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THEME
Cohomologie et déformations des Hom-bialgèbres et algèbres Hom-Hopf

Soutenue le : / /
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DEDICATION

To my late parents without whom I wouldn't be who I am today May Allah bless their souls and May they rest in peace

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Introduction

A brief history of Hopf algebras. The concept of a Hopf algebra arose in the 1940's in relation to the work of Heinz Hopf in algebraic topology and cohomology [25]. They also appeared, in some sense, in the theory of algebraic groups in the various works of Dieudonne, Cartier, and Hochschild. Beginning with the work of Milnor and Moore [42], a general theory of Hopf algebras was then developed in the 1960's and 1970's continuing in the works of Larson, Radford, Sweedler, Taft, and Wilson, among others. The first book on the subject of Hopf algebra was written by Sweedler and published in 1969 [43]. Later, these algebraic objects took on a prominent role in the theory of quantum groups that became popular in the 1980's and early 1990's. Hopf algebras now play an important role in many areas of mathematics and are linked with such topics as Lie algebras, Galois theory, conformal field theory, quantum mechanics, tensor categories, and combinatorics. For more information on the beginning history of Hopf algebras, see the survey of Andruskiewitsch and Ferrer [4].

A brief history of Hom-type algebras.

The first instance of Hom-type algebras appeared in various papers dealing with q-deformations of algebras of vector fields, mainly Witt and Virasoro algebras, which play an important rôle in Physics. In a theory with conformal symmetry, the Witt algebra W is a part of the complexified Lie algebra $Vect^{\mathbb{C}}(S) \times Vect^{\mathbb{C}}(S)$, where S is the unit circle, belonging to the classical conformal symmetry. The q-deformations of Witt and Virasoro algebras are obtained when the derivation is replaced by a σ-derivation, see for example [1]. Then Hartwig, Larsson and Silvestrov introduced and studied the concept of Hom-Lie algebra, which is a deformation of Lie algebra where the Jacobi identity is twisted by a homomorphism, see [23, 29]. The associative-type objects corresponding to Hom-Lie algebras, called Hom-associative algebras, have been introduced and studied by Makhlouf and Silvestrov in [37], where it is shown that usual functors between associative algebras and Lie algebras extend to Hom-type algebras. Moreover, Hom-analogues of coalgebras, bialgebras and Hopf algebras have been introduced in [34, 35]. The original definition of a Hom-bialgebra involved two linear maps, one twisting the associativity condition and the other one the coassociativity condition. Later, two directions of study on Hom-bialgebras were developed, one in which the two maps coincide, they are still called Hom-bialgebras, and another one, started in [6], where the two maps are assumed to be inverse to each other, they are called monoidal Hom-bialgebras. In the last years, many concepts and properties from classical algebraic theories have been extended to the framework of Hom-structures, see for instance [3, 10, 12, 13, 22, 33, 39, 40, 46, 49, 50, 51, 52, 53, 54, 60, 61].

The algebraic deformation theory, as first described by Gerstenhaber [17], studies pertubations of algebraic structures using cohomology and obstruction theory, which they define in [18], is a theory with coeficients, which we denote by $H^*_{GS}(M, N)$, for Hopf bimodules M and N over a Hopf algebra H.

Gerstenhaber's work has been extended in various directions. Many algebraic or geometric structures can be deformed, and for each kind of structure and deformation, one can associate a cohomology theory in order to study and control these deformations. In the case of associative algebras, the cohomology which appears is the Hochschild cohomology, for Lie algebras, it is the Chevalley-Eilenberg cohomology, for commutative algebras, it is Harrison cohomology. Other structures have been studied, see for example, Balavoine describes in [5] deformations of any algebra over a quadratic operad. In the same direction, Hinich [26] studies deformations of algebras over a differential graded operad. It is a natural problem to try to extend deformation theory to morphisms. The cohomology and deformations of Hom-associative algebra were initiated in [36] and then completed in [2].

Organization. Quantum groups or Hopf algebras are an exciting new generalisation of ordinary groups. They have a rich mathematical structure and numerous roles in situations where ordinary groups are not adequate. The main purpose of this thesis is to study the theory of Hom-Hopf algebras and define a cohomology complex for Hom-bialgebras, generalizing Gerstenhaber-Schack cohomology in [18, 19], and then study one-parameter formal deformations. It is organized as follows.

