission Problems

Chapter 2

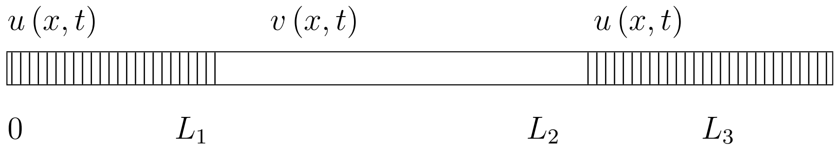
Decay for a transmission wave equations with delay

2.1 Introduction

In this chapter, we consider a transmission problem with a delay term of the form

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in (L_1, L_2) \times (0,+\infty), \end{cases} \tag{2.1.1}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0, L_1[\cup]L_2, L_3[$, a, b, μ_1 and μ_2 are positive constants and $\tau > 0$ is the delay.



System (2.1.1) is subjected to the following boundary conditions and transmission conditions:

$$\begin{cases} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), & i = 1,2 \\ au_x(L_i,t) = bv_x(L_i,t), & i = 1,2 \end{cases} \tag{2.1.2}$$

and the initial conditions:

$$\begin{cases} u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & x\in\Omega, \\ u(x,t-\tau)=f_0(x,t-\tau), & x\in\Omega,\ t\in[0,\tau], \\ v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), & x\in\,]L_1,L_2[\,. \end{cases} \tag{2.1.3}$$

For $\mu_2=0$, system (2.1.1)-(2.1.3) has been investigated in [8], for $\Omega=[0,L_1]$ and the authors showed the well-posedness and exponential stability of the total energy. Muñoz Rivera and Oquendo [82] studied the wave propagations over materials consisting of elastic and viscoelastic components, that is

$$\begin{cases} \rho_1 u_{tt}-\alpha_1 u_{xx}=0, & x\in\,]0,L_0[\,,\ t>0, \\ \rho_2 v_{tt}-\alpha_2 v_{xx}+\displaystyle\int_0^t g(t-s)\,v_{xx}(s)\,ds=0, & x\in\,]L_0,L[\,,\ t>0, \end{cases} \tag{2.1.4}$$

with the boundary and initial conditions

$$\begin{cases} u(0,t)=v(L,t)\,0, u(L_0,t)=v(L_0,t), & t>0, \\ \alpha_1 u_x(L_0,t)=\alpha_2 v_x(L_0,t)-\displaystyle\int_0^t g(t-s)\,v_x(s)\,ds, & t>0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & x\in[0,L_0], \\ v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), & x\in[L_0,L], \end{cases} \tag{2.1.5}$$

where ρ_1 and ρ_2 are densities of the materials and α_1,α_2 are elastic coefficients and g is a positive exponential decaying function. They showed that the dissipation produced by the viscoelastic part is strong enough to produce an exponential decay of the solution, no matter how small is its size. Ma and Oquendo [70] considered transmission problem involving two Euler−Bernoulli equations modeling the vibrations of a composite beam. By using just one boundary damping term in the boundary, they showed the global existence and decay property of the solution. Marzocchi et al. [73] investigated a 1−d semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero, no matter how small the damping subdomain is. A similar result has sheen shown by Messaoudi and Said-Houari [79], where a transmission problem in thermoelasticity of type III has been investigated. See also Marzocchi et al. [74] for a multidimensional linear thermoelastic transmission problem.

For $\mu_2>0$, problem (2.1.1) contains a delay term in the internal feedback. This delay term may destabilize system (2.1.1)-(2.1.3) which is exponentially stable in the absence of delays [8]. The effect of the delay in the stability of hyperbolic systems has been investigated by many people. See for instance [31, 32].

In [84] the authors examined a system of wave equation with a linear boundary damping term with a delay:

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \dfrac{\partial u}{\partial \nu}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau), & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = g_0(x,t-\tau), & x \in \Omega, \tau \in (0,1), \end{cases} \tag{2.1.6}$$

and proved under the assumption

$$\mu_2 < \mu_1, \tag{2.1.7}$$

that the solution is exponentially stable. On the contrary, if (2.1.7) does not hold, they found a sequence of delays for which the corresponding solution of (2.1.6) will be unstable. We also recall the result by Xu et al. [106], where the authors proved the same result as in [84] for the one space dimension by adopting the spectral analysis approach.

The aim of this part of this work is to study the well-posedness and asymptotic stability of system (2.1.1)-(2.1.3) provided that (2.1.7) is satisfied. This chapter is organized as follows. The well-posedness of the problem is analyzed in Section 2.2 by two methods first we use the Galarkin method, second we use the semigroup theory. In Section 2.3, we prove the exponential decay of the energy when time goes to infinity.

2.2 Well-posedness of the problem

2.2.1 First Method : Galerkin Method

The existence of the solution

We begin first to obtaining the vartional formulation of the problem. Let $\varphi \in H^1(\Omega)$ and $\psi \in H^1(L_1, L_2)$. Multiplying the first equation of (2.1.1) by φ and the second equation by ψ and integrate the first equation obtained on Ω and the second equation obtained on $]L_1, L_2[$, we obtain

$$\int_\Omega u_{tt}(x,t)\varphi dx - a\int_\Omega u_{xx}(x,t)\varphi dx + \mu_1\int_\Omega u_t(x,t)\varphi dx + \mu_2\int_\Omega u_t(x,t-\tau)\varphi dx = 0,$$

and

$$\int_{L_1}^{L_2} v_{tt}(x,t)\psi dx - b\int_{L_1}^{L_2} v_{xx}(x,t)\psi dx = 0.$$

Integration by parts, and using (Green's formula) we obtain

$$\begin{aligned}
&\int_\Omega u_{tt}(x,t)\varphi dx + a\int_\Omega u_x(x,t)\varphi_x dx - a\left[u_x(L_1,t)\varphi(L_1) - u_x(L_2,t)\varphi(L_2)\right]\\
&+\mu_1\int_\Omega u_t(x,t)\varphi dx + \mu_2\int_\Omega u_t(x,t-\tau)\varphi dx = 0,\\
&\int_{L_1}^{L_2} v_{tt}(x,t)\psi dx + b\int_{L_1}^{L_2} v_x(x,t)\psi_x dx - b\left[v_x(L_2,t)\psi(L_2) - v_x(L_1,t)\psi(L_1)\right] = 0.
\end{aligned}$$

Now, we consider the function $\omega \in H_0^1(0, L_3)$ defined by $\omega = \begin{cases} u \text{ on } \Omega, \\ v \text{ on }]L_1, L_2[\,. \end{cases}$

and choose the test functions φ, ψ such that $\begin{cases} \varphi(L_1) = \psi(L_1), \\ \varphi(L_2) = \psi(L_2) \end{cases}$

The previous system becomes, by adding the two equations,

$$\begin{aligned}
&\int_\Omega \omega_{tt}(x,t)\varphi dx + a\int_\Omega \omega_x(x,t)\varphi_x dx + \mu_1\int_\Omega \omega_t(x,t)\varphi dx\\
&\mu_2\int_\Omega \omega_t(x,t-\tau)\varphi dx + \int_{L_1}^{L_2}\omega_{tt}(x,t)\psi dx + b\int_{L_1}^{L_2}\omega_x(x,t)\psi_x dx = 0.
\end{aligned}$$

So the the variational formulation of the problem takes the form

$$\begin{aligned}
&\int_\Omega \omega_{tt}(x,t)\varphi dx + a\int_\Omega \omega_x(x,t)\varphi_x dx + \mu_1\int_\Omega \omega_t(x,t)\varphi dx\\
&\mu_2\int_\Omega \omega_t(x,t-\tau)\varphi dx + \int_{L_1}^{L_2}\omega_{tt}(x,t)\psi dx + b\int_{L_1}^{L_2}\omega_x(x,t)\psi_x dx = 0,
\end{aligned}$$

with initial conditions

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, t \in [0,\tau], \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in]L_1, L_2[. \end{cases}$$

Resolution of the variational problem

Let $(e_j)_{j\geq 1}$ be a special base of $H_0^1(0,L_3)$, there exists a sequence $(g_j)_{j\geq 1}$ such that $\omega = \sum_{j=1}^{+\infty} g_j e_j$. Then, we approach $(\omega_n)_{n\geq 1}$ defined by $\omega_n = \sum_{j=1}^{n} g_j(t) e_j(x)$ for all $n \geq 1$ and satisfying the approximate varational equation as follows

$$\int_\Omega (\omega_n)_{tt}(x,t)\varphi dx + a\int_\Omega (\omega_n)_x(x,t)\varphi_x dx + \mu_1 \int_\Omega (\omega_n)_t(x,t)\varphi dx$$

$$\mu_2 \int_\Omega (\omega_n)_t(x,t-\tau)\varphi dx + \int_{L_1}^{L_2} (\omega_n)_{tt}(x,t)\psi dx + b\int_{L_1}^{L_2} (\omega_n)_x(x,t)\psi_x dx = 0, \quad (2.2.1)$$

where

$$\begin{cases} (u_n)(x,0) = u_{n0}(x) \longrightarrow u_0(x) \ , (u_n)_t(x,0) = u_{n1}(x) \longrightarrow u_1(x), & x \in \Omega, \\ (u_n)(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, t \in [0,\tau], \\ (v_n)(x,0) = v_{n0}(x) \longrightarrow v_0(x), \ \ (v_n)_t(x,0) = v_{n1}(x) \longrightarrow v_1(x), & x \in]L_1, L_2[. \end{cases} \quad (2.2.2)$$

System (2.2.1)-(2.2.2) is a system of ordinary differential equations which has, according to the classical theory of ODE , a local solution on $[0,t_n]$. for some $t_n > 0$.

Multiplying (2.2.1) by $\frac{dg_j}{dt}$ and, summing on $j = 1, ..., n$, Then, by taking $\varphi = \omega_t$ and $\psi = \omega_t$ and integrating over $(0,t)$, such that $t \leq T \leq +\infty$, we obtain

$$\frac{1}{2}\int_\Omega |(\omega_n)_t(x,t)|^2 dx - \frac{1}{2}\int_\Omega |(\omega_n)_t(x,0)|^2 dx + \frac{a}{2}\int_\Omega |(\omega_n)_x(x,t)|^2 dx$$

$$-\frac{a}{2}\int_\Omega |(\omega_n)_x(x,0)|^2 dx + \mu_1 \int_0^t \int_\Omega |(\omega_n)_t(x,s)|^2 dxds$$

$$+\mu_2 \int_0^t \int_\Omega (\omega_n)_t(x,s-\tau)(\omega_n)_t(x,s) dxds + \frac{1}{2}\int_{L_1}^{L_2} |(\omega_n)_t(x,t)|^2 dx$$

$$-\frac{1}{2}\int_{L_1}^{L_2} |(\omega_n)_t(x,0)|^2 dx + \frac{b}{2}\int_{L_1}^{L_2} |(\omega_n)_x(x,t)|^2 dx - \frac{b}{2}\int_{L_1}^{L_2} |(\omega_n)_x(x,0)|^2 dx = 0.$$

Then, using the following Poincaré's inequality

Poincare's Inequality. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. There exist a positive constant C_Ω such that

$$\|u\|_{H_0^1(\Omega)} \leq C_\Omega \|\nabla\|_{L(\Omega)^2}, \forall u \in H_0^1(\Omega),$$

we obtain

$$\frac{1}{2}\left\|(\omega_n)_t(t)\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|u_{n1}\right\|^2_{L^2(\Omega)} + \frac{C_0}{2}\left\|(\omega_n)(t)\right\|^2_{H_0^1(0,L_3)} - \frac{a}{2}\int\limits_\Omega \left|(\omega_n)_x(x,0)\right|^2 dx$$
$$+\mu_1 \int\limits_0^t \left\|(\omega_n)_t(x,s)\right\|^2_{L^2(\Omega)} ds + \mu_2 \int\limits_0^t \int\limits_\Omega (\omega_n)_t(x,s-\tau)\,\omega_t(x,s)\,dxds$$
$$+\frac{1}{2}\left\|(\omega_n)_t(t)\right\|^2_{L^2(L_1,L_2)} - \frac{1}{2}\left\|u_{n1}\right\|^2_{L^2(L_1,L_2)} - \frac{b}{2}\int\limits_{L_1}^{L_2} \left|(\omega_n)_x(x,0)\right|^2 dx \leq 0,$$

such that $C_0 = \min(a,b)$.

Therfore, we deduce that

$$\left\|(w_n)_t(t)\right\|^2_{L^2(\Omega)} + C_0\left\|(\omega_n)(t)\right\|^2_{H_0^1(0,L_3)} + \left\|(v_n)_t(t)\right\|^2_{L^2(L_1,L_2)} + 2\mu_1 \int\limits_0^t \left\|(u_n)_t(s)\right\|^2_{L^2(\Omega)} ds$$
$$\leq \left\|u_{n1}\right\|^2_{L^2(\Omega)} + \left\|v_{n1}\right\|^2_{L^2(L_1,L_2)} + \left\|u_{n0}\right\|^2_{H^1(\Omega)} + \left\|v_{n0}\right\|^2_{H^1(L_1,L_2)}$$
$$-2\mu_2 \int\limits_0^t \int\limits_\Omega (\omega_n)_t(x,s-\tau)\,(\omega_n)_t(x,s)\,dxds. \tag{2.2.3}$$

Moreover,

$$\int\limits_0^t \int\limits_\Omega (\omega_n)_t(x,s-\tau)\,(\omega_n)_t(x,s)\,dxds$$
$$= \int\limits_0^\tau \int\limits_\Omega (u_n)_t(x,s-\tau)\,(u_n)_t(x,s)\,dxds + \int\limits_\tau^t \int\limits_\Omega (u_n)_t(x,s-\tau)\,(u_n)_t(x,s)\,dxds$$
$$= \int\limits_0^\tau \int\limits_\Omega (f_0)_t(x,s-\tau)\,(u_n)_t(x,s)\,dxds$$
$$+ \int\limits_\tau^t \int\limits_\Omega (u_n)_t(x,s-\tau)\,(u_n)_t(x,s)\,dxds. \tag{2.2.4}$$

We consider the new variable $\alpha = s - \tau$ in the second term of the right-side of the equation (2.2.4), we obtain

$$\int_0^t \int_\Omega (\omega_n)_t (x, s-\tau) (\omega_n)_t (x,s) \, dxds = \int_0^\tau \int_\Omega (u_n)_t (x, s-\tau) (u_n)_t (x,s) \, dxds$$
$$+ \int_\tau^t \int_\Omega (u_n)_t (x, s-\tau) (u_n)_t (x,s) \, dxds$$
$$= \int_0^\tau \int_\Omega (f_0)_t (x, s-\tau) (u_n)_t (x,s) \, dxds + \int_0^{t-\tau} \int_\Omega u_t (x,\alpha) (u_n)_t (x, \alpha+\tau) \, dxd\alpha$$
$$\leq \int_0^\tau \|(f_0)_t (s-\tau)\|_{L^2(\Omega)} \|(u_n)_t (s)\|_{L^2(\Omega)} \, ds$$
$$+ \int_0^{t-\tau} \|(u_n)_t (\alpha)\|_{L^2(\Omega)} \|(u_n)_t (\alpha+\tau)\|_{L^2(\Omega)} \, d\alpha$$

By using Hölder's inequality we get

$$\int_0^\tau \|(f_0)_t (s-\tau)\|_{L^2(\Omega)} \|(u_n)_t (s)\|_{L^2(\Omega)} \, ds$$
$$+ \int_0^{t-\tau} \|(u_n)_t (\alpha)\|_{L^2(\Omega)} \|(u_n)_t (\alpha+\tau)\|_{L^2(\Omega)} \, d\alpha$$
$$\leq \left(\|(f_0)_t (s-\tau)\|_{L^2(\Omega)} \right)^{\frac{1}{2}} \left(\int_0^\tau \|(u_n)_t (s)\|^2_{L^2(\Omega)} \, ds \right)^{\frac{1}{2}}$$
$$+ \int_0^t \|(u_n)_t (\alpha)\|^2_{L^2(\Omega)} \, d\alpha$$

By using Hölder's inequality for second time we obtain

$$\left(\|(f_0)_t (s-\tau)\|_{L^2(\Omega)} \right)^{\frac{1}{2}} \left(\int_0^\tau \|(u_n)_t (s)\|^2_{L^2(\Omega)} \, ds \right)^{\frac{1}{2}} + \int_0^t \|(u_n)_t (\alpha)\|^2_{L^2(\Omega)} \, d\alpha$$
$$\leq \left(\|(f_0)_t (s-\tau)\|_{L^2(\Omega)} \right)^{\frac{1}{2}} \left(\int_0^\tau \|(u_n)_t (s)\|^2_{L^2(\Omega)} \, ds \right)^{\frac{1}{2}}$$
$$+ \left(\int_0^t d\alpha \right)^{\frac{1}{2}} \left(\int_0^t \|(u_n)_t (\alpha)\|^2_{L^2(\Omega)} \, d\alpha \right)^{\frac{1}{2}}$$

By using Young's Inégalité we deduce

$$\left(\left\|(f_0)_t(s-\tau)\right\|_{L^2(\Omega)}\right)^{\frac{1}{2}}\left(\int_0^{\tau}\left\|(u_n)_t(s)\right\|^2_{L^2(\Omega)}ds\right)^{\frac{1}{2}}$$

$$+\left(\int_0^t d\alpha\right)^{\frac{1}{2}}\left(\int_0^t\left\|(u_n)_t(\alpha)\right\|^2_{L^2(\Omega)}d\alpha\right)^{\frac{1}{2}}$$

$$\leq c\left\|(f_0)_t\right\|_{L^2(\Omega\times[0,\tau])}+c\int_0^{\tau}\left\|(u_n)_t(s)\right\|^2_{L^2(\Omega)}ds+c. \tag{2.2.5}$$

By assuming that

$$\begin{cases} u_1\in L^2(\Omega),\\ v_1\in L^2(L_1,L_2),\\ u_0\in H^1(\Omega),\\ v_0\in H^1(L_1,L_2),\\ (f_0)_t\in L^2(\Omega\times[0,\tau]). \end{cases} \tag{2.2.6}$$

Then, from (2.2.1), (2.2.5) and (2.2.6), we deduce

$$\left\|(u_n)_t(t)\right\|^2_{L^2(\Omega)}+C_0\left\|(\omega_n)(t)\right\|^2_{H_0^1(0,L_3)}+\left\|(v_n)_t(t)\right\|^2_{L^2(L_1,L_2)}+2\mu_1\int_0^t\left\|(u_n)_t(s)\right\|^2_{L^2(\Omega)}ds$$

$$\leq c+c\int_0^t\left\|(u_n)_t(s)\right\|^2_{L^2(\Omega)}ds. \tag{2.2.7}$$

Moreover, we have in particular;

$$\left\|(u_n)_t(t)\right\|^2_{L^2(\Omega)}\leq c+c\int_0^t\left\|(u_n)_t(s)\right\|^2_{L^2(\Omega)}ds.$$

Which leads via the Gronwal lemma to

$$\left\|(u_n)_t(t)\right\|_{L^2(\Omega)}\leq c. \tag{2.2.8}$$

Then, by (2.2.7) and (2.2.8), we obtain for each $t\leq t_n$

$$\begin{aligned} &(u_n)\in L^\infty\left(0,T;H^1(\Omega)\right),\\ &(u_n)_t\in L^\infty\left(0,T;L^2(\Omega)\right),\\ &(v_n)\in L^\infty\left(0,T;H^1(L_1,L_2)\right),\\ &(v_n)_t\in L^\infty\left(0,T;L^2(L_1,L_2)\right). \end{aligned} \tag{2.2.9}$$

and the solution can be extended on the interval $[0, T]$ such that $0 \le t \le t_n \le T \le +\infty$.

From (2.2.9) we can extract a subsequence still denoted (u_n) such that :

$$\begin{array}{llll}
(u_n) \longrightarrow u & \text{in} & L^\infty\Big(0, T; H^1(\Omega)\Big) & \text{weak stare,} \\
(u_n)_t \longrightarrow u_t & \text{in} & L^\infty\Big(0, T; L^2(\Omega)\Big) & \text{weak stare,} \\
(v_n) \longrightarrow v & \text{in} & L^\infty\Big(0, T; H^1(L_1, L_2)\Big) & \text{weak stare,} \\
(v_n)_t \longrightarrow v_t & \text{in} & L^\infty\Big(0, T; L^2(L_1, L_2)\Big) & \text{weak stare.}
\end{array}$$

This allows us to pass to the limit in to deduce the existence of a weak solution (u, v) of (2.1.1)-(2.1.3) which has the regularity

$$\begin{aligned}
& u \in L^\infty\Big(0, T; H^1(\Omega)\Big), \\
& u_t \in L^\infty\Big(0, T; L^2(\Omega)\Big), \\
& v \in L^\infty\Big(0, T; H^1(L_1, L_2)\Big), \\
& v_t \in L^\infty\Big(0, T; L^2(L_1, L_2)\Big).
\end{aligned} \tag{2.2.10}$$

2.2.2 Second Method : The semi-group theory

In this section, we prove the local existence and the uniqueness of the solution of system (2.1.1)-(2.1.3) by using the semi-group theory. So let us introduce the following new variable [84]

$$y(x, \rho, t) = u_t(x, t - \tau\rho). \tag{2.2.11}$$

Then, we get

$$\tau y_t(x, \rho, t) + y_\rho(x, \rho, t) = 0, \qquad \text{in } \Omega \times (0, 1) \times (0, +\infty). \tag{2.2.12}$$

Therefore, problem (2.1.1) is equivalent to

$$\begin{cases}
u_{tt}(x, t) - a u_{xx}(x, t) + \mu_1 u_t(x, t) + \mu_2 y(x, 1, t) = 0, & (x, t) \in \Omega \times]0, +\infty[\\
v_{tt}(x, t) - b v_{xx}(x, t) = 0, & (x, t) \in]L_1, L_2[\times]0, +\infty[\\
\tau y_t(x, \rho, t) + y_\rho(x, \rho, t) = 0, & \text{in } \Omega \times (0, 1) \times (0, +\infty),
\end{cases} \tag{2.2.13}$$

which together with (2.1.3) can be rewritten as:

$$\begin{cases}
U' = \mathscr{A} U, \\
U(0) = (u_0, v_0, u_1, v_1, f_0(., -.\tau))^{\mathrm{T}},
\end{cases} \tag{2.2.14}$$

where the operator $\mathscr{A}$ is defined by

$$\mathscr{A}\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ y \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ au_{xx} - \mu_1\varphi - \mu_2 y(.,1) \\ bv_{xx} \\ -\frac{1}{\tau}y_\rho, \end{pmatrix} \tag{2.2.15}$$

with the domain

$$D(\mathscr{A}) = \left\{ \ (u,v,\varphi,\psi,y)^{\mathrm{T}} \in \mathscr{H}; \quad y(.,0) = \varphi \text{ on } \Omega \ \right\},$$

where

$$\mathscr{H} = \left\{\left(H^2(\Omega)\times H^2(L_1,L_2)\right)\cap X_*\right\}\times H^1(\Omega)\times H^1(L_1,L_2)\times L^2(0,1,H^1(\Omega)).$$

Here the space X_* is defined by

$$X_* = \left\{ \begin{array}{l} (u,v)\in H^1(\Omega)\cap H^1(L_1,L_2)|u(0,t)=u(L_3,t)=0, \\ u\left(L_i,t\right)=v\left(L_i,t\right),\ au_x\left(L_i,t\right)=bv_x\left(L_i,t\right),\ i=1,2. \end{array} \right\}$$

Now the energy space is defined by

$$\mathscr{K} = X_*\times L^2(\Omega)\times L^2(L_1,L_2)\times L^2((\Omega)\times(0,1)).$$

Let

$$U=(u,v,\varphi,\psi,y)^{\mathrm{T}}, \quad \overline{U}=(\overline{u},\overline{v},\overline{\varphi},\overline{\psi},\overline{y})^{\mathrm{T}}.$$

Then, for a positive constant ζ satisfying

$$\tau\mu_2 \le \zeta \le \tau(2\mu_1-\mu_2), \tag{2.2.16}$$

we define the inner product in $\mathscr{K}$ as follows:

$$(U,\overline{U})_{\mathscr{K}} = \int_\Omega\{\varphi\overline{\varphi}+au_x\overline{u}_x\}dx + \int_{L_1}^{L_2}\{\psi\overline{\psi}+bv_x\overline{v_x}\}dx + \zeta\int_\Omega\int_0^1 y(x,\rho)\overline{y}(x,\rho)d\rho dx.$$

The existence and uniqueness result is stated as follows;

Theorem 2.2.1. *For any $U_0\in\mathscr{K}$ there exists a unique solution $U\in C([0,+\infty[,\mathscr{K})$ of problem (2.2.14). Moreover, if $U_0\in D(\mathscr{A})$, then*

$$U\in C([0,+\infty[,D(\mathscr{A}))\cap C^1([0,+\infty[,\mathscr{K}).$$

Proof. In order to prove the result stated in Theorem 2.2.1, we use the semigroup theory, that is, we show that the operator $\mathscr{A}$ generates a C_0-semigroup in $\mathscr{K}$. In this step, we concern ourselves to prove that the operator $\mathscr{A}$ is dissipative. Indeed, for $U = (u, \varphi, v, \psi, y)^{\mathrm{T}} \in D(\mathscr{A})$, where $\varphi(L_2) = \psi(L_2)$ and ζ is a positive constant, we have

$$\begin{aligned}(\mathscr{A}U, U)_{\mathscr{K}} &= a\int_\Omega u_{xx}\varphi dx + b\int_{L_1}^{L_2} v_{xx}\psi dx - \mu_1\int_\Omega \varphi^2 dx\\ &\quad -\mu_2\int_\Omega y(.,1)\varphi dx - \frac{\zeta}{\tau}\int_\Omega\int_0^1 y(x,\rho)y_\rho(x,\rho)d\rho dx \\ &\quad +a\int_\Omega u_x\varphi_x dx + b\int_{L_1}^{L_2} v_x\psi_x dx.\end{aligned} \tag{2.2.17}$$

Looking now at the last term of the right-hand side of (2.2.17), we have

$$\begin{aligned}\zeta\int_\Omega\int_0^1 y(x,\rho)y_\rho(x,\rho)d\rho dx &= \zeta\int_\Omega \frac{1}{2}\frac{\partial}{\partial\rho}y^2(x,\rho)d\rho dx\\ &= \frac{\zeta}{2}\int_\Omega (y^2(x,1) - y^2(x,0))dx.\end{aligned} \tag{2.2.18}$$

Performing an integration by parts in (2.2.17), keeping in mind the fact that $y(x,0,t) = \varphi(x,t)$ and using (2.2.18), we have from (2.2.17)

$$\begin{aligned}(\mathscr{A}U, U)_{\mathscr{K}} &= a[u_x\varphi]_{\partial\Omega} + b[v_x\psi]_{L^1}^{L_2} \\ &\quad -\left(\mu_1 - \frac{\zeta}{2\tau}\right)\int_\Omega \varphi^2 dx - \mu_2\int_\Omega y(.,1)\varphi dx - \frac{\zeta}{2\tau}\int_\Omega y^2(x,1)dx.\end{aligned} \tag{2.2.19}$$

Using Young's inequality, (2.1.2) and the equality $\varphi(L_2) = \psi(L_2)$, we obtain from (2.2.19), that

$$(\mathscr{A}U, U)_{\mathscr{K}} \le -\left(\mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_\Omega \varphi^2 dx - \left(\frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_\Omega y^2(x,1)dx. \tag{2.2.20}$$

Consequently, using (2.2.16), then we deduce that $(\mathscr{A}U, U)_{\mathscr{K}} \le 0$. Thus, the operator $\mathscr{A}$ is dissipative.

Now to show that the operator $\mathscr{A}$ is maximal monotone, it is sufficient to show that the operator $\lambda I - \mathscr{A}$ is surjective for a fixed $\lambda > 0$. Indeed, given $(f_1, f_2, g_1, g_2, h)^{\mathrm{T}} \in \mathscr{K}$, we seek $U = (u, v, \varphi, \psi, y)^{\mathrm{T}} \in D(\mathscr{A})$ solution of

$$\begin{pmatrix} \lambda u - \varphi \\ \lambda v - \psi \\ \lambda\varphi - au_{xx} + \mu_1 y(.,0) + \mu_2 y(.,1) \\ \lambda\psi - bv_{xx} \\ \lambda y + \frac{1}{\tau}y_\rho \end{pmatrix} = \begin{pmatrix} f_1 \\ g_1 \\ f_2 \\ g_2 \\ h \end{pmatrix} \tag{2.2.21}$$

suppose we have find (u, v) with the appropriate regularity, then

$$\begin{aligned} \varphi &= \lambda u - f_1 \\ \psi &= \lambda v - g_1. \end{aligned} \tag{2.2.22}$$

It is clear that $\varphi \in H^1(\Omega)$ and $\psi \in H^1(L_1, L_2)$, furthermore, by (2.2.21), we can find y as $y(x, 0) = \varphi(x)$, $x \in \Omega$, using the approach as in Nicaise & Pignotti [84], we obtain, by using the equation in (2.2.21)

$$y(x, \rho) = \varphi(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau} \int_0^\rho h(x, \sigma)e^{\lambda\sigma\tau} d\sigma$$

From (2.2.22), we obtain

$$y(x, \rho) = \lambda u(x)e^{-\lambda\rho\tau} - f_1(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau} \int_0^\rho h(x, \sigma)e^{\lambda\sigma\tau} d\sigma$$

By using (2.2.21) and (2.2.22), the functions u, v satisfying the following equations:

$$\begin{aligned} \lambda^2 u - a u_{xx} + \mu_1 y(., 0) + \mu_2 y(., 1) &= f_2 + \lambda f_1 \\ \lambda^2 v - b v_{xx} &= g_2 + \lambda g_1 \end{aligned} \tag{2.2.23}$$

Since

$$\begin{aligned} y(x, 1) &= \varphi(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^1 h(x, \sigma)e^{\lambda\tau} d\sigma \\ &= \lambda u e^{-\lambda\tau} + y_0(x), \end{aligned}$$

for $x \in \Omega$, we have

$$y_0(x) = -f_1(x) + \tau e^{-\lambda\tau} \int_0^1 h(x, \sigma)e^{\lambda\tau} d\sigma$$

The problem (2.2.23) can be reformulated as

$$\begin{aligned} &\int_\Omega (\lambda^2 u - a u_{xx} + \mu_1 \lambda u + +\mu_2 \lambda u e^{-\lambda\tau})\omega_1 dx \\ = &\int_\Omega (f_2 + \lambda f_1 - \mu_2 \lambda y_0(x))\omega_1 dx, \\ &\int_{L_1}^{L_2} (\lambda^2 v - b v_{xx})\omega_2 dx \\ = &\int_{L_1}^{L_2} (g_2 + \lambda g_1)\omega_2 dx, \end{aligned} \tag{2.2.24}$$

for any $(\omega_1, \omega_2) \in X_*$.

Integrating the first equation in (2.2.24) by parts, we obtain

$$\begin{aligned}
&\int_\Omega (\lambda^2 u - a u_{xx} + \mu_1 u + +\mu_2 \lambda u e^{-\lambda\tau})\omega_1 dx \\
= &\int_\Omega \lambda^2 u\omega_1 dx - a\int_\Omega u_{xx}\omega_1 dx + \mu_1 \int_\Omega \lambda u dx + \mu_2 \int_\Omega \lambda u e^{-\lambda\tau}\omega_1 dx \\
= &\int_\Omega \lambda^2 u\omega_1 dx + a\int_\Omega u_x(\omega_1)_x dx - [a u_x \omega_1]_{\partial\Omega} + \mu_1 \int_\Omega \lambda u dx + \mu_2 \int_\Omega \lambda u e^{-\lambda\tau}\omega_1 dx \\
= &\int_\Omega (\lambda^2 + \mu_1\lambda + \mu_2\lambda e^{-\lambda\tau}) u\omega_1 dx + a\int_\Omega u_x(\omega_1)_x dx - [a u_x\omega_1]_{\partial\Omega} \qquad (2.2.25)
\end{aligned}$$

Integrating the second equation in (2.2.24) by parts, we obtain

$$\int_{L_1}^{L_2} (\lambda^2 v - b v_{xx})\omega_2 dx = \int_{L_1}^{L_2} \lambda^2 v\ \omega_2 dx + b\int_{L_1}^{L_2} v_x(\omega_2)_x dx - [b v_x\omega_2]_{L_1}^{L_2}. \qquad (2.2.26)$$

Using (2.2.25) and (2.2.26), the problem (2.2.24) is equivalent to the problem

$$\Phi((u,v),(\omega_1,\omega_2)) = l(\omega_1,\omega_2) \qquad (2.2.27)$$

where the bilinear form $\Phi : (X_* \times X_*) \to \mathbb{R}$ and the linear form $l : X_* \to \mathbb{R}$ are defined by

$$\begin{aligned}
\Phi((u,v),(\omega_1,\omega_2)) = &\int_\Omega (\lambda^2 + \mu_1\lambda + \mu_2\lambda e^{-\lambda\tau}) u\omega_1 dx + a\int_\Omega u_x(\omega_1)_x dx - [a u_x\omega_1]_{\partial\Omega} \\
&+ \int_{L_1}^{L_2} \lambda^2 v\ \omega_2 dx + b\int_{L_1}^{L_2} v_x(\omega_2)_x dx - [b v_x\omega_2]_{L_1}^{L_2}
\end{aligned}$$

and

$$l(\omega_1,\omega_2) = \int_\Omega (f_2 + \lambda f_1 - \mu_2\lambda y_0(x))\omega_1 dx + \int_{L_1}^{L_2} (g_2 + \lambda g_1)\omega_2 dx$$

Using the properties of the space X_*, it is clear that Φ is continuous and coerceive, and l is continuous. So applyiyg the Lax-Milgram theorem, we deduce that for all $(\omega_1,\omega_2) \in X_*$, problem (2.2.27) admits a unique solution $(u,v) \in X_*$. It follows from (2.2.25) and (2.2.26) that $(u,v) \in \left\{\left(H^2(\Omega) \times H^2(L_1,L_2)\right) \cap X_*\right\}$. Therefore, the operator $\lambda I - \mathscr{A}$ is dissipative for any $\lambda > 0$. Then the result in Theorem 2.2.1 follows from the Hille-Yoshida theorem. $\square$

2.3 Exponential decay of the solution

In this section we investigate the asymptotic of the system (2.1.1)-(2.1.3). For any regular solution of (2.1.1)-(2.1.3), we define the energy as:

$$E_1(t) = \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{a}{2}\int_\Omega u_x^2(x,t)dx \qquad (2.3.1)$$

and

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)dx. \tag{2.3.2}$$

The total energy is defined as:

$$E(t) = E_1(t) + E_2(t) + \frac{\zeta}{2}\int_\Omega \int_0^1 y^2(x,\rho,t)d\rho dx \tag{2.3.3}$$

where ζ is the positive constant defined in (2.2.16).

Our decay result reads as follows:

Theorem 2.3.1. *Let (u,v) be the solution of (2.1.1)-(2.1.3). Assume that $\mu_2 < \mu_1$ and*

$$\frac{a}{b} < \frac{L_3 + L_1 - L_2}{2(L_2 - L_1)}. \tag{2.3.4}$$

Then there exist two positive constantes C and d such that

$$E(t) \leq Ce^{-dt}, \qquad \forall t \geq 0. \tag{2.3.5}$$

Remark 2.3.2. The assumption (2.3.4) gives the relationship between the boundary regions and the transmission permitted. It can be also seen as a restriction on the wave speeds of the two equations and the damped part of the domain. It is known that for Timoshenko systems [105] and Bresse systems [1] that the wave speeds always control the decay rate of the solution. It is an interesting open question to show the behavior of the solution if (2.3.4) is not satisfied.

The proof of Theorem 2.3.1 will be done through somme lemmas;

Lemma 2.3.3. *Let (u,v,y) be the solution of (2.2.13), (2.1.3). Assume that $\mu_1 \geq \mu_2$. Then we have the following inequality*

$$\frac{dE(t)}{dt} \leq \left(-\mu_1 + \frac{\mu_2}{2} + \frac{\zeta}{2\tau}\right)\int_\Omega y^2(x,0,t)dx + \left(\frac{\mu_2}{2} - \frac{\zeta}{2\tau}\right)\int_\Omega y^2(x,1,t)dx. \tag{2.3.6}$$

Proof. We have from (2.3.3) that

$$\frac{dE_1(t)}{dt} = \int_\Omega u_{tt}(x,t)u_t(x,t)dx + a\int_\Omega u_{xt}(x,t)u_x(x,t)dx \tag{2.3.7}$$

Using system (2.2.13), and integrating by parts, we obtain

$$\frac{dE_1(t)}{dt} \quad = \quad a[u_x u_t]_{\partial\Omega} - \mu_1 \int_\Omega u_t^2\,(x,t) - \mu_2\int_\Omega u_t(x,t)y\,(x,1,t))dx \tag{2.3.8}$$

On the other hand, we have

$$\frac{dE_2(t)}{dt} = b[v_x v_t]_{L_1}^{L_2}. \tag{2.3.9}$$

Using the fact that

$$\begin{aligned}\frac{d}{dt}\frac{\zeta}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx &= \zeta\int_\Omega\int_0^1 y(x,\rho,t)y_t(x,\rho,t)d\rho dx\\ &= -\frac{\zeta}{\tau}\int_\Omega\int_0^1 y_\rho(x,\rho,t)y(x,\rho,t)d\rho dx\\ &= -\frac{\zeta}{2\tau}\int_\Omega\int_0^1 \frac{d}{d\rho}y^2(x,\rho,t)d\rho dx\\ &= -\frac{\zeta}{2\tau}\int_\Omega(y^2(x,1,t)-y^2(x,0,t))dx \qquad (2.3.10)\end{aligned}$$

Collecting (2.3.8), (2.3.9), (2.3.10), using (2.1.2and applying Young's inequality, then (2.3.6) holds. Thus, the proof of the Lemma 2.3.3 is completed. □

Following [4], we define the functional

$$I(t) = \int_\Omega\int_{t-\tau}^t e^{s-t}u_t^2(x,s)dsdx$$

and we have the following lemma.

Lemma 2.3.4. *Let (u,v) be the solution of (2.1.1)-(2.1.3). Then we have*

$$\frac{dI(t)}{dt} \le \int_\Omega u_t^2(x,t)dx - e^{-\tau}\int_\Omega u_t^2(x,t-\tau)dx - e^{-\tau}\int_\Omega\int_{t-\tau}^t u_t^2(x,s)dsdx. \tag{2.3.11}$$

The proof of Lemma 2.3.4 is straightforward, we omit the details.

Now, we define the functional $\mathscr{D}(t)$ as follows:

$$\mathscr{D}(t) = \int_\Omega uu_t dx + \frac{\mu_1}{2}\int_\Omega u^2 dx + \int_{L_1}^{L_2} vv_t dx. \tag{2.3.12}$$

Thus, we have the following estimate.

Lemma 2.3.5. *The functional $\mathscr{D}(t)$ satisfies the following estimate:*

$$\begin{aligned}\frac{d}{dt}\mathscr{D}(t) &\le -(a-\epsilon_0 c_0^2)\int_\Omega u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx\\ &\quad + \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx + C(\epsilon_0)\int_\Omega y^2(x,1,t)dx\end{aligned}$$

Proof. Taking the derivative of $\mathscr{D}(t)$ with respect to t and exploiting (2.1.1), we find

$$\begin{aligned} \frac{d}{dt}\mathscr{D}(t) &= \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx - a\int_\Omega u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx \\ &\quad -\mu_2 \int_\Omega u(x,t)y(x,1,t)dx + [au_x u]_{\partial\Omega} + [bv_x v]_{L_1}^{L_2}. \end{aligned} \tag{2.3.13}$$

Applying young's inequality and using the boundary conditions (2.1.2), we have

$$\begin{aligned} [au_x u]_{\partial\Omega} + [bv_x v]_{L_1}^{L_2} &= au_x(L_1,t)u(L_1,t) - au_x(L_2,t)u(L_2,t) \\ &\quad + bv_x(L_2,t)v(L_2,t) - bv_x(L_1,t)v(L_1,t) = 0. \end{aligned} \tag{2.3.14}$$

On the other hand, we have by Poincaré's inequality and Young's inequality

$$\mu_2 \int_\Omega u(x,t)y(x,1,t)dx \leq \epsilon_0 c_0^2 \int_\Omega u_x^2 dx + C(\epsilon_0)\int_\Omega y^2(x,1,t)dx \tag{2.3.15}$$

where ϵ_0 is a positive constants and c_0 is the Poincaé's constant. Consequently, plugging the above estimates into (2.3.13), we find (2.3.13). □

Now, inspired by [73], we introduce the functional

$$q(x) = \begin{cases} x - \dfrac{L_1}{2}, & x \in [0, L_1], \\ x - \dfrac{L_2 + L_3}{2}, & x \in [L_2, L_3], \\ \dfrac{L_2 - L_3 - L_1}{2(L_2 - L_1)}(x - L_1) + \dfrac{L_1}{2}, & x \in [L_1, L_2] \end{cases} \tag{2.3.16}$$

Next, we define the following functionals

$$\mathscr{F}_1(t) = -\int_\Omega q(x) u_x u_t dx$$

and

$$\mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x) v_x v_t dx.$$

Then, we have the following estimates:

Lemma 2.3.6. *For any $\epsilon_2 > 0$, we have the estimates:*

$$\begin{aligned} \frac{d}{dt}\mathscr{F}_1(t) &\leq C(\epsilon_2)\int_\Omega u_t^2 dx + \left(\frac{a}{2} + \epsilon_2\right)\int_\Omega u_x^2 dx + C(\epsilon_2)\int_\Omega y^2(x,1,t)\,dx \\ &\quad - \frac{a}{4}\left[(L_3 - L_2)u_x^2(L_2,t) + L_1 u_x^2(L_1,t)\right] \end{aligned} \tag{2.3.17}$$

and

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_2(t) &\leq \frac{L_2-L_3-L_1}{4\left(L_2-L_1\right)}\left(\int_{L_1}^{L_2} v_t^2 dx+\int_{L_1}^{L_2} b v_x^2 dx\right) \\ &+\frac{b}{4}\left(\left(L_3-L_2\right) v_x^2\left(L_2, t\right)+L_1 v_x^2\left(L_1, t\right)\right). \qquad (2.3.18)\end{aligned}$$

Proof. Taking the derivative of $\mathscr{F}_1(t)$ with respect to t and using equation (2.1.1), we get

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) &= -\int_\Omega q(x) u_{tx} u_t dx-\int_\Omega q(x) u_x u_{tt} dx \qquad (2.3.19)\\ &= -\int_\Omega q(x) u_{tx} u_t dx-\int_\Omega q(x) u_x\left(a u_{xx}(x, t)-\mu_1 u_t(x, t)-\mu_2 y(x, 1, t)\right) dx.\end{aligned}$$

Using integration by parts, we find

$$\int_\Omega q(x) u_{tx} u_t dx=-\frac{1}{2} \int_\Omega q'(x) u_t^2 dx+\frac{1}{2}\left[q(x) u_t^2\right]_{\partial\Omega}. \qquad (2.3.20)$$

On the other hand, we have

$$\int_\Omega a q(x) u_{xx} u_x dx=-\frac{1}{2} \int_\Omega a q'(x) u_x^2 dx+\frac{1}{2}\left[a q(x) u_x^2\right]_{\partial\Omega}. \qquad (2.3.21)$$

Inserting (2.3.20) and (2.3.21) into (2.3.19), we find

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) &= \frac{1}{2} \int_\Omega q'(x) u_t^2 dx+\frac{1}{2} \int_\Omega a q'(x) u_x^2 dx-\frac{1}{2}\left[q(x) u_t^2\right]_{\partial\Omega}-\frac{1}{2}\left[a q(x) u_x^2\right]_{\partial\Omega} \\ &+\int_\Omega q(x) u_x\left(\mu_1 u_t(x, t)+\mu_2 y(x, 1, t)\right) dx. \qquad (2.3.22)\end{aligned}$$

Exploiting Young's inequality and using (3.3.6), then (2.3.22) becomes (2.3.19), we find

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) &\leq C(\epsilon_2) \int_\Omega u_t^2 dx+\left(\frac{a}{2}+\epsilon_2\right) \int_\Omega u_x^2 dx-\frac{1}{2}\left[q(x) u_t^2\right]_{\partial\Omega}-\frac{a}{2}\left[q(x) u_x^2\right]_{\partial\Omega} \\ &+C(\epsilon_2) \int_\Omega y^2(x, 1, t) dx. \qquad (2.3.23)\end{aligned}$$

for any $\epsilon_2>0$. Since $q(L_1)>0$ and $q(L_2)<0$, we have by using the boundary conditions (2.1.2), that

$$\frac{1}{2}\left[q(x) u_t^2\right]_{\partial\Omega} \geq 0. \qquad (2.3.24)$$

Also, we have

$$\begin{aligned}-\frac{a}{2}\left[q(x) u_x^2\right]_{\partial\Omega} = &-\frac{a L_1}{4}\left[u_x^2(L_1, t)+u_x^2(0, t)\right] \qquad (2.3.25)\\ &-\frac{a(L_3-L_2)}{4}\left[u_x^2(L_3, t)+u_x^2(L_2, t)\right].\end{aligned}$$

Taking into account (2.3.24) and (2.3.25), then (2.3.23) gives (2.3.17).

By the same method, taking the derivative of $\mathscr{F}_2(t)$ with respect to t, we get

$$\begin{aligned} \frac{d}{dt}\mathscr{F}_2(t) &= -\int_{L_1}^{L_2} q(x)v_{tx}v_t dx - \int_{L_1}^{L_2} q(x)v_x v_{tt} \\ &= \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2 dx - \frac{1}{2}\left[q(x)v_t^2\right]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2 dx - \frac{b}{2}\left[q(x)v_x^2\right]_{L_1}^{L_2} \\ &\leq \frac{L_2-L_3-L_1}{4\left(L_2-L_1\right)}\left(\int_{L_1}^{L_2} v_t^2 dx + \int_{L_1}^{L_2} bv_x^2 dx\right) \\ &\quad + \frac{b}{4}\Big((L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t).\Big) \end{aligned} \tag{2.3.26}$$

which is exactly (2.3.18). □

Proof Theorem 2.3.1. We define the Lyapunov functional $\mathscr{L}(t)$ as follows

$$\mathscr{L}(t) = NE(t) + I(t) + \gamma_2\mathscr{D}(t) + \gamma_3\mathscr{F}_1(t) + \gamma_4\mathscr{F}_2(t) \tag{2.3.27}$$

where N, γ_2, γ_3 and γ_4 are positive constants that will be fixed later.