In the first chapter, we shall review an extensive amount of background material related to the study of Hopf algebras. Thus the familiar concept of algebra dualize to concept of coalgebra, and the structures of algebra and coalgebra combine to give the notion of a bialgebra. Incorporating antipodes (sometimes called conjugations), we obtain the notion of Hopf algebra. In the first section, we give an outline of basic definitions and theory related to algebras and coalgebras, and then proceed into the study of bialgebras, Hopf algebras in the second section. These are associative algebras equipped with additional structures such as a comultiplication, a counit and an antipode. In some appropriate sense, these structures and their axioms reflect the multiplication, the unit element and the inverse elements of a group and their corresponding properties. In fact, the group algebra $\Bbbk G$ of any group G is a Hopf algebra with multiplication induced by the group product and an antipode induced by the group inverse operation.

In the third section, we shall give a brief overview of the completed results in the classification of finite-dimensional bialgebras based on the structure of algebraic variety and a natural structure transport action which describes the set of isomorphic algebras. Solving such systems of polynomial equations leads to classifications of such structure. In particular, we give a complete classification of bialgebras over an algebraically closed field of characteristic zero in dimensions 2 and 3.

In chapter 2, we introduce the notions of Hom-algebra, Hom-coalgebra and Hom-bialgebra and describe some properties of those structures extending the classical structures of algebras, coalgebra and bialgebra. [37],[36]. Some other relevant properties of Hom-Hopf algebras which generalize Hopf algebras will be discussed in this chapter, including notions related to normal Hopf algebras and extensions of Hom-Hopf algebras, by strategically replacing the identity map by a twisting map α in the defining axioms.

A Hom-associative algebra A is given by a multiplication $\mu : A \otimes A \longrightarrow A$ and a linear self-map α such that the following α-twisted version of associativity holds: $\mu\left(\mu\left(x,y\right),\alpha\left(z\right)\right) = \mu\left(\alpha\left(x\right),\mu\left(y,z\right)\right)$. It is said to be multiplicative if, in addition, $\alpha \circ \mu = \mu \circ (\alpha \otimes \alpha)$. Ordinary associative algebras are multiplicative Hom-associative algebras with $\alpha = id_A,$. By dualization, in the sense that if we reverse all the arrows in the defining diagrams of Hom-associative algebra, we get the concept of a Hom-coassociative coalgebra. We will define a suitable notion of Hom-bialgebra, in which the comultiplication Δ satisfies an α-twisted version of coassociativity and is a morphism of Hom-associative algebras.

We made all the necessary preparations for constructing the dual Hom-associative algebra of a Hom-coassociative coalgebra. The fact that the dual of a finite-dimensional Hom-Hopf algebra will also be a Hom-Hopf algebra, the duality between of those structures also implies the relationship between homomorphisms structures. Some results of these chapter hold for the infinite-dimensional case, but we shall be concerned only with the finite-dimensional case in this thesis.

We now describe the main results of chapter 2 concerning Hom-type generalizations of Hopf algebras.

Proposition 1: *The dual of morphism of Hom-coassociative coalgebra, is a morphism of Hom-associative algebra, and the dual of morphism of Hom-associative algebra, is a morphism of Hom-coassociative coalgebra.*

Proposition 2: *The morphism of Hom bialgebra is a morphism of Hom-Hopf algebras*.

The next proposition gives some important properties of the antipode (see [6], [35]).

Proposition 3: We show that the antipode of a Hom-Hopf algebra is an anti-morphism of Hom-associative algebras and anti-morphism of Hom-coassociative coalgebras. This means that $S : H \longrightarrow H^{op}$ is a Hom-associative algebra morphism and $S : H \longrightarrow H^{cop}$ is a Hom-coassociative coalgebra morphism.

Proposition 4: *Let H be a finite dimensional Hom-Hopf algebra, with antipode S. Then the Hom-bialgebra H^* is a Hom-Hopf algebra, with antipode S^*.*

Chapter 3 is dedicated to the study of Hom-type version of module over algebras (resp. comodule over coalgebras), which will play an important role in Homological algebra and quantum group theory. We recall in this chapter the definitions of modules and comodules over Hom-associative algebras, these definitions of action and coactions are simply a polarisation of those of Hom-algebras and Hom-coalgebras. Moreover, we discuss their tensor products, the tensor products of bimodules can be endowed with bimodule structure. If M and N are bimodules over A, we shall consider two bimodule structures on $M \otimes N$ (dual of each other), which we will denote by $M\overline{\otimes}N$ and $M\underline{\otimes}N$ (see for example [48] for details). These notations will also be used for the tensor product of bimodules or bicomodules.