Now, it is clear from the boundary conditions (2.1.2), that

$$a^2u_x^2(L_i,t) = b^2v_x^2(L_i,t), \qquad i = 1, 2. \tag{2.3.28}$$

Taking the derivative of (2.3.27) with respect to t and making use of (2.3.6), (2.3.11), (2.3.13), (2.3.17), and taking into account (2.3.28), we obtain

$$\begin{aligned} \frac{d}{dt}\mathscr{L}(t) \leq\ & \left\{N\left(-\mu_1 + \frac{\mu_2}{2} + \frac{\zeta}{2\tau}\right) + 1 + \gamma_2 + \gamma_3 C(\epsilon_2)\right\}\int_\Omega u_t^2 dx \\ & + \left\{N\left(\frac{\mu_2}{2} - \frac{\zeta}{2\tau}\right) - e^{-\tau} + \gamma_2 C(\epsilon_0) + C(\epsilon_2)\gamma_3\right\}\int_\Omega y^2(x,1,t)dx \\ & + \left\{\gamma_2(-a + \epsilon_0 c_0^2) + \gamma_3\epsilon_2 + \frac{\gamma_3 a}{2}\right\}\int_\Omega u_x^2 dx \\ & + \left\{b\frac{L_2-L_3-L_1}{4(L_2-L_1)}\gamma_4 - \gamma_2 b\right\}\int_{L_1}^{L_2} v_x^2 dx \\ & + \left\{\frac{L_2-L_3-L_1}{4(L_2-L_1)}\gamma_4 + \gamma_2\right\}\int_{L_1}^{L_2} v_t^2 dx - e^{-\tau}\int_\Omega\int_{t-\tau}^{t} u_t^2(x,s)dsdx \\ & - \left(\gamma_3 - \frac{a}{b}\gamma_4\right)\frac{a(L_3-L_2)}{4}u_x^2(L_2,t) - \left(\gamma_3 - \frac{a}{b}\gamma_4\right)\frac{aL_1}{4}u_x^2(L_1,t). \end{aligned} \tag{2.3.29}$$

At this point, we choose our constants in (2.3.29), carefully, such that all the coefficients in (2.3.29) will be negative. Indeed, under the assumption (2.3.4), we can always find

γ_2, γ_3 and γ_4 such that

$$\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\gamma_4 + \gamma_2 < 0, \qquad \gamma_3 > \frac{a}{b}\gamma_4, \qquad \gamma_2 > \frac{\gamma_3}{2}. \tag{2.3.30}$$

Once the above constants are fixed, we may choose ϵ_2 and ϵ_0 small enough such that

$$\epsilon_0 c_0^2 + \gamma_3\epsilon_2 < a(\gamma_2 - \gamma_3/2).$$

Finally, keeping in mind (2.2.16) and choosing N large enough such that the first and the second coefficients in (2.3.29) are negatives.

Consequently, from above, we deduce that there exist a positive constant η_1, such that (2.3.29) becomes

$$\begin{aligned}\frac{d\mathscr{L}(t)}{dt} \quad &\leq \quad -\eta_1 \int_\Omega (u_t^2(x,t) + u_x^2(x,t) + u_t^2(x,t-\tau))dx \qquad (2.3.31)\\ &\quad -\eta_1 \int_{L_1}^{L_2} (v_t^2(x,t) + v_x^2(x,t))dx - \eta_1 \int_\Omega \int_{t-\tau}^{t} u_t^2(x,s)dsdx, \qquad \forall t \geq 0.\end{aligned}$$

Consequently, recalling (2.3.3), then, we deduce that there exist also $\eta_2 > 0$, such that

$$\frac{d\mathscr{L}(t)}{dt} \leq -\eta_2 E(t), \qquad \forall t \geq 0. \tag{2.3.32}$$

On the other hand, it is not hard to see that from (2.3.27) and for N large enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1\ E(t) \leq \mathscr{L}(t) \leq \beta_2 E(t), \qquad \forall t \geq 0. \tag{2.3.33}$$

Combining (2.3.32) and (2.3.33), we deduce that there exists $\Lambda > 0$ for which the estimate

$$\frac{d\mathscr{L}(t)}{dt} \leq -\Lambda\mathscr{L}(t), \qquad \forall t \geq 0, \tag{2.3.34}$$

holds. Integrating (2.3.34) over $(0,t)$ and using (2.3.32) once again, then (2.3.5) holds. Then, the proof of the Theorem 2.3.1 is completed. $\square$

Chapter 3

Well-posedness of a transmission problem with viscoelastic term and delay

3.1 Introduction

In this chapter, we consider a transmission problem in a bounded domain with a viscoelastic term and delay term. Under appropriate hypothesis on the relaxation function and the relationship between the weight of the damping and the weight of the delay, we prove the well-possedness result by using semi-group theory method. The prove of the general decay result for this problem is provided.
Viscoelastic wave equation with delay was first considered by Kirane and Said-Houari in [58] they studied the existence and asymptotic stability of viscoelastic wave equation with delay, they considered the following linear viscoelastic wave equation with a linear damping and a delay term:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds & \\ +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \geqslant 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \in \Omega, t \in (0,\tau), \end{cases} \tag{3.1.1}$$

where $u = u(x,t), t \geq 0, x \in \Omega, \triangle$ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of $\mathbb{R}^N$, $(N \geq 1)$, μ_1, μ_2 are positive

constants, $\tau > 0$ represents the time delay and u_0, u_1, f_0 are given functions belonging to suitable spaces. The purpose of their work was to study the existence and the asymptotic stability of problem (3.1.1) by relaxing the assumption in [84]. Introducing the delay term $\mu_2 u_t(x, t-\tau)$, made the problem different from those considered in the literature. First the authors used the Faedo-Galerkin approximations together with some energy estimates, and under some restriction on the parameters μ_1 and μ_2, they showed that the problem (3.1.1) is well-posed. Second under the hypothesis $\mu_1 < \mu_2$ between the weight of the delay term in the feedback and the weight of the term without delay, they proved a general decay result of the total energy of the problem (3.1.1).
Alabau et al in [2] treated a model combining viscoelastic damping and time-delay damping, they considered the problem

$$\begin{cases} u_{tt} - \Delta u(x,t) - \displaystyle\int_0^\infty g(s)\Delta u(x,t-s)ds + ku_t(x,t-\tau) = 0, & \text{in} \quad \Omega \times (0,+\infty), \\ u(x,t) = 0, & \text{on} \quad \partial\Omega \times (0,+\infty), \\ u(x,t) = u_0(x,t), u_t(x,0) = u_1(x) & \text{in} \quad \Omega \times (0,+\infty), \end{cases} \tag{3.1.2}$$

where $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a smooth boundary, the initial data u_0 belong to a suitable space, the constant $\tau > 0$ is the time delay, k is a real number and the memory kernel $\mu : [0,+\infty) \longrightarrow [0,+\infty)$ is a locally absolutely continuous function satisfying:

i) $g(0) = \mu_0 > 0$;

ii) $\displaystyle\int_0^{+\infty} g(t)dt = \tilde{\mu} < 1$;

iii) $g'(t) \leq -\alpha g(t)$, for some $\alpha > 0$.

For $k = 0$ the above problem is exponentiallay stable see [43]. The authors showed that an exponential stability result holds if the delay parameter k is small with respect to the memory kernel. That is for the energy $F(t)$ of problem (3.1.2), there exists a positive constant k_0 such that for $|k| < k_0$ there is $\sigma > 0$ such that

$$F(t) \leq F(0)e^{1-\sigma t}, \quad t \geq 0,$$

for every solution of problem (3.1.2). The constant k_0 depends only on the kernel $g(.)$ of the memory term, on the time delay τ and on the domain Ω. They observed that for

$\tau = 0$ and $k > 0$ the model (3.1.2) presents both viscoelastic and standard dissipative damping. They show that exponential stability may also occurs for $k < 0$, under a suitable smallness assumption on $|k|$. They noted that the term $ku_t(t)$ with $k < 0$ is a so-called anti-damping [38] namely a damping with an opposite sign with respect to the standard dissipative one and therefore it induces instability. Indeed, in absence of the viscoelastic damping, i.e for $\mu = 0$, the solution of the above problem, with $\tau = 0$ and $k < 0$, grows exponentially to infinity when t tends to infinity.
The stabilization problem for model (3.1.2) has been studied by Guesmia in [48] by using a different approach based on the construction of a suitable Lypunov functional. Let us considere the problem

$$\begin{cases} u_{tt} - au_{xx} - \displaystyle\int_0^{+\infty} g(s)u_{xx}(x, t-s)ds \\ +\mu_1 u_t(x,t) + |\mu_2|\, u_t(x, t-\tau) = 0, & (x,t) \in \Omega\times]0, +\infty[, \\ v_{tt} - bv_{xx} = 0, & (x,t) \in]L_1, L_2[\times]0, +\infty[, \end{cases} \tag{3.1.3}$$

where $0 < L_1 < L_2 < L_3$ and $\Omega =]0, L_1[\times]L_2, L_3[$, $\mu_1 \in \mathbb{R}_*^+$, and $\mu_2 \in \mathbb{R}$. System (3.1.3) is sudjected to the following boundary and transmission conditions:

$$\begin{cases} u(0,t) = u(L_3, t) = 0, \\ u(L_i, t) = v(L_i, t), & i = 1, 2 \\ au_x(L_i, t) - \displaystyle\int_0^{+\infty} g(s)u_{xx}(x, t-s)ds = bv_x(L_i, t), & i = 1, 2 \end{cases} \tag{3.1.4}$$

and the initial conditions:

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x, t-\tau) = f_0(x, t-\tau), & x \in \Omega,\ t \in [0, \tau], \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in]L_1, L_2[. \end{cases} \tag{3.1.5}$$

For the relaxation fubction g, we have the following assumptions:
$(B1)$ $g : [0, +\infty) \longrightarrow [0, +\infty)$ is C^1 function satisfying

$$g(0) > 0, \quad a - \int_0^{+\infty} g(t)dt = a - \tilde{\mu} = l > 0.$$

$(B2)$ There exists a non-increasing differentiable function $\xi : [0, +\infty) \longrightarrow [0, +\infty)$ such that

$$g'(t) \leq -\xi(t)g(t), \quad \forall t \geq 0 \quad \text{and} \quad \int_0^{+\infty} \xi(t)dt = +\infty. \tag{3.1.6}$$

Transmission problems related to (3.1.3)-(3.1.5) have been extensively studied. Bastos and Raposo in [8] investigated the transmission problem with frictional damping and showed the well-posedness and exponential stability of the total energy. Munos et al in [82] considered the transmission problem of viscoelastic waves and proved that the dissipation produced by the viscoelastic part can produce exponential decay of the solution, no matter how small its size is.
Motivated by the above results especially that in [2], we intend to study in this chapter the well-posedness and the decay result of problem (3.1.3)-(3.1.5), in which the infinite memory term $\int_0^{+\infty} g(s)u_{xx}(x,t-s)ds$ is involved. To obtain our goal, we use the semi-group theory to prove the well-possedness, and introduce a suitable Lyapunov functional to establish the decay result.

3.2 Well-posedness of the problem

In this section, we prove the local existence and the uniqueness of the solution of system (3.1.3)-(3.1.5) by using the semi-group theory. So let us introduce the following new variable [84]

$$y(x,\rho,t) = u_t(x,t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0. \tag{3.2.1}$$

and let us introduce another new variable [30]

$$\eta^t(x,s) = u(x,t) - u(x,t-s) \quad ,(x,t,s) \in \Omega \times]0,+\infty[\times]0,+\infty[. \tag{3.2.2}$$

Then we have

$$\tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0 \qquad x \in \Omega,\ \rho \in (0,1),\ t > 0, \tag{3.2.3}$$

and

$$\eta^t_t(x,s) = -\eta^t_s(x,s) + u_t(x,t) \qquad (x,t,s) \in \Omega \times]0,+\infty[\times]0,+\infty[. \tag{3.2.4}$$

Using (3.2.1) and (3.2.2) we can rewrite (3.1.3) as

$$\begin{cases} u_{tt} = lu_{xx} + \displaystyle\int_0^{+\infty} g(s)\eta^t_{xx}(x,t-s)ds & \\ -\mu_1 u_t(x,t) - |\mu_2|\, y(x,1,t), & (x,t) \in \Omega \times]0,+\infty[, \\ v_{tt} - bv_{xx} = 0, & (x,t) \in]L_1,L_2[\times]0,+\infty[, \\ \eta^t_t(x,s) = -\eta^t_s(x,s) + u_t(x,t), & (x,t,s) \in \Omega \times]0,+\infty[\times]0,+\infty[, \\ \tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0, & x \in \Omega,\ \rho \in (0,1), t > 0, \end{cases} \tag{3.2.5}$$

the boundary and transmission conditions (3.1.4) become

$$\begin{cases} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), & i=1,2,\ t\in(0,+\infty) \\ lu_x(L_i,t) + \displaystyle\int_0^{+\infty} g(s)\eta_x^t(L_i,s) = bv_x(L_i,t), & i=1,2,\ t\in(0,+\infty) \end{cases} \tag{3.2.6}$$

and the initial conditions (3.1.5) become

$$\begin{cases} u(,-t) = u_0(x,t),\ u_t(x,0) = u_1(x), & x\in\Omega, \\ y(x,0,t) = u_t(x,t),\ y(x,1,t) = f_0(x,t,-\tau), & (x,t)\in\Omega\times(0,+\infty), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x\in]L_1,L_2[. \end{cases} \tag{3.2.7}$$

It is clear that

$$\begin{cases} \eta^t(x,0) = 0, & \text{for all } x\in\Omega, \\ \eta^t(0,s) = \eta^t(L_3,s) = 0, & \text{for all } s>0, \\ \eta^0(x,s) = \eta_0(s), & \text{for all } s>0. \end{cases} \tag{3.2.8}$$

Let us denote by $U = (u,\varphi,v,\psi,w,y)^T$. Then we can rewrite the problem (3.2.5) with (3.1.5) as

$$\begin{cases} U' = A(U) \\ U(0) = (u_0,u_1,v_0,v_1,\eta_0,f_0(.,-.\tau))^T \end{cases} \tag{3.2.9}$$

where $\eta_0 = \eta^0(x,s)$ in $\partial\Omega\times]0,+\infty[$ and the operator A is defined by

$$A\begin{pmatrix} u \\ \varphi \\ v \\ \psi \\ w \\ y \end{pmatrix} = \begin{pmatrix} \varphi \\ lu_{xx} + \displaystyle\int_0^{+\infty} g(s)w_{xx}(s)ds - \mu_1 u_t - |\mu_2|\, y(.,1) \\ \psi \\ bv_{xx} \\ -w_s + \varphi \\ -\frac{1}{\tau}y_\rho \end{pmatrix} \tag{3.2.10}$$

with the domain

$$D(A) = \left\{(u,\varphi,v,\psi,\eta,y)^T \in H;\ y(.,0) = \varphi \text{ on } \Omega\right\},$$

where

$$H = \left\{ \begin{array}{l} (H^2(\Omega)\times H^2(L_1,L_2))\cap X_*)\times H^1(\Omega) \\ \times H^1(L_1,L_2)\times L_g^2((0,+\infty);H_0^1(\Omega))\times L^2(0,1,H^1(\Omega)) \end{array} \right\}.$$

Here the space X_* is defined by

$$X_* = \left\{ \begin{array}{l} (u,v) \in H^1(\Omega) \cap H^1(L_1,L_2) | u(0,t) = u(L_3,t) = 0,\ u(L_i,t) = v(L_i,t), \\ lu_x(L_i,t) + \int_0^{+\infty} g(s)\eta_x^t(L_i,s) = bv_x\left(L_i,t\right),\ i = 1,2 \end{array} \right\}.$$

Now the energy space is defined by

$$K = X_* \times L^2(\Omega) \times L^2(L_1,L_2) \times L_g^2((0,+\infty); H_0^1(\Omega)) \times L^2((\Omega) \times (0,1)),$$

where $L_g^2((0,+\infty); H_0^1(\Omega))$ is the Hilbert space of H^1-valued function on $(0,+\infty)$ endowed with the inner product

$$\langle \phi, \chi \rangle_{L_g^2((0,+\infty);H_0^1(\Omega))} = \int_\Omega \left(\int_0^{+\infty} g(s)\phi_x(x,s)\chi_x(x,s)ds \right) dx.$$

Let

$$U = (u,\varphi,v,\psi,w,y)^T \text{ and } \tilde{U} = (\tilde{u},\tilde{\varphi},\tilde{v},\tilde{\psi},\tilde{w},\tilde{y})^T.$$

Then, for a positive constant ζ satisfying

$$\tau\left|\mu_2\right| \leq \zeta \leq \tau(2\mu_1 - \left|\mu_2\right|), \tag{3.2.11}$$

we define the inner product in K as follows

$$\begin{aligned} \left\langle U,\tilde{U} \right\rangle_K &= a\int_\Omega [\varphi\tilde{\varphi}dx + lu_x\tilde{u}_x]dx + l\int_{L_1}^{L_2} [\psi\tilde{\psi} + bv_x\tilde{v}_x]dx \\ &\quad + a\int_\Omega \int_0^{+\infty} g(s)w_x(s)\tilde{w}_x(x,s)dsdx \\ &\quad + a\zeta\int_\Omega \int_0^1 y(x,\rho)\tilde{y}(x,\rho)d\rho dx. \end{aligned}$$

The existence and uniqueness results is stated as follows;

Theorem 3.2.1. *For any $U_0 \in K$ there exists a unique solution $U \in C([0,+\infty[,K)$ of the problem (3.2.9). Moreover, if $U_0 \in D(A)$, then*

$$U \in C([0,+\infty[,D(A)) \cap C^1([0,+\infty[,K).$$

Proof. In order to prove the result stated in Theorem 3.2.1, we use the semigroup theory, that is, we show that the operator A generates a C_0semi-group in K. In this step, we restrict ouselves to prove that the operator A is dissipative. Indeed, for $U=(u,\varphi,v,\psi,w,y)^T \in D(A)$ and a positive constant ξ , we have

$$\begin{aligned}
\langle AU,U\rangle_K &= al\int_\Omega u_{xx}\varphi dx + bl\int_{L_1}^{L_2} v_{xx}\psi dx - a\mu_1\int_\Omega \varphi^2 dx \\
&\quad -a\left|\mu_2\right|\int_\Omega y(.,1)\varphi dx + a\int_\Omega \left(\varphi\int_0^{+\infty} g(s)w_{xx}(s)ds\right)dx \\
&\quad -\frac{a\zeta}{\tau}\int_\Omega\int_0^1 y(x,\rho)y_\rho(x,\rho)d\rho dx \\
&\quad +a\int_\Omega\int_0^{+\infty} g(s)(-w_{sx}(x,s)+\varphi_x)w_x(x,s)dsdx \\
&\quad +al\int_\Omega u_x\varphi_x dx + bl\int_{L_1}^{L_2} v_x\psi_x dx. \qquad (3.2.12)
\end{aligned}$$

We also have

$$\begin{aligned}
\int_\Omega\int_0^1 y(x,\rho)y_\rho(x,\rho)d\rho dx &= \int_\Omega \frac{1}{2}\frac{\partial}{\partial\rho}y^2(x,\rho)d\rho dx \qquad (3.2.13)\\
&= \frac{1}{2}\int_\Omega (y^2(x,1)-y^2(x,0)dx.
\end{aligned}$$

Performing an integration by parts in (3.2.12), and using the third equation in (3.1.4) we get

$$\begin{aligned}
& al\int_\Omega u_{xx}\varphi dx + bl\int_{L_1}^{L_2} v_{xx}\psi dx + al\int_\Omega u_x\varphi_x dx + bl\int_{L_1}^{L_2} v_x\psi_x dx \\
=\ & [alu_x\varphi]_{\partial\Omega} + [blv_x\psi]_{L_1}^{L_2} - al\int_\Omega u_x\varphi_x dx \\
& -bl\int_{L_1}^{L_2} v_x\psi_x dx + al\int_\Omega u_x\varphi_x dx + bl\int_{L_1}^{L_2} v_x\psi_x dx \\
=\ & 0.
\end{aligned}$$

We have by integrating by parts using the fact $w(x,0)=0$,

$$-\int_\Omega\int_0^{+\infty} g(s)w_{sx}(x,s)w_x(x,s)dsdx = \int_\Omega\int_0^{+\infty} g'(s)\left|w_x(x,s)\right|^2 dsdx. \qquad (3.2.14)$$

Following [41], looking to the definition of $D(A)$ and using the fact that $y(x,0,t)=\varphi(x,t)$, we deduce

$$\begin{aligned}
\frac{1}{2}\langle AU,U\rangle_K &= -a(\mu_1-\frac{\zeta}{2\tau})\int_\Omega \varphi^2 dx - a\left|\mu_2\right|\int_\Omega y(.,1)\varphi dx - \frac{a\zeta}{2\tau}\int_\Omega y^2(x,1)dx \\
&\quad +\frac{a}{2}\int_\Omega\int_0^{+\infty} g'(s)\left|w_x(x,s)\right|^2 dsdx.
\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\left\langle AU,U\right\rangle_K \quad &\leq \quad -a(\mu_1-\frac{|\mu_2|}{2}-\frac{\zeta}{2\tau})\int_\Omega \varphi^2dx - a(\frac{\zeta}{2\tau}-\frac{|\mu_2|}{2})\int_\Omega y^2(x,1)dx \\ &\quad +\frac{a}{2}\int_\Omega\int_0^{+\infty} g'(s)\left|w_x(x,s)\right|^2 dsdx.\end{aligned}$$

Then, keeping in mind the fact that $g'(s)\leq -\xi(t)g(s)$ and $g(s)>0$, we obtain

$$\left\langle AU,U\right\rangle_K \leq 0.$$

Hence, the operator A is dissipative. Now we show that the operator A is surjective. Let $F=(f_1,f_2,f_3,f_4,f_5,f_6)^T\in H$, we prove that there exists $V=(u,\varphi,v,\psi,w,y)^T\in D(A)$ such that

$$(\lambda I-A)V=F, \tag{3.2.15}$$

which is equivalent to

$$\begin{cases}\lambda u-\varphi=f_1,\\ \lambda\varphi-(lu_{xx}+\displaystyle\int_0^{+\infty} g(s)w_{xx}(s)ds-\mu_1 u_t-\mu_2 y(.,1))=f_2,\\ \lambda v-\psi=f_3,\\ \lambda\psi-bv_{xx}=f_4,\\ \lambda w+w_s-\varphi=f_5,\\ \lambda y+\frac{1}{\tau}y_\rho=f_6.\end{cases} \tag{3.2.16}$$

Suppose that we have found u and v with the appropriate regularity. Therfore, the first and the third equations in (3.2.16) gives

$$\begin{cases}\varphi=\lambda u-f_1,\\ \psi=\lambda v-f_3.\end{cases} \tag{3.2.17}$$

It is clear that $\varphi\in H_0^1(\Omega)$ and $\psi\in H_0^1(L_1,L_2)$. furthermore, by using the sixth equation in (3.2.16) we can find y such that

$$y(x,0)=\varphi(x), \quad \text{for} \quad x\in\Omega. \tag{3.2.18}$$

Using the same approach as in Nicaise and Pignotti [84], we obtain, by using the sixth equation in (3.2.16)

$$y(x,1)=\varphi(x)e^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_0^1 f_6(x,\sigma)e^{\lambda\tau}d\sigma.$$

From (3.2.17), we find

$$\begin{aligned} y(x,1) &= \lambda u(x)e^{-\lambda\tau} - f_1(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 f_6(x,\sigma)e^{\lambda\tau}d\sigma \\ &= \lambda u(x)e^{-\lambda\tau} + y_0(x), \end{aligned} \tag{3.2.19}$$

with

$$y_0(x) = -f_1(x)e^{-\lambda\tau},! + \tau e^{-\lambda\tau}\int_0^1 f_6(x,\sigma)e^{\lambda\tau}d\sigma. \tag{3.2.20}$$

We note that the fifth equation in (3.2.16) with $w(x,0)=0$ has a unique solution

$$\begin{aligned} w(x,s) &= \left(\int_0^s e^{\lambda z}(f_5(x,z) + \varphi(x)dz\right)e^{-\lambda s} \\ &= \left(\int_0^s e^{\lambda z}(f_5(x,z) + \lambda u(x) - f_1(x)dz\right)e^{-\lambda s}. \end{aligned} \tag{3.2.21}$$

By using (3.2.16), (3.2.17) and (3.2.18) the functions u and v satisfy the folowing system

$$\begin{cases} \lambda^2 u - (lu_{xx} + \int_0^{+\infty} g(s)w_{xx}(s)ds - \mu_1 y(.,0) - |\mu_2|\, y(.,1)) = f_2 + \lambda f_1, \\ \lambda^2 v - bv_{xx} = f_4 + \lambda f_3. \end{cases} \tag{3.2.22}$$

Let

$$\tilde{l} = l + \lambda\int_0^{+\infty} g(s)e^{-\lambda s}\left(\int_0^s e^{\lambda z}dz\right)ds,$$

and

$$\tilde{f} = -\int_0^{+\infty} g(s)e^{-\lambda s}\left(\int_0^s e^{\lambda z}((f_5(x,z) - f_1(x))_{xx}dz\right)ds,$$

then the system (3.2.22) becomes

$$\begin{cases} \lambda^2 u - \tilde{l}u_{xx} + \tilde{f} + \mu_1 y(.,0) + |\mu_2|\, y(.,1) = f_2 + \lambda f_1, \\ \lambda^2 v - bv_{xx} = f_4 + \lambda f_3. \end{cases} \tag{3.2.23}$$

Using (3.2.18) we get

$$\begin{aligned} &\int_\Omega (\lambda^2 u - \tilde{l}u_{xx} + \mu_1\lambda u + |\mu_2|\,\lambda u e^{-\lambda\tau})g_1 dx \\ = &\int_\Omega (f_2 + \lambda f_1 - \tilde{f} - |\mu_2|\,\lambda y_0(x))g_1 dx, \qquad \forall g_1 \in H_0^1(\Omega) \\ &\int_{L_1}^{L_2} (\lambda^2 v - bv_{xx})g_2 dx \\ = &\int_{L_1}^{L_2} (f_4 + \lambda f_3)g_2 dx, \qquad \forall g_2 \in H_0^1(L_1,L_2) \end{aligned} \tag{3.2.24}$$

We have now to prove that (3.2.23) has a solution $(u, v) \in H^2(\Omega) \cap H_0^1(\Omega) \times H^2(L_1, L_2) \cap H_0^1(L_1, L_2)$ and replace it in (3.2.17), (3.2.18) and (3.2.20) to get $V = (u, \varphi, v, \psi, w, y)^T \in D(A)$ satisfying (3.2.14). To solve the problem (3.2.24) we consider

$$\Phi((u, v), (g_1, g_2)) = \Psi(g_1, g_2), \tag{3.2.25}$$

where the bilinear form $\Phi : (H_0^1(\Omega))^2 \times (H_0^1(L_1, L_2))^2 \longrightarrow \mathbb{R}$ and the linear form $\Psi : H_0^1(\Omega) \times H_0^1(L_1, L_2) \longrightarrow \mathbb{R}$ are defined by

$$\begin{aligned} \Phi((u, v), (g_1, g_2)) &= \int_\Omega (\lambda^2 u - \tilde{l} u_{xx} + \mu_1 \lambda u + |\mu_2| \, \lambda u e^{-\lambda \tau}) g_1 dx \\ &\quad + \int_{L_1}^{L_2} (\lambda^2 v - b v_{xx}) g_2 dx, \end{aligned}$$

and

$$\begin{aligned} \Psi(g_1, g_2) &= \int_\Omega (f_2 + \lambda f_1 - \tilde{f} - |\mu_2| \, \lambda y_0(x)) g_1 dx \\ &\quad + \int_{L_1}^{L_2} (f_4 + \lambda f_3) g_2 dx. \end{aligned}$$

It is clear that Φ is continuous and coercive and Ψ is continuous. So applying the Lax-Migram theorem (1.3.4), we deduce that for all $(g_1, g_2) \in H_0^1(\Omega) \times H_0^1(L_1, L_2)$, problem (3.2.25) admits a unique solution $(u, v) \in H_0^1(\Omega) \times H_0^1(L_1, L_2)$. Applying the clasical elliptic regularity, it follows from (3.2.24) that $(u, v) \in H^2(\Omega) \cap H_0^1(\Omega) \times H^2(L_1, L_2) \cap H_0^1(L_1, L_2)$. Therfore the operator $\lambda I - A$ is surjective for any $\lambda > 0$. then the result in Theorem 2.1 follows from the Hill-Yosida theorem (1.8.7). □

Remark. In the following section we are going to give only the theorem which gives the decay of the solution and the lemmas necessary for its proof, because this result has been studied by the authors Gang et al in [40].
For a solution u of problem (3.1.3)-(3.1.5) we define the energy

$$\begin{aligned} E(t) = \frac{1}{2} \int_\Omega [u_t^2(x, t) + l u_x^2(x, t)] dx + \frac{1}{2} \int_{L_1}^{L_2} [v_t^2(x, t) + b v_x^2(x, t)] dx \\ + \frac{1}{2} \int_\Omega \int_0^\infty g(s) |\eta_x^t(x, s)|^2 ds dx + \frac{\zeta}{2} \int_\Omega \int_0^1 z^2(x, \rho, t) d\rho dx, \end{aligned} \tag{3.2.26}$$

where ζ is the positive constant satisfying (3.2.11).

3.3 Decay of the solution

The decay result reads as follows

Theorem 3.3.1. *Let (u, v) be the solution of (3.1.3)-(3.1.5). Assume that $(B1)$, $(B2)$, that $|\mu_2| \leq \mu_1$, that for some $m_0 \geq 0$,*

$$\int_\Omega u_{0x}^2(x,s)\mathrm{d}x \leq m_0, \quad \forall s > 0 \tag{3.3.1}$$

and that

$$a > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2} l, \quad b > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2} l \tag{3.3.2}$$

hold, then there exists constants $d_0, d_2 > 0$ such that, for all $t \geq 0$ and for all $d_1 \in (0, d_0)$,

$$E(t) \leq d_2\Big(1 + \int_0^t (g(s))^{1-d_1} ds\Big) e^{-d_1 \int_0^t \xi(s)ds} + d_2 \int_t^{+\infty} g(s)ds. \tag{3.3.3}$$

For the proof of Theorem 3.3.1, we need the lemmas.

Lemma 3.3.2. *Let (u, v, z) be the solution of (3.2.5)-(3.2.7). Assume that $|\mu_2| < \mu_1$. Then we have the inequality*

$$\begin{aligned} \frac{dE(t)}{dt} \leq & -c_1\Big[\int_\Omega u_t^2(x,t)dx + \int_\Omega y^2(x,1,t)dx\Big] \\ & + \frac{1}{2}\int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2 ds dx. \end{aligned} \tag{3.3.4}$$

Now we define the functional

$$\mathscr{D}(t) = \int_\Omega u u_t dx + \frac{\mu_1}{2}\int_\Omega u^2 dx + \int_{L_1}^{L_2} v v_t dx.$$

Then we have the following estimate

Lemma 3.3.3. *The functional $\mathscr{D}(t)$ satisfies*

$$\begin{aligned} \frac{d\mathscr{D}(t)}{dt} \leq & \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx + (L^2\varepsilon + \varepsilon - l)\int_\Omega u_x^2 dx - \int_{L_1}^{L_2} b v_x^2 dx \\ & + \frac{g_0}{4\varepsilon}\int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 ds dx + \frac{\mu_2^2}{4\varepsilon}\int_\Omega z^2(x,1,t)dx, \end{aligned} \tag{3.3.5}$$

where $L = max(L_1, L_3 - L_2)$, and $\varepsilon > 0$.

Now, inspired by [73], we introduce the functional

$$q(x) = \begin{cases} x - \dfrac{L_1}{2}, & x \in [0, L_1], \\ x - \dfrac{L_2 + L_3}{2}, & x \in [L_2, L_3], \\ \dfrac{L_2 - L_3 - L_1}{2(L_2 - L_1)}(x - L_1) + \dfrac{L_1}{2}, & x \in [L_1, L_2] \end{cases} \tag{3.3.6}$$

It is easy to see that $q(x)$ is bounded: $|q(x)| \le M$, where $M = \max\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\}$.

We define the functionals

$$\mathscr{F}_1(t) = -\int_\Omega q(x)u_t\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\Big)dx, \quad \mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x)v_xv_tdx.$$

Then we have the following results.

Lemma 3.3.4. *The functionals $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$ satisfy*

$$\begin{aligned}
&\frac{d\mathscr{F}_1(t)}{dt} \\
&\le \Big(\frac{l+g_0}{2} + \frac{M^2\mu_1^2}{4\varepsilon_1} + \varepsilon_1 M^2\Big)\int_\Omega u_t^2dx + \big(l^2 + 2l^2\varepsilon_1\big)\int_\Omega u_x^2dx \\
&\quad + \frac{M^2\mu_2^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)dx + (g_0 + 2g_0\varepsilon_1)\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2dsdx \\
&- \frac{g(0)}{4\varepsilon_1}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}sdx - \Big[\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega} \\
&- \Big[\frac{q(x)}{2}\Big(lu_x(x,t) + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\Big]_{\partial\Omega}
\end{aligned} \tag{3.3.7}$$

and

$$\begin{aligned}
\frac{d\mathscr{F}_2(t)}{dt} \le &-\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\Big(\int_{L_1}^{L_2} v_t^2dx + \int_{L_1}^{L_2} bv_x^2dx\Big) + \frac{L_1}{4}v_t^2(L_1) \\
&+ \frac{L_3 - L_2}{4}v_t^2(L_2) + \frac{b}{4}\Big((L_3 - L_2)v_x^2(L_2,t) + L_1v_x^2(L_1,t)\Big).
\end{aligned} \tag{3.3.8}$$

Following [4], we define the functional

$$I(t) = \tau\int_\Omega\int_0^1 e^{-\tau\rho}y^2(x,\rho,t)d\rho dx,$$

then we have the following lemma.

Lemma 3.3.5. *The functional $I(t)$ satisfies*

$$\frac{dI(t)}{dt} \le -c_2\Big(\int_\Omega y^2(x,1,t)dx + \tau\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx\Big) + \int_\Omega u_t^2(x,t)dx.$$

Finally, for the proof of Theorem 3.1.1 the authors in [40] considered the following Lyapunov functional

$$L(t) = N_1E(t) + N_2\mathscr{D}(t) + \mathscr{F}_1(t) + N_4\mathscr{F}_2(t) + I(t), \tag{3.3.9}$$

where N_1, N_2, N_4 are positive constants. They proved the equivalence between $L(t)$ and $E(t)$. Then by choosing the constants one by one, they proved the existence of $d_0, , d_2 > 0$ such that for all $d_1 \in (d_0, d_2)$ (3.3.3) holds.

Chapter 4

Decay for a transmission problem with memory and time-varying delay

4.1 Introduction

In this chapter, we consider the transmission problem with a varying delay term,

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^t g(t-s)u_{xx}(x,s)ds + \mu_1 u_t(x,t) & \\ +|\mu_2|u_t(x,t-\tau(t)) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2) \times (0,+\infty), \end{cases} \tag{4.1.1}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0, L_1[\cup]L_2, L_3[$, a, b, μ_1, are positive constants, and μ_2 is a real number $\tau(t) > 0$ is the delay function.

System (4.1.1) is subjected to the following boundary and transmission conditions:

$$\begin{cases} u(0,t) = u(L_3,t) = 0, & \\ u(L_i,t) = v(L_i,t), & i = 1,2, \\ \left(a - \displaystyle\int_0^t g(s)ds\right) u_x(L_i,t) = bv_x(L_i,t), & i = 1,2, \end{cases} \tag{4.1.2}$$

and the initial conditions:

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t-\tau(t)) = f_0(x,t-\tau(t)), & x \in \Omega,\ t \in [0,\overline{\tau}], \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in]L_1,L_2[. \end{cases} \tag{4.1.3}$$

We assume, that there exist positive constants $\tau_0, \overline{\tau}$ such that

$$0 < \tau_0 \leq \tau(t) \leq \overline{\tau}. \tag{4.1.4}$$

Moreover, we assume that

$$\tau \in W^{2,\infty}([0,T]), \quad \forall T > 0, \tag{4.1.5}$$

and

$$\tau'(t) \leq d < 1, \quad \forall t > 0. \tag{4.1.6}$$

The problem (4.1.1)-(4.1.3) related to the wave propagation over a body composed of two different elastic materials and it is called a transmission problem, the body consists of an elastic part and a viscoelastic part. In recent years, many authors have investigated wave equations with viscoelastic damping and showed that the dissipation produced by the viscoelastic part can produce the decay of the solution see for instance [14, 15, 26] and the references therein.
Calvalcanti et al. in [26] studied the following equation,

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^\gamma u = 0, in \quad \Omega \times (0,\infty),$$

where $a : \Omega \to \mathbb{R}_+$ and satisfies $a(x) \geq a_0 > 0$ on $w \subset \Omega$, with w satisfying some geometry restrictions and

$$-\zeta_1 g(t) \leq g'(t) \leq -\zeta_2 g(t), \quad t \geq 0. \tag{4.1.7}$$

The authors showed the exponential decay of the solution using the perturbed energy method. Then Berrimi and Messaoudi [14] obtained the same result under weaker conditions on both a and g.
Kirane and Said-houari [58] considered the viscoelastic wave equation with delay

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(,x-s)d + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \text{ in } \Omega \times (0,\infty),$$

where μ_1 and μ_2 are positive constants. They established a general energy decay result under the condition that $0 \leq \mu_2 \leq \mu_1$. Later, Liu, [69] improved this result by considering the equation with a time-varying delay term where the cofficient μ_2 of the delay is not necessary positive. For $\mu_2 = 0$, system (4.1.1)-(4.1.3) has been investigated in [8]; for $\Omega = [0, L_1]$, the authors showed the well-posedness and exponential stability of the total energy. Muñoz Rivera and Oquendo [82] studied the wave propagations over materials consisting of elastic and viscoelastic components; that is, they considered the transmission problem

$$\begin{cases} \rho_1 u_{tt} - \alpha_1 u_{xx} = 0, & x \in]0, L_0[, \ t > 0, \\ \rho_2 v_{tt} - \alpha_2 v_{xx} + \int_0^t g(t-s) v_{xx}(s) ds = 0, & x \in]L_0, L[, \ t > 0, \end{cases} \tag{4.1.8}$$

with the boundary and initial conditions:

$$\begin{cases} u(0,t) = v(L,t), \quad u(L_0,t) = v(L_0,t), & t > 0, \\ \alpha_1 u_x(L_0,t) = \alpha_2 v_x(L_0,t) - \int_0^t g(t-s) v_x(s) ds, & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in [0, L_0], \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in [L_0, L]. \end{cases} \tag{4.1.9}$$

Here ρ_1 and ρ_2 are densities of the materials and α_1, α_2 are elastic coefficients and g is positive and exponentially decaying function. They showed that the dissipation produced by the viscoelastic part is strong enough to yield an exponential decay of the solution, no matter how small is its size. Ma and Oquendo [70] considered a transmission problem involving two Euler-Bernoulli equations modeling the vibration of a composite beam. By using just one boundary damping term in the boundary conditions, they showed the global existence and decay property of the solution. Marzocchi et al [73] investigated a 1-D semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero, no matter how small the damping subdomain is. A similar result has sheen shown by Messaoudi and Said-Houari [79], where a transmission problem in thermoelasticity of type III has been investigated. See also Marzocchi et al [74] for a multidimensional linear thermoelastic transmission problem.

For $\mu_2 > 0$, problem (4.1.1) has a delay term in the internal feedback. This delay term may destabilize system (4.1.1)-(4.1.3) that is exponentially stable in the absence of delays [8]. The effect of the delay in the stability of hyperbolic systems has been investigated by many people. See for instance [31, 32].

In [84], the authors examined a system of wave equations with a linear boundary damping term with a delay

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau), & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau) = g_0(x, t-\tau), & x \in \Omega,\ \tau \in (0,1), \end{cases} \tag{4.1.10}$$

and under the assumption

$$\mu_2 < \mu_1, \tag{4.1.11}$$

they proved that the solution is exponentially stable. On the contrary, if (4.1.11) does not hold, they found a sequence of delays for which the corresponding solution of (4.1.10) will be unstable. We also recall the result by Xu et al [106], where the authors proved the same result as in [84] for the one-dimentional space by adopting the spectral analysis approach. Motivite by the above results, we intend to consider the well-posedness and the general decay of problem (4.1.1)-(4.1.3). The main difficulty we enconter her arises from the simultanous appearance of the viscoelastic term and the varying delay. The aim of this chapter is to study the asymptotic stability of system (4.1.1)-(4.1.3) provided that the condition $|\mu_2| < \mu_1$ is satisfied. The chapter is organized as follows. In section 2, we give some materials needed for our work and state our main results. The general decay result is proved in section 3.

4.2 Preliminaries and main results

In this section, we present some materials that will be used in order to prove our main results.
Let us first introduce the following notaton:

$$(g * h)(t) := \int_0^t g(s-t)h(s)ds,$$
$$(g \diamond h)(t) := \int_0^t g(s-t)|h(t) - h(s)|ds,$$
$$(g \Box h)(t) := \int_0^t g(t-s)|h(t) - h(s)|^2 ds.$$

Note that the sign of $(g\Box h)(t)$ depends on the sign of g. We see that the above operators satisfy

$$\begin{aligned}(g*h)(t) &:= \left(\int_0^t g(s)\right) h(t)ds - (g\diamond h)(t),\\ |(g\diamond h)(t)|^2 &\leq \left(\int_0^t |g(s)|ds\right) (|g|\Box h)(t),\end{aligned}$$

Lemma 4.2.1. *For any $g, h \in C^1(\mathbb{R})$, the following identity holds*

$$2[g*h]h' = g'\Box h - g(t)|h|^2 - \frac{d}{dt}\left\{g\Box h - \left(\int_0^t g(s)ds\right)|h|^2\right\}.$$

For the relaxation function g, we assume

(G1) $g :\to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \quad 0 < \beta(t) := a - \int_0^t g(s)ds \quad \textit{and} \quad 0 < \beta_0 := a - \int_0^\infty g(s)ds.$$

(G2) There exists a nonincreasing differential function $\xi(t) : \mathbb{R} \to \mathbb{R}_+$ such that

$$g'(t) \leq -\xi(t)g(t), \, \forall t \geq 0 \quad \textit{and} \quad \int_0^\infty \xi(t) = +\infty.$$

These hypotheses imply that

$$\beta_0 \leq \beta(t) \leq a. \tag{4.2.1}$$

We introduce the following new variable [84]

$$y(x,\rho,t) = u_t(x, t-\tau(t)\rho), x \in \Omega, \rho \in (0,1), t > 0. \tag{4.2.2}$$

Then, we obtain

$$\tau(t)y_t(x,\rho,t) + (1-\tau'(t)\rho)y_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{4.2.3}$$

Therefore, problem (4.1.1) is equivalent to the system

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + g*u_{xx} + \mu_1 u_t(x,t) + |\mu_2|y(x,1,t) = 0, & (x,t) \in \Omega\times]0,+\infty[,\\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in]L_1,L_2[\times]0,+\infty[,\\ \tau(t)y_t(x,\rho,t) + (1-\tau'(t)\rho)y_\rho(x,\rho,t) = 0, & \text{in } \Omega\times(0,1)\times(0,+\infty), \end{cases} \tag{4.2.4}$$

and the boundary and transmission conditions become

$$\begin{cases} u(0,t) = u(L_3,t) = 0, & & (x,t) \in \Omega \times [O,+\infty), \\ u(L_i,t) = v(L_i,t), & i = 1,2, & t \in (0,+\infty), \\ \left(a - \int_0^t g(s)ds\right) u_x(L_i,t) = bv_x(L_i,t), & i = 1,2, & t \in (0,+\infty), \\ y(x,0,t) = u_t(x,t), & & (x,t) \in \Omega \times (0,+\infty) \\ y(x,\rho,t) = f_0(x,t-\tau(t)), & & (x,t) \in \Omega \times (0,\overline{\tau}). \end{cases} \tag{4.2.5}$$

Similar to [95, 11], we denote the Hilbert space X_* defined by

$$X_* = \Big\{(u,v) \in H^1(\Omega) \cap H^1(L_1,L_2) : u(0,t) = u(L_3,t) = 0,$$
$$u(L_i,t) = v(L_i,t),\ \left(a - \int_0^t g(s)ds\right) u_x(L_i,t) = bv_x(L_i,t),\ i = 1,2\Big\}.$$

and

$$\mathcal{L}^2 = L^2(\Omega) \times L^2(L_1,L_2).$$

We now state, without a proof, a well-posedness result, which can be established by combining the results in [58, 39].

Lemma 4.2.2. *Assume that $|\mu_2| \leq \mu_1, (G1)$ and $(G2)$ hold. Then given $(u_0,v_0) \in X_*, (u_1,v_1) \in \mathcal{L}^2$ and $f_0 \in L^2((0,1),H^1(\Omega))$, there exists a weak solution (u,v,y) of problem (4.2.4)- (4.2.5) such that*

$$(u,v) \in C(\mathbb{R}_+;X_\star) \cap C^1(\mathbb{R}_+;\mathcal{L}^2),$$
$$y \in C(\mathbb{R}_+;L^2(0,1),H^1(\Omega)).$$

For any regular solution of (4.1.1)-(4.1.3), we define the following energies

$$E_1(t) = \frac{1}{2}\int_\Omega u_t^2(x,t)\,dx + \frac{\beta(t)}{2}\int_\Omega u_x^2(x,t)dx + \frac{1}{2}\int_\Omega (g\Box u_x)dx, \tag{4.2.6}$$

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)\,dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)\,dx\,. \tag{4.2.7}$$

The total energy is defined as

$$E(t) = E_1(t) + E_2(t) + \frac{\zeta}{2}\int_\Omega \int_{t-\tau(t)}^t u_t^2(x,s)dsdx, \tag{4.2.8}$$

where ζ is the positive constant defined by

$$\frac{|\mu_2|}{\sqrt{1-d}} \leq \zeta \leq 2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}. \tag{4.2.9}$$

Our decay result reads as follows:

Theorem 4.2.3. *Let* (u,v) *be the solution of* (4.1.1)-(4.1.3)*. Assume that* $|\mu_2| < \mu_1$ *and*

$$b > \frac{4(L_2 - L_1)}{L_1 + L_3 - L_2}\beta_0, \quad a > \frac{4(L_2 - L_1)}{L_1 + L_3 - L_2}\beta_0. \tag{4.2.10}$$

Then there exist two positive constants C *and* d_0*, such that for all* $t \in \mathbb{R}_+$ *and for all* $d_1 \in (0, d_0)$

$$E(t) \leq Ce^{-d_1 \int_0^t \xi(s)ds}. \tag{4.2.11}$$

4.3 General decay of the solution

In this section we study the asymptotic behavior of the system (4.1.1)-(4.1.3). We use the following lemmas.