A more precise version of the following result is proved in chapter 3.

Proposition 5: *A right A-module is nothing else that a left module over the opposite unital Hom-associative algebra A^{op}*, and *a right C-comodule is the same as a left comodule over the opposite counital Hom-coassociative coalgebra C^{cop}.*

Theorem 1: *Let C be a counital Hom-coassociative coalgebra. Then for any right C-comodule M, M^* is a left C^*module. Conversely, let A be a finite-dimensional unital Hom-associative algebra. If N is a left A-module, N^* is a right A^*-comodule, and, if L^** is a left A-module, then L is a right A^*-comodule.

Proposition 6: *Let M* and *N are an H-Hom-bimodule and H*-Hom-bicomodule, respectively*.* The n-fold interior (bimodule) tensor power of M, $M^{\overline{\otimes}n}$ *is an H-bimodule , and* the n-fold interior (bicomodule) tensor power of N, $N^{\underline{\otimes}n}$ is the interior H-bicomodule.

The purpose of Chapter 4 is to construct cochain complex $\mathcal{C}^{p,q}_{Hom} = Hom_{\Bbbk}\left(B^{\otimes q}, B^{\otimes p}\right)$ of a multiplicative and comultiplicative Hom-bialgebra B with coefficients in B that defines a cohomology $H^n_{Hom}\left(B, B\right)$. The second cohomology group play an important role in deformation theory, it is the space of infinitesimal deformations. Our theory gives a natural identification between the underlying $\Bbbk$-modules of the original and the deformed Hom-bialgebra. Moreover, we compute the second cohomology group of Hom-type Taft-Sweedler bialgebra. We show that, in this case the second cohomology group is not trivial.

In the last chapter, we define formal algebraic deformation for a Hom-bialgebra, in section 1, 2 and 3. Our theory gives a natural identification between the underlying $\Bbbk$-modules of the original and the deformed Hom-bialgebra. In section 4 we begin the discussion of infinitesimal methods, which are an essential part of any deformation theory. We describe the infinitesimal of a deformation and the relationship with cohomology, emphasizing several special cases and the way in which deformation theories determine cohomology theories in low dimensions. An important aspect of the infinitesimal theory is the study of obstructions which is our topic in section 5. A basic principle is that the obstructions should be described using a structure on the cochain complex governing the deformation problem.

In section 6 and 7, we discuss the connection between the twistings of Hom-bialgebras (see Proposition 2.4.6) and their formal deformations, and unitality and counitality of Hom-bialgebra deformations, and show that every nontrivial formal deformation is equivalent to a unital and counital deformation with the same unit and counit. Deformation preserves the existence of antipodes, a deformation of a Hom-Hopf algebra as a Hom-bialgebra is automatically a Hom-Hopf algebra.

A more precise version of the following result is proved in chapter 4 and 5,

Theorem 2: Let $B = (B, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra and $\delta^{p,q}_{Hom,H} : \mathcal{C}^{p,q}_{Hom} \longrightarrow \mathcal{C}^{p,q+1}_{Hom}$, $\delta^{p,q}_{Hom,C} : \mathcal{C}^{p,q}_{Hom} \longrightarrow \mathcal{C}^{p+1,q}_{Hom}$ the operators defined in (4.1), (4.2) then $\left(\mathcal{C}^{p,q}_{Hom}, \delta^{p,q}_{Hom,H}, \delta^{p,q}_{Hom,C}\right)$ is a bicomplex,

Proposition 7: The integrability of (μ_1, Δ_1) depends only on its cohomology class.

Proposition 8: If $H^2_{Hom}(B, B) = 0$ then all deformations of Hom-bialgebra B are equivalent to a trivial deformation.

We fix some conventions and notations. In this thesis $\Bbbk$ denotes an algebraically closed field of characteristic zero, even if the general theory does not require it. Vector spaces, tensor products, and linearity are all meant over $\Bbbk$, unless otherwise specified. We denote by $\tau_{i,j} : V_1 \otimes ... \otimes V_i \otimes ... \otimes V_j \otimes ... \otimes V_n \longrightarrow V_1 \otimes ... \otimes V_j \otimes ... \otimes V_i \otimes ... \otimes V_n$ the flip isomorphism where $\tau_{i,j}(x_1 \otimes x_2 \otimes ... \otimes x_i \otimes ... \otimes x_j \otimes ... \otimes x_n) = (x_1 \otimes x_2 \otimes ... \otimes x_j \otimes ... \otimes x_i \otimes ... \otimes x_n)$.