Lemma 4.3.1. *Let (u,v,y) be the solution of problem (*(4.1.1)-(4.1.3)*). Assume that* $|\mu_2| < \mu_1$*. Then we have*

$$\begin{aligned}\frac{dE(t)}{dt} &\leq -(\mu_1 - \frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\zeta}{2}) \int_\Omega u_t^2(x,t)\,dx \\ &-\Big(\frac{\zeta(1-d)}{2} - \frac{|\mu_2|}{2\sqrt{1-d}}\Big) \int_\Omega u_t^2(x,t-\tau(t))dx \\ &+\frac{1}{2}\int_\Omega (g'\Box u_x)(t)dx.\end{aligned} \tag{4.3.1}$$

Proof. Multiplying the first equation of (4.2.4) by u_t, integating by parts and (4.2.5), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega [u_t^2(x,t) + au_x^2(x,t)]dx\right\} = -\mu_1 \int_\Omega u_t^2(x,t)dx \\ -|\mu_2|\int_\Omega u_t(x,t)u_t(x,t-\tau(t))dx + \int_0^t g(t-s)\int_\Omega u_x(s)u_{xt}(t)dsdx.\end{aligned} \tag{4.3.2}$$

From Lemma 2.1, the last term in the right-hand side of (4.3.2) can be rewritten as

$$\begin{aligned} &\int_0^t g(t-s)\int_\Omega u_x(s)u_{xt}(t)dsdx + \frac{1}{2}g(t)\int_\Omega u_x^2 dx \\ &= \frac{1}{2}\left\{\int_0^t g(s)\int_\Omega u_x^2(x,t) - \int_\Omega (g\Box u_x)(t)dx\right\} + \frac{1}{2}\int_\Omega (g'\Box u_x)(t)dx. \end{aligned} \tag{4.3.3}$$

So (4.3.2) becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega [u_t^2(x,t)+\beta(t)u_x^2(x,t)]dx\right\} + \frac{1}{2}\int_\Omega (g\Box u_x)(t)dx \\ &= -\mu_1\int_\Omega u_t^2(x,t)dx - |\mu_2|\int_\Omega u_t(x,t)u_t(x,t-\tau(t))dx \\ &\qquad -\frac{1}{2}g(t)\int_\Omega u_x^2 dx + \frac{1}{2}\int_\Omega (g'\Box u_x)(t)dx. \end{aligned} \tag{4.3.4}$$

On the other hand, we have

$$\frac{dE_2(t)}{dt} = b[v_x v_t]_{L_1}^{L_2}. \tag{4.3.5}$$

Using the fact that

$$\frac{d}{dt}\int_\Omega\int_{t-\tau(t)}^t u_t^2(x,s)\,ds\,dx = \int_\Omega u_t^2(x,t)dx - (1-\tau'(t)\int_\Omega u_t^2(x,t-\tau(t))\,dx, \tag{4.3.6}$$

and collecting (4.3.2), (4.3.3), (4.3.4) (4.3.5), using (4.1.2) and applying Young's inequality, we show that (4.3.1) holds. The proof is complete. □

We note

$$c_0 = \min\left\{\mu_1 - \frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\zeta}{2}, \frac{\zeta(1-d)}{2} - \frac{|\mu_2|}{2\sqrt{1-d}}\right\}, \tag{4.3.7}$$

then (4.3.1) becomes

$$\frac{dE(t)}{dt} \leq -c_0\left[\int_\Omega u_t^2(x,t)\,dx + \int_\Omega u_t^2(x,t-\tau(t))dx\right] + \frac{1}{2}\int_\Omega (g'\Box u_x)(t)dx. \tag{4.3.8}$$

Following [4], we define the functional

$$I(t) = \int_\Omega\int_{t-\tau(t)}^t e^{(s-t)}u_t^2(x,s)\,ds\,dx$$

and state the following lemma.

Lemma 4.3.2. *Let (u, v) be the solution of* (4.1.1)-(4.1.3). *Then*

$$\frac{dI(t)}{dt} \leq \int_\Omega u_t^2(x,t)\,dx - (1-d)e^{-\overline{\tau}} \int_\Omega u_t^2(x,t-\tau(t))dx - e^{-\overline{\tau}} \int_\Omega \int_{t-\tau(t)}^t u_t^2(x,s)\,ds\,dx. \tag{4.3.9}$$

Proof. By differentiating $I(t)$ and using (4.1.4)-(4.1.6), we obtain

$$\begin{aligned}
\frac{dI(t)}{dt} &= \int_\Omega u_t^2(x,t)\,dx - \int_\Omega e^{-\tau(t)} u_t^2(x,t-\tau(t))(1-\tau'(t))\,dx \\
&\quad - \int_\Omega \int_{t-\tau(t)}^t e^{(s-t)} u_t^2(x,s)\,ds\,dx. \\
&\leq \int_\Omega u_t^2(x,t)\,dx - (1-d)e^{-\overline{\tau}} \int_\Omega u_t^2(x,t-\tau(t))dx \\
&\quad - e^{-\overline{\tau}} \int_\Omega \int_{t-\tau(t)}^t u_t^2(x,s)\,ds\,dx
\end{aligned}$$

□

We denote by

$$c_1 = \min\left\{(1-d)e^{-\overline{\tau}}, e^{-\overline{\tau}}\right\}, \tag{4.3.10}$$

then (4.3.9) becomes

$$\frac{dI(t)}{dt} \leq \int_\Omega u_t^2(x,t)\,dx - c_1 \left\{\int_\Omega u_t^2(x,t-\tau(t))dx + \int_\Omega \int_{t-\tau(t)}^t u_t^2(x,s)\,ds\,dx\right\}. \tag{4.3.11}$$

Now, we define the functional $\mathscr{D}(t)$ as follows

$$\mathscr{D}(t) = \int_\Omega u u_t\,dx + \frac{\mu_1}{2} \int_\Omega u^2\,dx + \int_{L_1}^{L_2} v v_t\,dx. \tag{4.3.12}$$

Then, we have the following estimate.

Lemma 4.3.3. *The functional $\mathscr{D}(t)$ satisfies*

$$\begin{aligned}
\frac{d}{dt}\mathscr{D}(t) &\leq (c^\star \epsilon + \epsilon - \beta(t)) \int_\Omega u_x^2 dx - b \int_{L_1}^{L_2} v_x^2 dx \\
&\quad + \frac{1}{4\epsilon}(a - \beta(t)) \int_\Omega (g \square u_x) dx + \frac{\mu_2^2}{4\epsilon} \int_\Omega u_t^2(x,t-\tau(t))dx \\
&\quad + \int_\Omega u_t^2\,dx + \int_{L_1}^{L_2} v_t^2\,dx.
\end{aligned} \tag{4.3.13}$$

Proof. Taking the derivative of $\mathscr{D}(t)$ with respect to t and using (4.2.4) , we find that

$$\begin{aligned}\frac{d}{dt}\mathscr{D}(t) &= \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2\, dx - \int_\Omega (au_x - g\star u_x)u_x, dx - b\int_{L_1}^{L_2} v_x^2\, dx \\ &\quad - |\mu_2|\int_\Omega u(x,t)u_t(x,t-\tau(t))dx \\ &= \int_\Omega u_t^2 - \beta(t)\int_\Omega u_x^2 - \int_\Omega (g\diamond u_x)u_x dx - |\mu_2|\int_\Omega u(x,t)u_t(x,t-\tau(t))dx.\end{aligned} \tag{4.3.14}$$

On the other hand, we have by Poincaré's inequality and Young's inequality,

$$|\mu_2|\int_\Omega u(x,t)u_t(x,t-\tau(t))\, dx \le \frac{\mu_2^2}{4\epsilon}\int_\Omega u_t(x,t-\tau(t))dx + c^\star\epsilon\int_\Omega u_x^2\, dx, \tag{4.3.15}$$

where $\epsilon > 0$.
Young's inequality and (G1) imply that

$$\begin{aligned}\int_\Omega (g\diamond u_x)u_x dx &\le \epsilon\int_\Omega u_x^2\, dx + \frac{1}{4\epsilon}\int_\Omega (g\diamond u_x)^2 dx \\ &\le \epsilon\int_\Omega u_x^2 dx + \frac{1}{4\epsilon}(a-\beta(t))\int_\Omega (g\Box u_x)dx.\end{aligned} \tag{4.3.16}$$

Inserting the estimates (4.3.15, and(4.3.16) into (4.3.14), then (4.3.13) is fulfilled. The proof is completed.

□

Now, inspired by [73], we introduce the function

$$q(x) = \begin{cases} x - \dfrac{L_1}{2}, & x\in[0,L_1], \\ x - \dfrac{L_2+L_3}{2}, & x\in[L_2,L_3], \\ \dfrac{L_2-L_3-L_1}{2(L_2-L_1)}(x-L_1) + \dfrac{L_1}{2}, & x\in[L_1,L_2] \end{cases} \tag{4.3.17}$$

It is easy to see that $q(x)$ is bounded, that is $|q(x)| \le M$, where $M = \max\left(\frac{L_1}{2}, \frac{L_3-L_2}{2}\right)$ is a positive constant.
Next, we define the functionals

$$\mathscr{F}_1(t) = -\int_\Omega q(x)u_t(au_x - g\star u_x)\, dx, \quad \mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x)v_x v_t\, dx.$$

Then, we have the following estimates.

Lemma 4.3.4. *For any $\epsilon_1 > 0$, we have the estimates:*

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) \leq & \left[-\frac{q(x)}{2}(au_x - g\star u_x)^2\right]_{\partial\Omega} - \left[\frac{a}{2}q(x)u_t^2\right]_{\partial\Omega} + \left[\frac{a}{2} + \frac{\mu_1^2}{2\epsilon_1} + \frac{M^2}{4\epsilon_1}\right]\int_\Omega u_t^2\,dx \\ & + \left[\epsilon_1 M^2 a^2 + \beta^2(t) + 2M^2\epsilon_1(a-\beta(t))^2 + c^2\epsilon_1\right]\int_\Omega u_x^2 dx + \frac{\mu_2^2}{2\epsilon_1}\int_\Omega u_t^2(x,t-\tau(t))dx \\ & + \Big(1 + 2M^2\epsilon_1)(a-\beta(t)\Big)\int_\Omega (g\Box u_x)dx + (a-\beta(t))\epsilon_1\int_\Omega (g'\Box u_x)dx.\end{aligned} \tag{4.3.18}$$

and

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_2(t) \leq & \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\Big(\int_{L_1}^{L_2} v_t^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx\Big) \\ & + \frac{b}{4}\Big((L_3 - L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\Big).\end{aligned} \tag{4.3.19}$$

Proof. Taking the derivative of $\mathscr{F}_1(t)$ with respect to t and using (4.1.1), we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) = & -\int_\Omega q(x)u_{tt}(au_x - g\star u_x)dt\,dx - \int_\Omega q(x)u_t(au_{xt} - g(t)u_x(t) + (g'\diamond u_x)(t))\,dx \\ = & \left[\frac{q(x)}{2}(au_x - g\star u_x)^2\right]_{\partial\Omega} + \frac{1}{2}\int_\Omega q'(x)(au_x - u_x)^2 dx - \left[\frac{a}{2}q(x)u_t^2\right]_{\partial\Omega} dx \\ & + \frac{a}{2}\int_\Omega q'(x)u_t^2 dx - \int_\Omega q(x)(\mu_1 u_t(x,t) + |\mu_2|u_t(x,t-\tau(t))(g\star u_x)dx \\ & + \int_\Omega q(x)au_x(\mu_1 u_t(x,t) + |\mu_2|u_t(x,t-\tau(t))dx \\ & - \int_\Omega q(x)u_t[(g'\diamond u_x)(t) - g(t)u_x]dx.\end{aligned} \tag{4.3.20}$$

We note that

$$\begin{aligned}\frac{1}{2}\int_\Omega q'(x)(au_x - g\star u_x)^2 dx = & \frac{1}{2}\int_\Omega \left[\left(a - \int_0^t g(s)ds\right)u_x + g\diamond u_x\right]^2 dx \\ \leq & \int_\Omega |\beta(t)|^2 u_x^2 dx + \int_\Omega |g\diamond u_x|^2 dx \\ \leq & \int_\Omega |\beta(t)|^2 u_x^2 dx + (a-\beta(t))\int_\Omega (g\Box u_x)dx.\end{aligned} \tag{4.3.21}$$

Young's inequality gives, for any $\epsilon_1 > 0$

$$\begin{aligned}& \int_\Omega q(x)au_x(\mu_1 u_t(x,t) + |\mu_2|u_t(x,t-\tau(t))dx \\ & \leq \epsilon_1 M^2 a^2\int_\Omega u_x^2 dx + \frac{\mu_1^2}{4\epsilon_1}\int_\Omega u_t^2 dx + \frac{\mu_2^2}{4\epsilon_1}\int_\Omega u_t^2(x,t-\tau(t))dx,\end{aligned} \tag{4.3.22}$$

and

$$\begin{aligned}
&\int_\Omega q(x)u_x(\mu_1 u_t(x,t) + |\mu_2| u_t^2(x,t-\tau(t))(g\star u_x)dx \\
&\leq \epsilon_1 M^2 \int_\Omega (g\star u_x)^2 dx + \frac{\mu_1^2}{4\epsilon_1}\int_\Omega u_t^2 dx + \frac{\mu_2^2}{4\epsilon_1}\int_\Omega u_t^2(x,t-\tau(t))dx \\
&\leq 2\epsilon_1 M^2 (a-\beta(t))^2 \int_\Omega u_x^2 dx + 2\epsilon_1 M^2 (a-\beta(t)) \int_\Omega (g\Box u_x)dx + \int_\Omega u_t^2 dx \\
&+ \frac{\mu_2^2}{4\epsilon_1}\int_\Omega u_t^2(x,t-\tau(t))dx,
\end{aligned} \tag{4.3.23}$$

in addition

$$\begin{aligned}
&\int_\Omega q(x)u_t[(g'\diamond u_x)(t) - g(t)u_x]dx \\
&\leq \frac{M^2}{4\epsilon_1}\int_\Omega u_t^2 dx + c^2\epsilon_1 \int_\Omega u_x^2 dx + (a-\beta(t))\epsilon_1 \int_\Omega (g'\Box u_x)dx.
\end{aligned} \tag{4.3.24}$$

Inserting (4.3.21)-(4.3.24), in (4.3.20) we get (4.3.18).

By the same method, taking the derivative of $\mathscr{F}_2(t)$ with respect to t, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathscr{F}_2(t) &= -\int_{L_1}^{L_2} q(x)v_{tx}v_t\,dx - \int_{L_1}^{L_2} q(x)v_x v_{tt} \\
&= \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2\,dx - \frac{1}{2}[q(x)v_t^2]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2\,dx - \frac{b}{2}[q(x)v_x^2]_{L_1}^{L_2} \\
&\leq \frac{L_2-L_3-L_1}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx\Big) \\
&\quad + \frac{b}{4}\Big((L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\Big).
\end{aligned} \tag{4.3.25}$$

which is exactly (4.3.19). □

We denote by

$$c_2 = |a-\beta(t)|, \quad c_3 = \max((1+M^2\epsilon_1)c_2, \frac{1}{4\epsilon}c_2). \tag{4.3.26}$$

Proof of Theorem 2.3

We define the Lyapunov functional

$$\mathscr{L}(t) = \gamma_1 E(t) + I(t) + \gamma_2\mathscr{D}(t) + \gamma_3\mathscr{F}_1(t) + \gamma_4\mathscr{F}_2(t), \tag{4.3.27}$$

where $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are positive constants that will be fixed later.

Taking the derivative of (4.3.27) with respect to t and making use of the above lammas, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathscr{L}(t) \leq & \Big\{ -\gamma_1 c_0 + 1 + \gamma_2 + \gamma_3(\frac{a}{2} + \frac{\mu_1^2}{2\epsilon_1} + \frac{M^2}{4\epsilon_1})\Big\} \int_\Omega u_t^2\, dx \\
& + \Big\{ -\gamma_1 c_0 - c_1 + \frac{\mu_2^2\gamma_2}{4\epsilon} + \frac{\mu_2^2\gamma_3}{2\epsilon_1}\Big\} \int_\Omega u_t^2(x, t-\tau(t))\, dx \\
& + \Big\{ -\gamma_2((\beta(t) - \epsilon(c^\star + 1)) \\
& \quad + \gamma_3(\epsilon_1 M^2 a^2 + \beta^2(t) + 2M^2\epsilon_1(a-\beta(t))^2 + c^2\epsilon_1)\Big\} \int_\Omega u_x^2\, dx \\
& + \Big\{ b\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\gamma_4 - \gamma_2 b\Big\} \int_{L_1}^{L_2} v_x^2\, dx \\
& + \Big\{ \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\gamma_4 + \gamma_2\Big\} \int_{L_1}^{L_2} v_t^2\, dx \\
& + (-\gamma_4 - a\gamma_3)\left[\frac{L_4}{4} v_x^2(L_1, t) + \frac{L_3 - L_2}{4} v_x^2(L_2, t)\right] \\
& + c_3(\gamma_4 + \gamma_3)\int_\Omega (g\Box u_x) dx + (\frac{\gamma_1}{2} - c_2\gamma_3)\int_\Omega (g'\Box u_x) dx.
\end{aligned} \tag{4.3.28}$$

At this point, we choose our constants in (4.3.28) carefully, such that all the coefficients in (4.3.28) will be negative except the last two.
Indeed, under the assumption (4.2.10), we can always find γ_2, γ_3 and γ_4 such that

$$\gamma_2 < \frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\gamma_4, \quad \gamma_4 > a\gamma_3, \quad \gamma_2 > \gamma_3\beta_0. \tag{4.3.29}$$

Once the above constants are fixed, we may choose ϵ and ϵ_1 small enough such that

$$\gamma_2\epsilon(c^\star + 1) + \gamma_3(\epsilon_1 M^2 a^2 + 2M^2\epsilon_1(a - \beta(t))^2 + c^2\epsilon_1) < \gamma_2 - \gamma_3\beta(t).$$

Now we choose γ_1 large enough such that the coefficients of the two first terms in (4.3.28) are negative and the last coefficient is positive. Then we deduce that there exists two positive constants α_1 and α_2 such that (4.3.28 becomes

$$\frac{d}{dt}\mathscr{L}(t) \leq -\alpha_1 E(t) + \alpha_2 \int_\Omega (g\Box u_x) dx. \tag{4.3.30}$$

On the other hand, by the definition of the functionals $\mathscr{D}(t), \mathscr{F}_1(t), \mathscr{F}_2(t), I(t)$, and $E(t)$, for N large enough, there exists a positive constant α_3 satisfaying

$$|\gamma_2\mathscr{D}(t) + \gamma_3\mathscr{F}_1(t) + \gamma_4\mathscr{F}_2(t) + I(t)| \leq \alpha_3 E(t), \tag{4.3.31}$$

which implies that

$$(\gamma_1 - \alpha_3)E(t) \leq \mathscr{L}(t) \leq (\gamma_1 + \alpha_3)E(t). \tag{4.3.32}$$

On the other hand we have, using $(G1)$ and (4.3.1), we have

$$\xi(t)\int_\Omega (g\Box u_x)dx \leq \int_\Omega ((\xi g)\Box u_x))dx \leq -\int_\Omega (g'\Box u_x)dx \leq -2\frac{d}{dt}E(t). \tag{4.3.33}$$

We define the functional $L(t)$ as

$$L(t) = \xi(t)\mathscr{L}(t) + 2\alpha_2 E(t).$$

The fact that $\mathscr{L}$ and $E(t)$ are equivalent and $(G2)$ imply that, for some positive constants η_1, and η_2,

$$\eta_1 E(t) \leq L(t) \leq \eta_2 E(t). \tag{4.3.34}$$

Using (4.3.33), (4.3.34) and with $(G2)$, we get

$$\begin{aligned}
\frac{d}{dt}L(t) &= \xi'(t)\mathscr{L}(t) + \xi(t)\frac{d}{dt}\mathscr{L}(t) + 2\alpha_2\frac{d}{dt}E(t) \\
&\leq \xi(t)\left(-\alpha_1 E(t) + \alpha_2\int_\Omega (g\Box u_x)dx\right) + 2\alpha_2\frac{d}{dt}E(t) \\
&\leq -\alpha_1\xi(t)E(t) \\
&\leq -d_0\xi(t)L(t),
\end{aligned} \tag{4.3.35}$$

where $d_0 = \dfrac{\alpha_1}{\eta_2}$ we conclude that, for any $d_1 \in (0, d_0)$,

$$\frac{d}{dt}L(t) \leq -d_1\xi(t)L(t). \tag{4.3.36}$$

Integration of (4.3.36) over $(0,t)$ we obtain

$$L(t) \leq L(0)e^{-d_1\int_0^t \xi(s)ds}, \ \forall t \geq 0. \tag{4.3.37}$$

Then (4.2.11) holds. The proof of theorem 2.3 is complete.

Part II

Boundary Value Problem

Chapter 5

An Evolutionary Boundary Value Problem

5.1 Introduction

In these last few years the theory of variational inequalities, is being developped very fast, having as model the variational theory of boundary value problems for partial differential equations. The theory of varitional inequalities represents, in very natural generalization of theory of boundary value problems and allows us to consider new problems arising from many fields of applied Mathematics, such as Mechanics, Physics, the Theory of convex programming and the Theory of control, and in engineering science.

While the variational theory of boundary value problems has its starting point in the method of orthogonal projection, the theory of variational inequalities has its starting point in the projection on a convex set. The first existence theorem for variational inequalities was proved in connection with the theory of second order equations with discontinuous coefficients in order to bring together again, as it was at the bigining, potential theory and theory of elliptic partial differential equations.

For instance, general results on the analysis of variational inequalities, including existence and uniqueness results, can be found in [7, 18, 44, 45, 49, 57, 68]. Phenomena of contact between deformable bodies or between deformable and rigid bodies abound in industry and everyday life. Contact of, braking pads with wheels, tires with roads, pistons with skirts are just few simple examples. Common industrial processes such as metal forming, metal extrusion, involve contact evolutions. Owing to their inherent complexity, contact phenomena lead to mathematical models expressed in terms of strongly nonlinear elliptic or evolutionary boundary value problems. For this reason, considerable progress has been achieved recently in modelling mathematical analysis. To this end, it uses various mathematical concepts which include both variational and hemivariational inequalities and multivalued inclusions. An excelent reference on analysis of contact problems involving elastic materials with or without friction is in [7, 18, 44, 45, 49, 57, 68]. The variational analysis of various contact problems can be found in [102]. Existence, uniqueness and regularity results in the study of a new class of variational inequalities were proved in [35, 50, 52, 49, 56, 86, 88, 97, 102, 101, 103]. We shall use a method called penalization. We want to emphasize that this method, often used to demonstrate the regularity of solutions, may sometimes be used to obtain this existence when the assumptions of general theorems do not apply. The method of penalization consists in substiuting the variational inequality by a family of nonlinear boundary value problems and demonstrating that their solutions converge to the solution of the variational inequality. The main difficulty lies in obtaining suitable priori estimate. We stress that there are different choices of penalization. Some recent works in optimization theory have shown some relationships between the optimality conditions, the notion of a gap function, and the solution of variational inequalities. The results established in [6, 104] explicitly refer to a ralationship between the gap function in optimization theory and a variational view. On way to solve the constrained optimization problems is to approximate the problem with a function which includes a penality term; see [9, 83]. In other words, in the case that the variational inequality formulation of equilibrium conditions underlying a specific problem is characterized by a function with a symetric Jacobian, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same.

The study of the dual of the elastic problems with given normal stress and Comlomb's law of dry was a subject to several works that the dualization can be used in deriving criteria for a given problem indirectly by considering one of the alternative formulation. Also, a given problem can often be solved more easily by way of a dual method, see [102] pages 155-161. The purpose of the second part of this thesis is, to introduce the reader to a mathematical theory of contact problems involving deformable bodies. This concerning the mathematical modeling and the variational analysis of the models, including existence, uniqueness and convergence results. The contact is frictionless and modeled with various conditions, including normal compliance and memory term.

Our aim in this chapter is to provide the variational analysis of an initial boundary value problem by using arguments of evolutionary variational inequalities and history-dependent operators. Recently, there is an interest in the study of a special class of inequalities, the so-called history-dependent variational inequalities. There are inequalities in which various functions or operators depend on the history of the solution. Their study is motivated by important application in problems involving constitutive laws for materials with memory, total slip or total slip rate friction laws.

The problem we are interested in this chapter leads, in a primal variational formulation, to an evolutionary variational inequality. In contrast, its dual variational formulation is in a form of a histroy-dependent variational inequality. To introduce this problem let L, h and T be given positive constants and denote $\Omega = (0, L) \times (-h, h)$. Everywhere below we use the notation (x, y) for a generic point in Ω and the subscripts x and y will represent the partial derivative with respect to the variables. The problem under consideration is the following.
Denote $\Omega = (0, L) \times (-h, h)$ where $L > 0$ and $h > 0$. Let $T > 0$. We consider the following boundary value problem.

Problem $\mathcal{P}$. *Find the functions* $u = u(x, y, t) : [0, L] \times [-h, h] \times [0, T] \to \mathbb{R}$ *and* $w = (x, t) : [0, L] \times [0, T] \to \mathbb{R}$ *such that*

$$\lambda \dot{u}_{xx} + E u_{xx} + \mu \dot{u}_{yy} + G u_{yy} + q_B = 0 \qquad \text{for all } \ (x, y) \in \Omega, \ t \in [0, T], \tag{5.1.1}$$

$$\mu \dot{w}_{xx} + G w_{xx} + (\lambda - \mu) \dot{u}_{xy} + (E - G) u_{xy} + f_B = 0 \qquad \text{for all } \ (x, y) \in \Omega, \ t \in [0, T], \tag{5.1.2}$$

$$u(0, y, t) = w(0, t) = 0 \quad \text{for all } \ y \in [-h, h], \ t \in [0, T], \tag{5.1.3}$$

$$\lambda \dot{u}_x(L,y,t) + E u_x(L,y,t) = 0 \tag{5.1.4}$$
$$\text{for all} \quad y \in [-h,h], \quad t \in [0,T],$$

$$\mu(\dot{u}_y(L,y,t) + \dot{w}_x(L,y,t)) + G(u_y(L,y,t) + w_x(L,y,t)) = 0 \tag{5.1.5}$$
$$\text{for all} \quad y \in [-h,h], \quad t \in [0,T].$$

$$\mu(\dot{u}_y(x,h,t) + \dot{w}_x(x,t)) + G(u_y(x,h,t) + w_x(x,t)) = q_N(x,t) \tag{5.1.6}$$
$$\text{for all} \quad x \in [0,L], \quad t \in [0,T],$$

$$(\lambda - 2\mu)\dot{u}_x(x,h,t) + (E - 2G)u_x(x,h,t) = f_N(x,t) \tag{5.1.7}$$
$$\text{for all} \quad x \in [0,L], \quad t \in [0,T].$$

$$|(\lambda - 2\mu)(\dot{u}_x(x,-h,t) + (E - 2G)(u_x(x,-h,t)| \leq g, \tag{5.1.8}$$
$$-(\lambda - 2\mu)(\dot{u}_x(x,-h,t) - (E - 2G)(u_x(x,-h,t) = g\frac{\dot{w}(x,t)}{|\dot{w}(x,t)|}$$
$$\text{if } \dot{w}(x,t) \neq 0, \quad \text{for all} \quad x \in [0,L], \quad t \in [0,T],$$
$$\mu(\dot{u}(x,-h,t) + \dot{w}(x,t)) + G(u_y(x,-h,t) + w_x(x,t)) = 0 \tag{5.1.9}$$

$$u(x,y,0) = u_0(x,y), \qquad w(x,0) = w_0(x), \tag{5.1.10}$$
$$\text{for all} \quad x \in [0,L], \; y \in [-h,h].$$

Problem $\mathcal{P}$ describes the equilibrium of a viscoelastic plate submitted to the action of body forces and tractions and to unilateral constraints on the boundary. Here Ω represents the cross section of the plate, u is the horizontal displacement and w is the vertical displacement. λ and μ are positive viscosity coefficients and E and G are positive elastic constant.

A brief description of equations and boundary condition in in Problem $\mathcal{P}$, including their mechanical significance, follows.

First, equations (5.1.1) and (5.1.2) represents the equilibrium equation in which The functions $q_B = q_B(x,y,t) : \Omega \times [0,T] \to \mathbb{R}$ and $f_B = f_B(x,y,t) : \Omega \times [0,T] \to \mathbb{R}$ are the horizontal and the vertical components of the body forces.

Condition (5.1.3) shows that the plate is fixed on the boundary $x = 0$ and conditions (5.1.4), (5.1.5) show that the boundary $x = L$ is free of tractions. Next, conditions

(5.1.6), (5.1.7) represent the traction conditions. Here, the functions $q_N = q_N(x,t) : [0,L] \times [0,T] \to \mathbb{R}$ and $f_N = f_N(x,t) : [0,L] \times [0,T] \to \mathbb{R}$ denote the horizontal and the vertical components of the traction forces which act on the top $y = h$ of the plate.

Condition (5.1.8) represents the bilateral contact condition on the bottom $x = -h$ and condition (5.1.9) represents the friction law, in which $g \geq 0$ is given.

Finally, (5.1.10) represents the initial condition in which the functions u and w are the initial horizontal and vertical displacement, respectively.

The rest of chapter is structured as follows. In Section 5.2 we we list the assumptions on the data and derive the variational formulation of problem $\mathcal{P}$. In Section 5.3 we state and prove our main result, Theorem 5.3.1, which states the unique weak solvability of the problem. The proof is based on arguments of evolutionary variational inequalities. In Section 5.4 we state and prove a convergence result, Theorem 5.4.1. It states the continuous dependence of the solution with respect to the data. Finally, in Section 5.5 we introduce the dual variational formulation of Problem $\mathcal{P}$ for which we prove an existence, uniqueness and equivalence result, Theorem 5.5.2.

5.2 Variational formulation

We start with some notation and preliminaries. Thus, for any real Hilbert space Y we denote by $\langle\cdot,\cdot\rangle_Y$ its inner product and by $\|\cdot\|_Y$ the associate norm, i.e. $\|y\|_Y^2 = \langle u, u\rangle_Y$ for all $y \in Y$. For a normed space Y we denote by $C([0,T];Y)$ the space of the continuous functions defined on $[0,T]$ with values to Y, equiped with the canonic norm. Moreover, $\|\cdot\|_{\mathcal{L}(Y,Z)}$ denotes the norm in the space of linear continuous operators on Y with values on the normed space Z

Everywhere below we use the standard notation for Lebesgue and Sobolev spaces. In addition, recalling that $\Omega = (0,L) \times (-h,h)$, we introduce the spaces

$$V = \{u \in H^1(\Omega) : u(0,\cdot) = 0\}, \quad W = \{w \in H^1(0,L) : w(0) = 0\}. \tag{5.2.1}$$

Note that equalities $u(0,\cdot) = 0$ and $w(0) = 0$ in the definitions of the spaces V and W are understood in the sense of traces. The spaces V and W are real Hilbert spaces with the cannonical inner products defined by

$$\langle u,\psi\rangle_V = \iint_\Omega (u\psi + u_x\psi_x + u_y\psi_y)\,dxdy \qquad \forall\, u,\psi \in V, \tag{5.2.2}$$

$$\langle w,\varphi\rangle_W = \int_0^L (w\varphi + w_x\varphi_x)\,dx \qquad \forall\, w,\varphi \in W. \tag{5.2.3}$$

We also consider the product space $X = V \times W$ equipped with the cannonical inner product given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_X = \langle u, \psi \rangle_V + \langle w, \varphi \rangle_W \qquad \forall\, \mathbf{u} = (u, w),\ \mathbf{v} = (\psi, \varphi) \in X, \tag{5.2.4}$$

We also consider the Hilbert space $U = L^2(0, L)$ endowed with its cannonical inner product.

On the data of Problem $\mathcal{P}$ we make the following hypothesis.

$$\lambda > 0, \quad E > 0, \quad \mu > 0, \quad G > 0. \tag{5.2.5}$$

$$f_B \in L^2(0, T; L^2(\Omega)), \quad q_B \in L^2(0, T; L^2(\Omega)). \tag{5.2.6}$$

$$f_N \in L^2(0, T; U), \quad q_N \in L^2(0, T; U). \tag{5.2.7}$$

$$g \geq 0. \tag{5.2.8}$$

$$u_0 \in V, \quad w_0 \in W. \tag{5.2.9}$$

Under these assumptions we define the operators $A, B : X \to X$ the functional $j : X \to \mathbb{R}$ and the function $\boldsymbol{f} : [0, T] \to X$ by equalities

$$\langle A\mathbf{u}, \mathbf{v} \rangle_X = \lambda \iint_\Omega u_x \psi_x \, dxdy + \mu \iint_\Omega (u_y + w_x)(\psi_y + \varphi_x) \, dxdy, \tag{5.2.10}$$

$$\langle B\mathbf{u}, \mathbf{v} \rangle_X = E \iint_\Omega u_x \psi_x \, dxdy + G \iint_\Omega (u_y + w_x)(\psi_y + \varphi_x) \, dxdy, \tag{5.2.11}$$

$$j(\mathbf{v}) = g \int_0^L |\varphi| \, dx, \tag{5.2.12}$$

$$\langle \boldsymbol{f}(t), v \rangle_X \tag{5.2.13}$$

$$= \iint_\Omega q_B(t)\psi \, dxdy + \iint_\Omega f_B(t)\varphi \, dxdy + \int_0^L q_N(t)\psi \, dx + \int_0^L f_N(t)\varphi \, dx$$

$$\forall\, \mathbf{u} = (u, w),\ \mathbf{v} = (\psi, \varphi),\ \mathbf{v} = (\psi, \varphi) \in X,\ t \in [0, T]$$

We also consider the initial data $\mathbf{u}_0 \in X$ given by

$$\mathbf{u}_0 = (u_0, v_0). \tag{5.2.14}$$

Note that the definitions above we do not specify the dependence of various functions on the variables x and y.

With these preliminaries we are in a position to derive the variational formulation of the Problems $\mathcal{P}$. We proceed formally. We assume in what follows that $\mathbf{u} = (u(x,y,t), w(x,t))$ reprents a solution to the problem $\mathcal{P}$ and let $\mathbf{v} = (\psi(x,y), \varphi(x)) \in \mathbf{X}, t \in [0,T]$ be fixed. Then multiplying (5.1.1) by $\psi - \dot{u}$ and integrating over Ω, we obtain

$$\begin{aligned}
&\iint_\Omega \lambda \dot{u}_{xx}(x,y,t)(\psi(x,y) - \dot{u}(x,y,t))\,dxdy \\
&\quad + \iint_\Omega E u_{xx}(x,y,t)(\psi(x,y) - \dot{u}(x,y,t))\,dxdy \\
&\quad + \iint_\Omega \mu \dot{u}_{yy}(x,y,t)(\psi(x,y) - \dot{u}(x,y,t)\,dxdy. \\
&\quad + \iint_\Omega G u_{yy}(x,y,t)(\psi(x,y) - \dot{u}(x,y,t)\,dxdy + \iint_\Omega q_B dxdy = 0.
\end{aligned} \tag{5.2.15}$$

Using Green's formulla we have,

$$\begin{aligned}
&\iint_\Omega E u_{xx}(x,y,t)(\psi(x,y) - \dot{u}(x,y,t))\,dxdy \\
&\quad = E \int_{-h}^{h} u_x(L,y,t)(\psi(L,y) - \dot{u}(L,y,t))\,dy \\
&\quad\quad - E \int_{-h}^{h} u_x(0,y,t)(\psi(0,y) - \dot{u}(0,y,t))\,dy \\
&\quad\quad\quad - E \iint_\Omega u_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy.
\end{aligned} \tag{5.2.16}$$

Similar arguments show that

$$\begin{aligned}
&\iint_\Omega G u_{yy}(x,y,t)(\psi(x,y) - \dot{u}(x,y,t))\,dxdy \\
&\quad = -G \int_0^L u_y(x,-h,t)(\psi(x,-h) - \dot{u}(x,-h,t))\,dx \\
&\quad\quad + G \int_0^L u_y(x,h,t)(\psi(x,h) - \dot{u}(x,h,t))\,dx \\
&\quad\quad\quad - G \iint_\Omega u_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy.
\end{aligned} \tag{5.2.17}$$

$$\begin{aligned}
&\iint_\Omega \lambda \dot{u}_{xx}(x,y,t)(\psi(x,y) - \dot{u}(x,y,t))\,dxdy \qquad (5.2.18)\\
&= \lambda \int_{-h}^{h} \dot{u}_x(L,y,t)(\psi(L,y) - \dot{u}(L,y,t))\,dy\\
&\quad -\lambda \int_{-h}^{h} \dot{u}_x(0,y,t)(\psi(0,y) - \dot{u}(0,y,t))\,dy\\
&\quad -\lambda \iint_\Omega \dot{u}_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy.
\end{aligned}$$

$$\begin{aligned}
&\iint_\Omega \mu \dot{u}_{yy}(x,y,t)(\psi(x,y) - \dot{u}(x,y,t))\,dxdy \qquad (5.2.19)\\
&= -\mu \int_0^L \dot{u}_y(x,-h,t)(\psi(x,-h) - \dot{u}(x,-h,t))\,dx\\
&\quad +\mu \int_0^L \dot{u}_y(x,h,t)(\psi(x,h) - \dot{u}(x,h,t))\,dx\\
&\quad -\mu \iint_\Omega \dot{u}_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy.
\end{aligned}$$

We now add the equalities (5.2.16)–(5.2.19), then we use the boundary conditions (5.1.3), (5.1.4) and the definition (5.2.1) of the space V to deduce that

$$\begin{aligned}
&E \iint_\Omega u_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy\\
&\quad +G \iint_\Omega u_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy\\
&\quad +\lambda \iint_\Omega \dot{u}_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy.\\
&\quad +\mu \iint_\Omega \dot{u}_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy.\\
&\quad = \int_0^L (Gu_y(x,h,t) + \mu\dot{u}_y(x,h,t))(\psi(x,h) - \dot{u}(x,h,t))\,dy\\
&\quad - \int_0^L (Gu_y(x,-h,t) + \mu\dot{u}_y(x,-h,t))(\psi(x,-h) - \dot{u}(x,-h,t))\,dx\\
&\quad + \iint_\Omega q_B(t)(\psi(x,t) - \dot{u}(x,y,t))dxdy \qquad (5.2.20)
\end{aligned}$$

Assume now that $x \in [0,L]$ is fixed. We integrate equation (5.1.2) with respect to y on $[-h,h]$ and deduce that

$$\begin{aligned}
&2h\mu \dot{w}_{xx}(x,t) + 2hGw_{xx}(x,t)\\
&+ (\lambda - \mu) \int_{-h}^{h} \dot{u}_{xy}(x,y,t)\,dy + (E - G) \int_{-h}^{h} u_{xy}(x,y,t)\,dy + \int_{-h}^{h} f_B(t)dy = 0. \qquad (5.2.21)
\end{aligned}$$

Then, we write

$$\int_{-h}^{h} u_{xy}(x,y,t)\,dy = u_x(x,h,t) - u_x(x,-h,t)$$

and

$$\int_{-h}^{h} \dot{u}_{xy}(x,y,t)\,dy = \dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t)$$

To facilate the calculus we put

$$\sigma = (\lambda - 2\mu)\dot{u}_x(x,-h,t) + (E-2G)u_x(x,-h,t)$$

Using the (5.1.7) and (5.1.8) we obtain

$$\begin{aligned}
&(\lambda-\mu)\int_{-h}^{h} \dot{u}_{xy}(x,y,t)\,dy + (E-G)\int_{-h}^{h} u_{xy}(x,y,t)\,dy \qquad (5.2.22)\\
&\quad = (\lambda-\mu)(\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t))\\
&\quad\quad +(E-G)(u_x(x,h,t) - u_x(x,-h,t))\\
&\quad\quad = (\lambda-2\mu)(\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t))\\
&\quad\quad +(E-2G)(u_x(x,h,t) - u_x(x,-h,t))\\
&\quad\quad +\mu(\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t)) + G(u_x(x,h,t) - u_x(x,-h,t))\\
&\quad\quad = -\sigma + (\lambda-2\mu)\dot{u}_x(x,h,t) + (E-2G)u_x(x,h,t)\\
&\quad\quad +\mu(\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t)) + G(u_x(x,h,t) - u_x(x,-h,t))\\
&\quad\quad = f_N(x,t) - \sigma + \mu(\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t))\\
&\quad\quad\quad +G(u_x(x,h,t) - u_x(x,-h,t))
\end{aligned}$$

Next, we substract equalities (5.2.22) and (5.2.21) we deduce

$$\begin{aligned}
&-2hGw_{xx}(x,t) - 2h\mu\dot{w}_{xx}(x,t) \qquad (5.2.23)\\
&\quad = f_N(x,t) - \sigma + \mu(\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t))\\
&\quad\quad +G(u_x(x,h,t) - u_x(x,-h,t)) + \int_{-h}^{h} f_B(t)dy.
\end{aligned}$$

Consider now an element $\xi = \xi(x) \in W$. We multiply equality (5.2.23) with ξ, then we integrate the result on $[0,L]$ to obtain that

$$\begin{aligned}
&-2hG\int_0^L w_{xx}(x,t)\xi(x)\,dx - 2h\mu\int_0^L \dot{w}_{xx}(x,t)\xi(x)\,dx \qquad (5.2.24)\\
&= \int_0^L -\sigma\xi(x)dx + \mu\int_0^L (\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t))\xi(x)dx\\
&\quad +G\int_0^L (u_x(x,h,t) - u_x(x,-h,t))\xi(x)dx\\
&\quad +\int_0^L f_N\xi(x)dx + \iint_\Omega f_B(t)\xi(x)dxdy
\end{aligned}$$

Next, we perform an integration by parts and use equality $\xi(0) = 0$ to see that

$$\begin{aligned}
&-2hG\int_0^L w_{xx}(x,t)\xi(x)\,dx \\
&\quad = -2hGw_x(L,t)\xi(L) + 2hG\int_0^L w_x(x,t)\xi_x(x)\,dx, \\
&-2h\mu\int_0^L \dot{w}_{xx}(x,t)\xi(x)\,dx \\
&\quad = -2h\mu\dot{w}_x(L,t)\xi(L) + 2h\mu\int_0^L \dot{w}_x(x,t)\xi_x(x)\,dx,
\end{aligned} \tag{5.2.25}$$

Inserting (5.2.25) in (5.2.24) to obtain,

$$\begin{aligned}
&2hG\int_0^L w_x(x,t)\xi_x(x)\,dx + 2h\mu\int_0^L \dot{w}_x(x,t)\xi_x(x)\,dx \\
&\quad = \int_0^L -\sigma\xi(x)dx + \mu\int_0^L (\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t))\xi(x) \\
&\qquad +G\int_0^L u_x(x,h,t) - u_x(x,-h,t))\xi(x)dx \\
&\qquad +2hGw_x(L,t)\xi(L) + 2h\mu\dot{w}_x(L,t)\xi(L) \\
&\qquad +\int_0^L f_N\xi(x)dx + \iint_\Omega f_B\xi(x)dxdy
\end{aligned} \tag{5.2.26}$$

On the other hand we have the identity

$$2hG\int_0^L w_x(x,t)\xi_x(x)dx = G\iint_\Omega w_x(x,t)\xi_x(x)dxdy \tag{5.2.27}$$

Then (5.2.26) becomes

$$\begin{aligned}
&G\iint_\Omega w_x(x,t)\xi_x(x)\,dxdy + \mu\iint_\Omega \dot{w}_x(x,t)\xi_x(x)\,dxdy, \\
&\quad = \int_0^L -\sigma\xi(x)dx + \mu\int_0^L (\dot{u}_x(x,h,t) - \dot{u}_x(x,-h,t))\xi(x) \\
&\qquad +G\int_0^L u_x(x,h,t) - u_x(x,-h,t))\xi(x)dx \\
&\qquad +2hGw_x(L,t)\xi(L) + 2h\mu\dot{w}_x(L,t)\xi(L) \\
&\qquad +\int_0^L f_N\xi(x)dx + \iint_\Omega f_B\xi(x)dxdy
\end{aligned} \tag{5.2.28}$$

Now, we add equalities (5.2.20) and (5.2.28 taking $\xi = \varphi - \dot{w}$ to obtain

$$
\begin{aligned}
&E \iint_\Omega u_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy \qquad (5.2.29)\\
&\quad +G \iint_\Omega u_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy\\
&\quad\quad +\lambda \iint_\Omega \dot{u}_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy.\\
&\quad\quad +\mu \iint_\Omega \dot{u}_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy.\\
&\quad\quad\quad +G \iint_\Omega w_x(x,t)(\varphi_x(x) - \dot{w}_x(x,t))\,dxdy\\
&\quad\quad\quad +\mu \iint_\Omega \dot{w}_x(x,t)(\varphi_x(x,t) - \dot{w}_x(x,t))\,dxdy,
\end{aligned}
$$

$$
\begin{aligned}
&= \iint_\Omega q_B(t)(\psi(x,t) - \dot{u}(x,y,t))dxdy\\
&\quad + \int_0^L f_N(\varphi(x,t) - \dot{w}(x,t))dx\\
&\quad + \iint_\Omega f_B(\varphi(x,t) - \dot{w}(x,t))dxdy\\
&\quad\quad - \int_0^L \sigma(\varphi(x,t) - \dot{w}(x,t))dx\\
&\quad\quad +\mu \int_0^L \dot{u}_x(x,h,t)(\varphi(x,t) - \dot{w}(x,t))dx\\
&\quad\quad\quad +G \int_0^L u_x(x,h,t)(\varphi(x,t) - \dot{w}(x,t))dx\\
&\quad\quad\quad +G \int_0^L u_y(x,h,t)(\psi(x,h) - \dot{u}(x,h,t))\\
&\quad\quad\quad\quad +\mu \int_0^L \dot{u}_y(x,h,t))(\psi(x,h) - \dot{u}(x,h,t))\,dx\\
&\quad\quad\quad\quad -\mu \int_0^L \dot{u}_x(x,-h,t)(\varphi(x,t) - \dot{w}(x,t))dx\\
&\quad\quad\quad\quad\quad -G \int_0^L u_x(x,-h,t)(\varphi(x,t) - \dot{w}(x,t))dx\\
&\quad\quad\quad\quad\quad -G \int_0^L u_y(x,-h,t)(\psi(x,t) - \dot{u}(x,-h,t))\\
&\quad\quad\quad\quad\quad\quad -\mu \int_0^L \dot{u}_y(x,-h,t))(\psi(x,t) - \dot{u}(x,-h,t))\,dx\\
&\quad\quad\quad\quad\quad\quad +2hGw_x(L,t)\xi(L) + 2h\mu\dot{w}_x(L,t)\xi(L)
\end{aligned}
$$

On the other hand we have

$$\mu\int_0^L \dot{u}_x(x,h,t)(\varphi(x,t)-\dot{w}(x,t))dx \tag{5.2.30}$$
$$=\mu\dot{u}(L,h,t)(\varphi(L,t)-\dot{w}(L,t))-\mu\int_0^L \dot{u}(x,h,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dx.$$
$$G\int_0^L u_x(x,h,t)(\varphi(x,t)-\dot{w}(x,t))dx$$
$$=Gu(L,h,t)(\varphi(L,t)-\dot{w}(L,t))-G\int_0^L u(x,h,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dx.$$
$$-\mu\int_0^L \dot{u}_x(x,-h,t)(\varphi(x,t)-\dot{w}(x,t))dx$$
$$=-\mu\dot{u}(L,-h,t)(\varphi(L,t)-\dot{w}(L,t))-\mu\int_0^L \dot{u}(x,-h,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dx.$$
$$-G\int_0^L u_x(x,h,t)(\varphi(x,t)-\dot{w}(x,t))dx$$
$$=-Gu(L,-h,t)(\varphi(L,t)-\dot{w}(L,t))-G\int_0^L u(x,-h,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dx.$$
$$2h\mu\dot{w}_x(L,t)(\varphi(L,t)-\dot{w}(L,t))=\mu\int_{-h}^h \dot{w}_x(L,t)(\varphi(L,t)-\dot{w}(L,t))dy$$
$$2hGw_x(L,t)(\varphi(L,t)-\dot{w}(L,t))=G\int_{-h}^h w_x(L,t)(\varphi(L,t)-\dot{w}(L,t))dy$$

Adding this six equalities, we get

$$\Sigma=\mu\int_0^L \dot{u}_x(x,h,t)(\varphi(x,t)-\dot{w}(x,t))dx+G\int_0^L u_x(x,h,t)(\varphi(x,t)-\dot{w}(x,t))dx$$
$$-\mu\int_0^L \dot{u}_x(x,-h,t)(\varphi(x,t)-\dot{w}(x,t))dx-G\int_0^L u_x(x,h,t)(\varphi(x,t)-\dot{w}(x,t))dx$$
$$+2h\mu\dot{w}_x(L,t)(\varphi(L,t)-\dot{w}(L,t))+2hGw_x(L,t)(\varphi(L,t)-\dot{w}(L,t))$$

$$=\mu(\dot{u}_x(L,h,t)-\dot{u}_x(L,-h,t))(\varphi(L,t)-\dot{w}(L,t))dx$$
$$+G(u_x(L,h,t)-u_x(L,-h,t))(\varphi(L,t)-\dot{w}(L,t))dx$$
$$-\mu\int_0^L \dot{u}(x,h,t)-\dot{u}(x,-h,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dx$$
$$-G\int_0^L u(x,h,t)-u(x,-h,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dx$$
$$+\mu\int_{-h}^h \dot{w}_x(L,t)(\varphi(L,t)-\dot{w}(L,t))dy$$
$$+G\int_{-h}^h w_x(L,t)(\varphi(L,t)-\dot{w}(L,t))dy$$

On the other hand we have again

$$
\begin{aligned}
&G(u(L,h,t)-u(L,h,t))(\varphi(L,t)-\dot{w}(L,t))\\
&=G\int_{-h}^{h}u_y(L,y,t)(\varphi(L,t)-\dot{w}(L,t))dy\\
&\mu(\dot{u}(L,h,t)-\dot{u}(L,-h,t)(\varphi(L,t)-\dot{w}(L,t))dx\\
&=\mu\int_{-h}^{h}\dot{u}_y(L,y,t)(\varphi(L,t)-\dot{w}(L,t))dy\\
&-\mu\int_0^L(\dot{u}(x,h,t)-\dot{u}(x,-h,t))(\varphi_x(x,t)-\dot{w}_x(x,t))dx\\
&=-\mu\int_0^L\int_{-h}^{h}\dot{u}_y(x,y,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dxdy.\\
&-G\int_0^L(u(x,h,t)-u(x,-h,t))(\varphi_x(x,t)-\dot{w}_x(x,t))dx\\
&=-G\int_0^L\int_{-h}^{h}u_y(x,y,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dxdy.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Sigma &=\mu\int_{-h}^{h}\dot{u}_y(L,y,t)(\varphi(L,t)-\dot{w}(L,t))dy+G\int_{-h}^{h}u_y(L,y,t)(\varphi(L,t)-\dot{w}(L,t))dy\\
&-\mu\int_0^L\int_{-h}^{h}\dot{u}_y(x,y,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dxdy.\\
&-G\int_0^L\int_{-h}^{h}u_y(x,y,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dxdy.\\
&+\mu\int_{-h}^{h}\dot{w}_x(L,t)(\varphi(L,t)-\dot{w}(L,t))dy\\
&+G\int_{-h}^{h}w_x(L,t)(\varphi(L,t)-\dot{w}(L,t))dy\\
&=\int_{-h}^{h}\left[\mu(\dot{u}_y(L,y,t)+\dot{w}_x(L,t))+G(u_y(L,y,t)+w_x(L,t))\right](\varphi(L,t)-\dot{w}(L,t))dy\\
&-\mu\iint_\Omega\dot{u}_y(x,y,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dxdy-G\int_0^L u_y(\varphi_x(x,t)-\dot{w}_x(x,t))dxdy
\end{aligned}
$$

Using (5.1.5) we obtain

$$
\Sigma=-\mu\iint_\Omega\dot{u}_y(x,y,t)(\varphi_x(x,t)-\dot{w}_x(x,t))dxdy-G\int_0^L u_y(\varphi_x(x,t)-\dot{w}_x(x,t))dxdy \tag{5.2.31}
$$

Inserting (5.2.31) in (5.2.29) we get

$$
\begin{aligned}
&E \iint_\Omega u_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy \\
&\quad +G \iint_\Omega u_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy \\
&\qquad +\lambda \iint_\Omega \dot{u}_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy. \\
&\qquad +\mu \iint_\Omega \dot{u}_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy. \\
&\qquad\quad +G \iint_\Omega w_x(x,t)(\varphi_x(x) - \dot{w}_x(x,t))\,dxdy \\
&\qquad\quad +\mu \iint_\Omega \dot{w}_x(x,t)(\varphi_x(x,t) - \dot{w}_x(x,t))\,dxdy,
\end{aligned}
\tag{5.2.32}
$$

$$
\begin{aligned}
&+G \iint_\Omega w_x(x,t)(\varphi_x(x) - \dot{w}_x(x,t))\,dxdy \\
&\quad +\mu \iint_\Omega \dot{w}_x(x,t)(\varphi_x(x,t) - \dot{w}_x(x,t))\,dxdy, \\
&\quad = \iint_\Omega q_B(t)(\psi(x,t) - \dot{u}(x,y,t))dxdy \\
&\quad + \int_0^L f_N(\varphi(x,t) - \dot{w}(x,t))dx \\
&\quad + \iint_\Omega f_B(\varphi(x,t) - \dot{w}(x,t))dxdy \\
&\qquad - \int_0^L \sigma(\varphi(x,t) - \dot{w}(x,t))dx \\
&\qquad +\mu \int_0^L \dot{u}_y(x,h,t))(\psi(x,h) - \dot{u}(x,h,t))\,dx \\
&\qquad +G \int_0^L u_y(x,h,t)(\psi(x,h) - \dot{u}(x,h,t)) \\
&\qquad\quad -\mu \int_0^L \dot{u}_y(x,-h,t))(\psi(x,t) - \dot{u}(x,-h,t))\,dx \\
&\qquad\quad -G \int_0^L u_y(x,-h,t)(\psi(x,t) - \dot{u}(x,-h,t))
\end{aligned}
$$

From (5.1.6) we get

$$\mu\dot{u}_y(x,h,t) + Gu_y(x,h,t) = q_N(x,t) - \mu\dot{w}_x(x,t) - Gw_x(x,h,t)$$

Then

$$\begin{aligned}
&\mu \int_0^L \dot{u}_y(x,h,t))(\psi(x,t) - \dot{u}(x,h,t))\,dx \\
&+G \int_0^L u_y(x,h,t)(\psi(x,h) - \dot{u}(x,h,t)) \\
&= \int_0^L q_N(x,t)(\psi(x,t) - \dot{u}(x,h,t)dx - \mu \int_0^L \dot{w}_x(x,t)(\psi(x,t) - \dot{u}(x,h,t)dx \\
&\quad -G \int_0^L w_x(x,h,t)(\psi(x,t) - \dot{u}(x,h,t)dx
\end{aligned} \tag{5.2.33}$$

From (5.1.9) we get

$$-\mu\dot{u}_y(x,-h,t) - Gu_y(x,-h,t) = \mu\dot{w}_x(x,t) + Gw_x(x,h,t)$$

Then

$$\begin{aligned}
&-\mu \int_0^L \dot{u}_y(x,-h,t))(\psi(x,t) - \dot{u}(x,-h,t))\,dx \\
&-G \int_0^L u_y(x,-h,t)(\psi(x,t) - \dot{u}(x,h,t)) \\
&= \mu \int_0^L \dot{w}_x(x,t)(\psi(x,t) - \dot{u}(x,-h,t)dx \\
&\quad +G \int_0^L w_x(x,-h,t)(\psi(x,t) - \dot{u}(x,-h,t)dx
\end{aligned} \tag{5.2.34}$$

We combine (5.2.32),(5.2.33), and (5.2.34) to obtain

$$\begin{aligned}
&E \iint_\Omega u_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy \\
&\quad +G \iint_\Omega u_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy \\
&\qquad +\lambda \iint_\Omega \dot{u}_x(x,y,t)(\psi_x(x,y) - \dot{u}_x(x,y,t))\,dxdy. \\
&\qquad +\mu \iint_\Omega \dot{u}_y(x,y,t)(\psi_y(x,y) - \dot{u}_y(x,y,t))\,dxdy. \\
&\qquad\quad +G \iint_\Omega w_x(x,t)(\varphi_x(x) - \dot{w}_x(x,t))\,dxdy \\
&\qquad\quad +\mu \iint_\Omega \dot{w}_x(x,t)(\varphi_x(x,t) - \dot{w}_x(x,t))\,dxdy, \\
&\qquad\quad +G \iint_\Omega w_x(x,t)(\varphi_x(x) - \dot{w}_x(x,t))\,dxdy \\
&\qquad\quad +\mu \iint_\Omega \dot{w}_x(x,t)(\varphi_x(x,t) - \dot{w}_x(x,t))\,dxdy,
\end{aligned} \tag{5.2.35}$$

$$
\begin{aligned}
&= \iint_\Omega q_B(t)(\psi(x,t) - \dot{u}(x,y,t))dxdy \\
&+ \int_0^L f_N(\varphi(x,t) - \dot{w}(x,t))dx \\
&+ \iint_\Omega f_B(\varphi(x,t) - \dot{w}(x,t))dxdy \\
&\quad - \int_0^L \sigma(\varphi(x,t) - \dot{w}(x,t))dx \\
&\qquad + \int_0^L q_N(x,t)(\psi(x,t) - \dot{u}(x,h,t)dx \\
&\qquad -\mu \int_0^L \dot{w}_x(x,t)(\psi(x,t) - \dot{u}(x,h,t)dx \\
&\qquad -G \int_0^L w_x(x,t)(\psi(x,t) - \dot{u}(x,h,t)dx \\
&\qquad +\mu \int_0^L \dot{w}_x(x,t)(\psi(x,t) - \dot{u}(x,-h,t)dx \\
&\qquad +G \int_0^L w_x(x,-h,t)(\psi(x,t) - \dot{u}(x,-h,t)dx
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
(\overrightarrow{f}, \overrightarrow{v}) &= \iint_\Omega q_B(t)(\psi(x,t) - \dot{u}(x,y,t))dxdy \\
&+ \int_0^L f_N(\varphi(x,t) - \dot{w}(x,t))dx \\
&+ \iint_\Omega f_B(\varphi(x,t) - \dot{w}(x,t))dxdy \\
&\qquad + \int_0^L q_N(x,t)(\psi(x,t) - \dot{u}(x,h,t)dx \\
&\qquad \forall \overrightarrow{v} = (\psi, \varphi) \in X
\end{aligned}
\tag{5.2.36}
$$

We have again

$$
\begin{aligned}
&-\mu \int_0^L \dot{w}_x(x,t)(\psi(x,t) - \dot{u}(x,h,t)dx \\
&+\mu \int_0^L \dot{w}_x(x,t)(\psi(x,t) - \dot{u}(x,-h,t)dx \\
&= -\mu \int_0^L \dot{w}_x(x,t)((\psi(x,t) - \dot{u}(x,h,t) - (\psi(x,t) - \dot{u}(x,-h,t))dx \\
&= -\mu \int_0^L \int_{-h}^h \dot{w}_x(x,t)(\psi_y(x,t) - \dot{u}_y(x,y,t))dxdy \\
&= -\mu \iint_\Omega \dot{w}_x(x,t)(\psi_y(x,t) - \dot{u}_y(x,y,t))dxdy
\end{aligned}
\tag{5.2.37}
$$

and

$$
\begin{aligned}
&-G\int_0^L w_x(x,t)(\psi(x,t)-\dot{u}(x,h,t)dx \\
&+G\int_0^L w_x(x,t)(\psi(x,t)-\dot{u}(x,-h,t)dx \\
&= -G\int_0^L w(x,t)((\psi(x,t)-\dot{u}(x,h,t)-(\psi(x,t)-\dot{u}(x,-h,t))dx \\
&= -G\int_0^L\int_{-h}^h w_x(x,t)(\psi_y(x,t)-\dot{u}_y(x,y,t))dxdy \\
&= -G\iint_\Omega w_x(x,t)(\psi_y(x,t)-\dot{u}_y(x,y,t))dxdy
\end{aligned} \tag{5.2.38}
$$

Inserting (5.2.36),(5.2.37) ,and (5.2.38) in (5.2.35) with $v = \mathbf{v} - \dot{\mathbf{u}}(t)$ we obtain

$$
\begin{aligned}
&\langle A\dot{\mathbf{u}}(t), \mathbf{v}-\dot{\mathbf{u}}(t)\rangle_X + \langle B\mathbf{u}(t), \mathbf{v}-\dot{\mathbf{u}}(t)\rangle_X + \int_0^L \sigma(\varphi(x,t)-\dot{w}(x,t))dx \\
&\quad = \langle \boldsymbol{f}(t), \mathbf{v}-\dot{\mathbf{u}}(t)\rangle_X \qquad \text{for all } \mathbf{v}\in X, \quad t\in[0,T].
\end{aligned} \tag{5.2.39}
$$

On the other hand we have

$$
\int_0^L \sigma(\varphi(x,t)-\dot{w}(x,t))dx \le \int_0^L g|\varphi| - g|\dot{w}| = j(\mathbf{v}) - j(\dot{\mathbf{u}}) \tag{5.2.40}
$$

We now combine equality (5.2.39) with inequality (5.2.40) and then use the initial conditions (5.1.10) and notation (5.2.14). As a result we obtain the variational formulation of problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. *Find a function* $\mathbf{u} : [0,T] \to X$ *such that*

$$
\begin{aligned}
&\langle A\dot{\mathbf{u}}(t), \mathbf{v}-\dot{\mathbf{u}}(t)\rangle_X + \langle B\mathbf{u}(t), \mathbf{v}-\dot{\mathbf{u}}(t)\rangle_X + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \\
&\qquad \ge \langle \boldsymbol{f}(t), \mathbf{v}-\dot{\mathbf{u}}(t)\rangle_X \qquad \text{for all } \mathbf{v}\in X, \quad t\in[0,T],
\end{aligned} \tag{5.2.41}
$$

$$
\mathbf{u}(0) = \mathbf{u}_0. \tag{5.2.42}
$$

Note that Problem $\mathcal{P}_V$ represents an evolutionary variational inequality. Its unique solvability will presented in the next section. Here we restrict ourselves to mention that the solution of this inequality will be called a *weak solution* to Problem $\mathcal{P}$. We also mention that in Section 5.5 we provide a second variational formulation of Problem $\mathcal{P}$, the so-called dual variational formulation, which, in fact, is equivalent with Problem $\mathcal{P}_V$.

5.3 Existence and uniqueness

Our existence and uniqueness result in the study of Problem $\mathcal{P}_V$ is the following.

Theorem 5.3.1. *Assume* (5.2.5)–(5.2.9). *Then Problem $\mathcal{P}_V$ has a unique solution with regularity $\mathbf{u} \in C^1([0,T];X)$.*

The proof is carried out in several steps. The first one consists to investigate the properties of the operators A and B and, with this concern, we have the following results.

Lemma 5.3.2. *Assume that* (5.2.5) *holds. Then the operator A is linear, symmetric continuous and coercive, i.e. it satisfies*

$$\langle A\mathbf{v},\mathbf{v}\rangle_X \geq m_A\|\mathbf{v}\|_X^2 \qquad \text{for all } \mathbf{v}\in X, \quad \text{with} \quad m_A > 0. \tag{5.3.1}$$

Lemma 5.3.3. *Assume that* (5.2.5) *holds. Then the operator B is linear, symmetric and coercive, i.e. it satisfies*

$$\langle B\mathbf{v},\mathbf{v}\rangle_X \geq m_B\|\mathbf{u}\|_X^2 \qquad \text{for all } \mathbf{v}\in X, \quad \text{with} \quad m_B > 0. \tag{5.3.2}$$

The proof of Lemmas 5.3.2 and 5.3.3 are identical and are based on standard arguments. Nevertheless, for the convenience of the reader we present, for instance, the proof of Lemma 5.3.2.

Proof. The linearity and symmetry of the operator A are obvious. Moreover, an elementary computation shows that

$$\langle A\mathbf{v},\mathbf{v}\rangle_X \leq (\lambda+2\mu)\,\|\mathbf{u}\|_X\|\mathbf{v}\|_X \qquad \forall\,\mathbf{u},\,\mathbf{v}\in X. \tag{5.3.3}$$

which implies that A is continuous. Inequality (5.3.1) is a direct consequence of the two-dimensional version of Korn's inequality. Indeed, consider an arbitrary element $\mathbf{v} = (\psi(x,y),\varphi(x)) \in X$. Then, the small strain tensor associated to the two-dimensional displacement field $\mathbf{v}$ is given by

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \begin{pmatrix} \psi_x & \frac{1}{2}\left(\psi_y+\varphi_x\right) \\ \frac{1}{2}\left(\psi_y+\varphi_x\right) & 0 \end{pmatrix}.$$

We have

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 = \boldsymbol{\varepsilon}(\mathbf{v})\cdot\boldsymbol{\varepsilon}(\mathbf{v}) = \psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2 \qquad \text{a.e. on } \Omega. \tag{5.3.4}$$

Note also that the function $\mathbf{v}$ vanishes on the boundary $x = 0$ of the rectangle Ω which is, obviously, of positive one-dimensional measure and, in addition, since X can be identified as a subspace of $H^1(\Omega)^2$, we have $\mathbf{v} \in H^1(\Omega)^2$. Therefore, using Korn's inequality we obtain that there exists a constand $c_K > 0$ which depends on h such that

$$\iint_\Omega \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 \, dxdy \geq c_K \, \|\mathbf{v}\|^2_{H^1(\Omega)^2}. \tag{5.3.5}$$

We now combine (5.3.4) and (5.3.5) to deduce that

$$\iint_\Omega \left(\psi_x^2 + \frac{1}{2}(\psi_y + \varphi_x)^2\right) dxdy \geq c_K \iint_\Omega \left(\psi^2 + \psi_x^2 + \psi_y^2 + \varphi^2 + \varphi_x^2\right) dxdy$$

and then, using (5.2.2)–(5.2.4), we obtain that

$$\iint_\Omega \left(\psi_x^2 + \frac{1}{2}(\psi_y + \varphi_x)^2\right) dxdy \geq \widetilde{c}_K \|\mathbf{v}\|_X^2. \tag{5.3.6}$$

where $\widetilde{c}_K$ depends on c_K and L. On the other hand, using the definition (5.2.10) of the operator A and inequality (5.3.6) we deduce that

$$\langle A\mathbf{v}, \mathbf{v}\rangle_X \geq \min(\lambda, 2\mu) \iint_\Omega \left(\psi_x^2 + \frac{1}{2}(\psi_y + \varphi_x)^2\right) dxdy. \tag{5.3.7}$$

We now combine (5.3.6), (5.3.7) and assumption (5.2.5) to see that inequality (5.3.1) holds with $m_A = \widetilde{c}_K \min(\lambda, 2\mu) > 0$, which concludes the proof. □

We are now in a position to provide the proof of Theorem 5.3.1.

Proof. Using assumption (5.2.8) it is easy to see that the functional j is a continuous seminorm on the space X. Therefore, it follows from here that j is a convex lower semicontinuous function on X. In addition, assumptions (5.2.6), (5.2.7) and definition (5.2.13) imply that $\boldsymbol{f} \in C([0,T]; X)$. Moreover, assumption (5.2.9) shows that the initial data satisfy $\mathbf{u}_0 \in V$. Finally, Lemma 5.3.2 shows that $A : X \to X$ is a strongly monotone Lipschitz continuous operator and Lemma 5.3.3 implies that $B : X \to X$ is Lipschitz continuous operator. Theorem 5.3.1 is now a direct consequence of Theorem 3.11 in [102]. □

5.4 A continuous dependence result

In this section we study the dependence of the solution with respect the parametres E, G and g. To this end we assume that (5.2.5)–(5.2.9) hold and we consider some positive constants E_ρ, G_ρ and g_ρ which represent a perturbation of E, G and g, respectively. Here ρ denotes a positive parameter which will converge to zero. We define the operator B_ρ and the function j_ρ by equalities

$$\langle B_\rho \mathbf{u}, \mathbf{v}\rangle_X = E_\rho \iint_\Omega u_x \psi_x \, dxdy + G_\rho \iint_\Omega (u_y + w_x)(\psi_y + \varphi_x) \, dxdy, \tag{5.4.1}$$

$$j_\rho(\mathbf{v}) = g_\rho \int_0^L |\varphi(x)| \, dx \tag{5.4.2}$$

for all $\mathbf{u} = (u, w)$, $\mathbf{v} = (\psi, \varphi) \in X$. Then, we consider the following variational problem.

Problem $\mathcal{P}_V^\rho$. *Find a function $\mathbf{u}_\rho : [0,T] \to X$ such that*

$$\begin{aligned} &\langle A\dot{\mathbf{u}}_\rho(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t)\rangle_X + \langle (B_\rho \mathbf{u}_\rho)(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t)\rangle_X \\ &\quad + j_\rho(\mathbf{v}) - j_\rho(\dot{\mathbf{u}}_\rho(t)) \geq \langle \boldsymbol{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t)\rangle_X \quad \text{for all } \mathbf{v} \in X, \quad t \in [0,T]. \end{aligned} \tag{5.4.3}$$

$$\mathbf{u}_\rho(0) = \mathbf{u}_0. \tag{5.4.4}$$

Using Theorem 5.3.1 it follows that Problem $\mathcal{P}_V$ has a unique solution $\mathbf{u} \in C^1(0,T;X)$ and, in addition, Problem $\mathcal{P}_V^\rho$ has a unique solution $\mathbf{u}_\rho \in C^1([0,T];X)$. Our main result in this section is the following.

Theorem 5.4.1. *Assume* (5.2.5)–(5.2.9) *and, moreover, assume that*

$$E_\rho \to E, \quad G_\rho \to G, \quad g_\rho \to g \quad \text{as} \ \ \rho \to 0. \tag{5.4.5}$$

Then the solution $\mathbf{u}_\rho$ of problem $\mathcal{P}_V^\rho$ converges to the solution $\mathbf{u}$ of the problem $\mathcal{P}_V$ i.e

$$\mathbf{u}_\rho \longrightarrow \mathbf{u} \quad \text{in} \quad C^1([0,T];X) \quad \text{as} \ \ \rho \to 0. \tag{5.4.6}$$

Proof. Let $\rho > 0$ and let $t \in [0,T]$ be given. We use inequalities (5.2.41) and (5.4.3) to deduce that

$$\begin{aligned} &\langle A\dot{\mathbf{u}}(t), \dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\rangle_X + \langle B\mathbf{u}(t), \mathbf{u}_\rho(t) - \dot{\mathbf{u}}(t)\rangle_X \\ &\quad + j(\dot{\mathbf{u}}_\rho(t)) - j(\dot{\mathbf{u}}(t)) \geq \langle \boldsymbol{f}(t), \dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\rangle_X, \end{aligned}$$

$$\begin{aligned} &\langle A\dot{\mathbf{u}}_\rho(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t)\rangle_X + \langle B_\rho \mathbf{u}_\rho(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t)\rangle_X \\ &\quad + j_\rho(\dot{\mathbf{u}}(t)) - j_\rho(\dot{\mathbf{u}}_\rho(t)) \geq \langle \boldsymbol{f}(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t)\rangle_X. \end{aligned}$$

We now add these inequalities and use the property (5.3.1) of the operator A to obtain that

$$m_A\|\dot{\mathbf{u}}_\rho(t)-\dot{\mathbf{u}}(t)\|_X^2 \le \langle B_\rho\mathbf{u}_\rho(t)-B\mathbf{u}(t),\dot{\mathbf{u}}(t)-\dot{\mathbf{u}}_\rho(t)\rangle_X \tag{5.4.7}$$
$$+j_\rho(\dot{\mathbf{u}}(t))-j_\rho(\dot{\mathbf{u}}_\rho(t))+j(\dot{\mathbf{u}}_\rho(t))-j(\dot{\mathbf{u}}(t)).$$

Next, we use the definitions (5.4.2) and (5.2.12) to see that

$$j_\rho(\dot{\mathbf{u}}(t))-j_\rho(\dot{\mathbf{u}}_\rho(t))+j(\dot{\mathbf{u}}_\rho(t))-j(\dot{\mathbf{u}}(t)) \tag{5.4.8}$$
$$\le c\,|g_\rho-g|\,\|\dot{\mathbf{u}}_\rho(t)-\dot{\mathbf{u}}(t)\|_X$$

where, here and below, c represents a constant wich does not depend on ρ and whose value may change from line to line. We now combine inequalities (5.4.7) and (5.4.8) to find that

$$m_A\|\dot{\mathbf{u}}_\rho(t)-\dot{\mathbf{u}}(t)\|_X \le \|B_\rho\mathbf{u}_\rho(t)-B\mathbf{u}(t)\|_X + c\,|g_\rho-g| \tag{5.4.9}$$

On the other hand, using definitions (5.2.11) and (5.4.1) it is easy to see that

$$\|B_\rho\mathbf{u}_\rho(t)-B\mathbf{u}(t)\|_X \le (E_\rho+G_\rho)\|\mathbf{u}_\rho(t)-\mathbf{u}(t)\|_X \tag{5.4.10}$$
$$+(|E_\rho-E|+|G_\rho-G|)\|\mathbf{u}(t)\|_X.$$

It follows now from assumption (5.4.5) that $E_\rho+G_\rho\le c$ and, therefore, inequalities (5.4.9), (5.4.10) imply

$$\|\dot{\mathbf{u}}_\rho(t)-\dot{\mathbf{u}}(t)\|_X \le c\,\|\mathbf{u}_\rho(t)-\mathbf{u}(t)\|_X \tag{5.4.11}$$
$$+(|E_\rho-E|+|G_\rho-G|)\max_{r\in[0,T]}\|\mathbf{u}(r)\|_X + c\,|g_\rho-g|.$$

Next, we use the initial conditions (5.2.42) and (5.4.3) to see that

$$\|\mathbf{u}_\rho(t)-\mathbf{u}(t)\|_X \le \int_0^t \|\dot{\mathbf{u}}_\rho(s)-\dot{\mathbf{u}}(s)\|_X\,ds, \tag{5.4.12}$$

then we substitute this inequality in (5.4.11) and use the Gronwall's Lemma to obtain that

$$\|\dot{\mathbf{u}}_\rho(t)-\dot{\mathbf{u}}(t)\|_X \tag{5.4.13}$$
$$\le c\,(|E_\rho-E|+|G_\rho-G|)\max_{r\in[0,T]}\|\mathbf{u}(r)\|_X + |g_\rho-g|).$$

The convergence (5.4.6) follows now from inequalities (5.4.12), (5.4.13) and assumption (5.4.5). □

5.5 Dual variational formulation

In this section we introduce and study a second variational formulation of Problem $\mathcal{P}$, the so-called *dual variational formulation.* It is obtained by operating the change of variable $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$ in Problem $\mathcal{P}_V$. Dual variational formulations of boundary problems originate in Contact Mechanics, as explained in [51, 97, 100]. The main idea is to introduce a new variational formulation expressed in terms of the stress field, equivalent with the primal variational formulation which, in turn, is expressed in term of displacement.

Everywhere below we assume that (5.2.5)–(5.2.9) hold and we denote by A^{-1} the inverse of the operator A, whose existence is guaranteed by Lemma 5.3.2. Note also that Lemmas 5.3.2 and 5.3.3 imply that the operators A^{-1} and B a linear continuous operators, and we shall use this results in various places below. We start with the following result.

Lemma 5.5.1. *Then there exists an operator $\mathcal{R} : C([0,T];X) \longrightarrow C([0,T],X)$ such that, for any functions $\boldsymbol{\sigma} \in C([0,T];X)$ and $\mathbf{u} \in C^1([0,T];X)$ with $\mathbf{u}(0) = \mathbf{u}_0$, the following equivalence holds:*

$$\boldsymbol{\sigma}(t) = A\dot{\mathbf{u}}(t) + B\mathbf{u}(t) \quad \forall\, t \in [0,T] \tag{5.5.1}$$

if and only if

$$\dot{\mathbf{u}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t) \quad \forall\, t \in [0,T]. \tag{5.5.2}$$

Note that in (5.5.2) and we use the short hand notation $\mathcal{R}\boldsymbol{\sigma}(t)$ instead of $(\mathcal{R}\boldsymbol{\sigma})(t)$. We shall use this notation in many places below, when no confusion arises.

Proof. Let $\boldsymbol{\sigma} \in C([0,T];X)$ and define the operator $\Lambda_\sigma : C([0,T];X) \longrightarrow C([0,T];X)$ by equality

$$(\Lambda_\sigma\boldsymbol{\theta})(t) = -A^{-1}B\Big(\int_0^t (\boldsymbol{\theta}(s) + A^{-1}\boldsymbol{\sigma}(s))ds + \mathbf{u}_0\Big) \tag{5.5.3}$$

$$\forall\, \boldsymbol{\theta} \in C([0,T],X),\ t \in [0,T].$$

We shall prove that Λ_σ has a unique fixed point, denoted $\mathcal{R}\boldsymbol{\sigma}$. To this end consider two functions $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in C([0,T];X)$ and let $t \in [0,T]$. Then, using the properties of the operators A and B it is easy to see that

$$\|(\Lambda_\sigma\boldsymbol{\theta}_1)(t) - (\Lambda_\sigma\boldsymbol{\theta}_2)(t)\|_X \le c\int_0^t \|\boldsymbol{\theta}_1(s) - \boldsymbol{\theta}_2(s))\|_X\, ds,$$

where c denotes a positive constant which depends on A and B. This inequality shows that the operator Λ_σ is a history-dependent operator and, using Theorem 3.1 in [102] we deduce that there exists a unique element $\mathcal{R}\boldsymbol{\sigma} \in C([0,T];X)$ such that

$$\mathcal{R}\boldsymbol{\sigma}(t) = \Lambda_\sigma(\mathcal{R}\boldsymbol{\sigma})(t). \tag{5.5.4}$$

We now compare equalities (5.5.3) and (5.5.4) to deduce that

$$(\mathcal{R}\boldsymbol{\sigma})(t) = -A^{-1}B\Big(\int_0^t (\mathcal{R}\boldsymbol{\sigma}(s) + A^{-1}\boldsymbol{\sigma}(s))ds + \mathbf{u}_0\Big).$$

Assume now that (5.5.1) holds. Then it is easy to see that

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = -A^{-1}B\mathbf{u}(t)$$

and, since $\mathbf{u}(t) = \displaystyle\int_0^t \dot{\mathbf{u}}(s)\,ds + \mathbf{u}_0$, we deduce that

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = -A^{-1}B\Big(\int_0^t \dot{\mathbf{u}}(s)\,ds + \mathbf{u}_0\Big)$$

which shows that

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = -A^{-1}B\Big(\int_0^t (\dot{\mathbf{u}}(s) - A^{-1}\boldsymbol{\sigma}(s) + A^{-1}\boldsymbol{\sigma}(s))\,ds + \mathbf{u}_0\Big). \tag{5.5.5}$$

We now combine (5.5.3) and (5.5.5) to see that $\dot{\mathbf{u}} - A^{-1}\boldsymbol{\sigma}$ is a fixed point for the operator Λ_σ. On the other hand, recall that this operator has a unique fixed point, denoted $\mathcal{R}\boldsymbol{\sigma}$. Therefore $\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = \mathcal{R}\boldsymbol{\sigma}(t)$, which shows that (5.5.2) holds.

Conversely, assume that (5.5.2) holds. Then, since $\mathcal{R}\boldsymbol{\sigma}$ is the unique fixed point of the operator Λ_σ, we have the equalities

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = \mathcal{R}\boldsymbol{\sigma}(t) = \Lambda_\sigma(\mathcal{R}\boldsymbol{\sigma})(t) = \Lambda_\sigma(\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t)).$$

We use now the definition (5.5.3) to deduce that (5.5.5) holds. Next, since

$$\mathbf{u}(t) = \int_0^t \dot{\mathbf{u}}(s)\,ds + \mathbf{u}_0,$$

equality (5.5.5) implies that

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = -A^{-1}B\mathbf{u}(t).$$

This shows that equality (5.5.1) holds which concludes the proof. □

Next, for each $t \in [0,T]$ we define the set $\Sigma(t) \subset V$ by equality

$$\Sigma(t) = \{\, \boldsymbol{\tau} \in V : \langle \boldsymbol{\tau}, \mathbf{v} \rangle_X + j(\mathbf{v}) \geq \langle \boldsymbol{f}(t), \mathbf{v} \rangle_X \quad \forall\, \mathbf{v} \in V \,\}. \tag{5.5.6}$$

Then, we consider the following variational problem.

Problem $\mathcal{P}_V^D$. *Find a function* $\boldsymbol{\sigma} : [0,T] \to X$ *such that*

$$\boldsymbol{\sigma} \in \Sigma(t), \quad \Big\langle A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t) \Big\rangle_X \geq 0 \quad \text{for all } \boldsymbol{\tau} \in \Sigma(t),\ t \in [0,T]. \tag{5.5.7}$$

We refer in what follows to Problem $\mathcal{P}_V^D$ as the dual formulation of Problem $\mathcal{P}_V$. The link between the variational problems $\mathcal{P}_V$ and $\mathcal{P}_V^D$ is given by the following result.

Theorem 5.5.2. *Assume that* (5.2.5)–(5.2.9) *hold and let* $\mathbf{u} \in C^1([0,T];X)$, $\boldsymbol{\sigma} \in C([0,T];X)$. *Consider the following statements:*

(a) $\mathbf{u}$ *is solution to problem* $\mathcal{P}_V$.

(b) $\boldsymbol{\sigma}$ *is solution of problem* $\mathcal{P}_V^D$.

(c) $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$ *and* $\mathbf{u}(0) = \mathbf{u}_0$.

Then, if two of the statements above hold, the reminder one holds, too.

Proof. The proof is based on the implications (a) and (c) $\Longrightarrow$ (b), (a) and (b) $\Longrightarrow$ (c), (b) and (c) $\Longrightarrow$ (a) which will be proved in the three steps below.

1) (a) and (c) $\Longrightarrow$ (b). We assume in what follows that $\mathbf{u}$ solution of $\mathcal{P}_V$, $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$ and let $t \in [0,T]$ be given. Then, Lemma 5.5.1 implies that $\dot{\mathbf{u}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)$ and, substituting this inequality in (5.2.41) we have

$$\langle \boldsymbol{\sigma}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \langle \boldsymbol{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X. \tag{5.5.8}$$

Next, testing in (5.5.8) with $\mathbf{v} = 2\dot{\mathbf{u}}(t)$ and $\mathbf{v} = \mathbf{0}_X$ we succesively obtain

$$\langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t) \geq \langle \boldsymbol{f}(t), \dot{\mathbf{u}}(t) \rangle_X,$$

$$\langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t)) \leq \langle \boldsymbol{f}(t), \dot{\mathbf{u}}(t) \rangle_X$$

which imply that

$$\langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t)) = \langle \boldsymbol{f}, \dot{\mathbf{u}}(t) \rangle_X. \tag{5.5.9}$$

We use (5.5.8), (5.5.9) and the definition (5.5.6), to see that

$$\boldsymbol{\sigma}(t) \in \Sigma(t). \tag{5.5.10}$$

Moreover, we note that (5.5.6) and (5.5.9) yield

$$\langle \boldsymbol{\tau} - \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t)\rangle_X = \langle \boldsymbol{\tau}, \dot{\mathbf{u}}(t)\rangle_X + j(\dot{\mathbf{u}}(t)) - \langle \boldsymbol{f}(t), \dot{\mathbf{u}}(t)\rangle_X \geq 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma(t)$$

and, using equality $\dot{\mathbf{u}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)$ we obtain that

$$\Big\langle \boldsymbol{\tau} - \boldsymbol{\sigma}(t), A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)\Big\rangle_X \geq 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma(t). \tag{5.5.11}$$

We now gather (5.5.10) and (5.5.11) to see that $\boldsymbol{\sigma}$ is solution of Problem $\mathcal{P}_V^D$, i.e. (b) holds.

2) (a) and (b) $\Longrightarrow$ (c). We assume in what follows that $\mathbf{u}$ is a solution of $\mathcal{P}_V$ and $\boldsymbol{\sigma}$ is solution of $\mathcal{P}_V^D$. Denote

$$\widetilde{\boldsymbol{\sigma}} = A\dot{\mathbf{u}} + B\mathbf{u} \tag{5.5.12}$$

and let $t \in [0, T]$. Then using the implication (a) and (c) $\Longrightarrow$ (b), it follows that $\widetilde{\boldsymbol{\sigma}}$ is solution of $\mathcal{P}_V^D$. Since both $\boldsymbol{\sigma}$ and $\widetilde{\boldsymbol{\sigma}}$ are solution to Problem $\mathcal{P}_V^D$ we have

$$\Big\langle A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t), \widetilde{\boldsymbol{\sigma}}(t) - \boldsymbol{\sigma}(t)\Big\rangle_X \geq 0,$$

$$\Big\langle A^{-1}\widetilde{\boldsymbol{\sigma}}(t) + \mathcal{R}\widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t) - \widetilde{\boldsymbol{\sigma}}(t)\Big\rangle_X \geq 0$$

and, adding these inequalities, we obtain that

$$\Big\langle A^{-1}\widetilde{\boldsymbol{\sigma}}(t) - A^{-1}\boldsymbol{\sigma}(t), \boldsymbol{\sigma}(t) -, \widetilde{\boldsymbol{\sigma}}(t)\Big\rangle_X \leq \langle \mathcal{R}\widetilde{\boldsymbol{\sigma}}(t) - \mathcal{R}\boldsymbol{\sigma}(t), \boldsymbol{\sigma}(t) - \widetilde{\boldsymbol{\sigma}}(t)\rangle_X .$$

This inequality combined with the properties of A^{-1} yields

$$\|\widetilde{\boldsymbol{\sigma}}(t) - \boldsymbol{\sigma}(t)\|_X \leq c\, \|\mathcal{R}\widetilde{\boldsymbol{\sigma}}(t) - \mathcal{R}\boldsymbol{\sigma}(t)\|_X \tag{5.5.13}$$

where, here and below, c denotes a given positive constant whose value will change from line to line. On the other hand, by the definition of the operator $\mathcal{R}$ we have

$$\mathcal{R}\widetilde{\boldsymbol{\sigma}}(t) = -A^{-1}B\Big(\int_0^t (\mathcal{R}\widetilde{\boldsymbol{\sigma}}(t)(s) + A^{-1}\widetilde{\boldsymbol{\sigma}}(t)(s))ds + \mathbf{u}_0\Big),$$

$$\mathcal{R}\boldsymbol{\sigma}(t) = -A^{-1}B\Big(\int_0^t (\mathcal{R}\boldsymbol{\sigma}(s) + A^{-1}\boldsymbol{\sigma}(s))ds + \mathbf{u}_0\Big).$$

Therefore,

$$\|\mathcal{R}\widetilde{\boldsymbol{\sigma}}(t) - \mathcal{R}\boldsymbol{\sigma}(t)\|_X$$
$$\leq c\Big(\int_0^t \|\widetilde{\boldsymbol{\sigma}}(s) - \boldsymbol{\sigma}(s)\|_X ds + \int_0^t \|R\widetilde{\boldsymbol{\sigma}}(s) - R\boldsymbol{\sigma}(s)\|_X ds\Big)$$

and, applying Gronwall's lemma yields

$$\|\mathcal{R}\widetilde{\boldsymbol{\sigma}}(t) - \mathcal{R}\boldsymbol{\sigma}(t)\|_X \leq c \int_0^t \|\widetilde{\boldsymbol{\sigma}}(s) - \boldsymbol{\sigma}(s)\|_X \, ds. \tag{5.5.14}$$

We now combine inequalities (5.5.13) and (5.5.14) then we apply Gronwal's lemma, again, to deduce that

$$\widetilde{\boldsymbol{\sigma}}(t) = \boldsymbol{\sigma}(t). \tag{5.5.15}$$

It follows now from (5.5.12) and (5.5.15) that $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$ and, therefore (c) holds.

3) (b) and (c) $\Longrightarrow$ (a). We assume that $\boldsymbol{\sigma}$ a solution to problem $\mathcal{P}_V^D$ and, in addition, $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$ and $\mathbf{u}(0) = \mathbf{u}_0$. Let $t \in [0, T]$. Then, Lemma 5.5.1 implies that $\dot{\mathbf{u}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)$. We substitute this equality in (5.5.7) to obtain

$$\langle \boldsymbol{\tau} - \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t)\rangle_X \geq 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma(t) \tag{5.5.16}$$

Let $\boldsymbol{d}(t) \in X$ be a subgradient of j in the point $\dot{\mathbf{u}}(t)$. Then

$$j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \langle \boldsymbol{d}(t), \mathbf{v} - \dot{\mathbf{u}}(t)\rangle_X \quad \forall\, \mathbf{v} \in X \tag{5.5.17}$$

and taking succesively $\mathbf{v} = \dot{\mathbf{u}}(t)$, $\mathbf{v} = \mathbf{0}_V$ in this inequality we find that

$$\langle \boldsymbol{d}(t), \dot{\mathbf{u}}(t)\rangle_X = j(\dot{\mathbf{u}}(t)). \tag{5.5.18}$$

We now combine (5.5.17) and (5.5.18) to see that

$$j(\mathbf{v}) \geq \langle \boldsymbol{d}(t), \mathbf{v}\rangle_X \quad \forall\, \mathbf{v} \in X. \tag{5.5.19}$$

This inequality shows that $\boldsymbol{f}(t) - \boldsymbol{d}(t) \in \Sigma(t)$ and, therefore, we are allowed to test in (5.5.16) with $\boldsymbol{\tau} = \boldsymbol{f}(t) - \boldsymbol{d}(t)$. As a result we find that

$$\langle \boldsymbol{f}(t), \dot{\mathbf{u}}(t)\rangle_X \geq \langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t)\rangle_X + \langle \boldsymbol{d}(t), \dot{\mathbf{u}}(t)\rangle_X . \tag{5.5.20}$$

and, using (5.5.18) yields

$$\langle \boldsymbol{f}(t), \dot{\mathbf{u}}(t)\rangle_X \geq \langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t)\rangle_X + j(\dot{\mathbf{u}}(t)). \tag{5.5.21}$$

Note that the converse inequality also holds, since $\boldsymbol{\sigma}(t) \in \Sigma(t)$. Therefore, we conclude from above that

$$\langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t)\rangle_X + j(\dot{\mathbf{u}}(t)) = \langle \boldsymbol{f}(t), \dot{\mathbf{u}}(t)\rangle_X . \tag{5.5.22}$$

Now, since $\boldsymbol{\sigma}(t) \in \Sigma(t)$ we have

$$\langle \boldsymbol{\sigma}(t), \mathbf{v} \rangle_X + j(\mathbf{v}) \geq \langle \boldsymbol{f}(t), \mathbf{v} \rangle_X \qquad \forall\, \mathbf{v} \in X \tag{5.5.23}$$

and, using (5.5.21) we deduce that

$$\langle \boldsymbol{\sigma}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \langle \boldsymbol{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X \qquad \forall\, \mathbf{v} \in X.$$

Finally, using the equalities $\boldsymbol{\sigma}(t) = A\dot{\mathbf{u}}(t) + B\mathbf{u}(t)$, $\mathbf{u}(0) = \mathbf{u}_0$ we deduce that $\mathbf{u}$ is a solution of Problem $\mathcal{P}_V$, which concludes the proof. $\square$

A carefully examination of Problems $\mathcal{P}_V$ and $\mathcal{P}_V^D$ lead to the conclusion that these problems have a different feature. First, Problem $\mathcal{P}_V$ is an evolutionary variational inequality, since the derivative of the unknown $\mathbf{u}$ appears in its statment. Therefore, an initial condition, (5.2.42), is required. Moreover, it does not involve any contraint on the solution. In contrast, Problem $\mathcal{P}_V^D$ is a history-dependent inequality with constraints. Indeed, this inequality is governed by the operator $\mathcal{R}$ which satisfies inequality (5.5.14) and, therefore, is a history-dependent operator. Moreover, the inequality is governed by the set of constraints $\Sigma(t)$, which is a time-dependent convex set. Nevertheless, despite these different feature, Problems $\mathcal{P}_V$ and $\mathcal{P}_V^D$ are equivalent, as stated in Theorem 5.5.2. Moreover, combining Theorem 5.5.2 with Theorem 5.3.1 we deduce the unique solvability of Problem $\mathcal{P}_V^D$, under the assumptions (5.2.5)–(5.2.9).

Conclusion

In this thesis we have studied the stability of some transmission problems with delay, our attention has focused on the equations of the waves. In the Part I, first we considered a transmission problem of waves and wave equations where a damping term and a delay term appear in the first equation of the system on a one-dimensional domain on which conditions have been imposed at the extrimity points. We have proved the existence and the uniqueness of the solution and then, using the lyapunov method, we have proved the exponential decay of the energy by assuming that the weight of the damping is greater than the weight of the delay. A functional Lyapunov was built. In the second axis we have added to the first system a viscoelastic term, in the presence of this term we have also proved the decrease of the energy of the solution under the same hypothesis, that is to say the weight of the damping Is greater than the weight of the delay term. In the third axis we considered the same previous system but here the delay is a function in time. Under some assumptions on the delay function and on the hypothesis between the weights cited above, the energy deccreases. The last axis of our study concerns a problem of partial derivative equations of evolution modeling a mechanical phenomenon, after having derived a variational formulation in the form of a variational inequality of the problem, we proved the existence and the Uniqueness of the weak solution of the latter. An equivalent formulation was obtained such that its solution converged towards the solution of the initial problem. In Part II, we studied a mathematical models of contact. We introduced a mathematical model that describes the evolution of a viscoelastic plate in frictional contact with foundation, we derived the variational inequality for the displacement field, then we established the existence of a unique weak solution to the model. A research perspective focused on the study of the stability of the wave equation in a domain of $\mathbb{R}^n$ with a delay term.

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