We use in the sequel Sweedler's notation for the comultiplication, $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, and sometimes the multiplication is denoted by a dot for simplicity and when there is no confusion.

Chapter 1

Bialgebras and Hopf algebras

The aim of this chapter is to provide some classical definitions of algebraic structures by use of dual commutative diagrams. Thus the familiar concept of associative algebra dualize to concept of coassociative coalgebra, and the structures of associative algebra and coassociative coalgebra combine to give the notion of a bialgebra. Incorporating antipodes (sometimes called conjugations), we obtain the notion of a Hopf algebra. In the cocommutative case, bialgebras and Hopf algebras can be viewed as monoids and groups in the symmetric monoidal category of cocommutative coalgebras.

1.1 Algebras and coalgebras

We begin with the definition of an associative algebra over the field $\mathbb{k}$.

Definition 1.1.1 *An* ***associative algebra with unit*** *is a vector space A over $\mathbb{k}$ together with two linear maps $\mu : A \otimes A \longrightarrow A$, called the multiplication or the product, and $\eta : \mathbb{k} \longrightarrow A$, called the unit, such that*

$$\mu \circ (\mu \otimes id_A) = \mu \circ (id_A \otimes \mu) \tag{1.1}$$

$$\mu \circ (\eta \otimes id_A) = \mu \circ (id_A \otimes \eta) = id_A \tag{1.2}$$

Given such an unital associative algebra $A = (A, \mu, \eta)$, and the mapping η is determined by its value $\eta(1_{\mathbb{k}}) \in A$, which is the unit element of A. Both definitions of an unital associative algebra are easily seen to be equivalent. Equation (1.1) is the associativity law, while (1.2) says that $\eta(1_{\mathbb{k}}) = 1_A$ is a unit element of A using the identification of $\mathbb{k} \otimes A$ and $A \otimes \mathbb{k}$ with A.

The associativity (1.1) of the multiplication μ means that the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \overset{\mu \otimes id_A}{\longrightarrow} & A \otimes A \\ {\scriptstyle id_A \otimes \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\ A \otimes A & \overset{\mu}{\longrightarrow} & A \end{array}$$

is commutative. Likewise, the condition (1.2) of the unit can be expressed by the commutativity of the following diagram:

$$\begin{array}{ccccc} \Bbbk \otimes A & \overset{\eta \otimes id_A}{\longrightarrow} & A \otimes A & \overset{id_A \otimes \eta}{\longleftarrow} & A \otimes \Bbbk \\ \cong & & \downarrow {\scriptstyle \mu} & & \cong \\ A & \overset{id_A}{\longrightarrow} & A & \overset{id_A}{\longleftarrow} & A \end{array}$$

Proposition 1.1.2 *If a unit exists, it is unique.*

Proof. Let η, η' be two units. Then

$$\mu \circ (\eta (1_{\Bbbk}) \otimes x) = \mu \circ (x \otimes \eta (1_{\Bbbk})) = x, \ \forall x \in A$$

and

$$\mu \circ \left(\eta' (1_{\Bbbk}) \otimes y\right) = \mu \circ \left(y \otimes \eta' (1_{\Bbbk})\right) = y, \ \forall y \in A$$

for $x = \eta' (1_{\Bbbk})$ and $y = \eta (1_{\Bbbk})$, we have

$$\eta' (1_{\Bbbk}) = \mu \circ \left(\eta (1_{\Bbbk}) \otimes \eta' (1_{\Bbbk})\right) = \mu \circ \left(\eta' (1_{\Bbbk}) \otimes \eta (1_{\Bbbk})\right) = \eta (1_{\Bbbk}),$$

which completes the proof. ■

Example 1.1.3 *Here are some examples of associative algebras over $\Bbbk$:*

1. *The field $\Bbbk$, with the canonical structure, is an associative algebra.*

2. *the set of polynomials in variables $x_1, ..., x_n$, $k[x_1, ..., x_n]$ is an associative algebra.*

3. *The algebra $End\,(V)$ of endomorphisms of a vector space V over $\Bbbk$. The multiplication is given by composition of operators.*

4. *The vector space $M_n(\Bbbk)$ of all $n \times n$ matrices over $\Bbbk$ is an n^2-dimensional associative algebra over $\Bbbk$ with the usual matrix multiplication and matrix identity element I. It is a standard result that $M_n(\Bbbk)$ is an unital associative algebra.*

Let A and A' be unital associative algebras. A $\Bbbk$-linear mapping $f : A \longrightarrow A'$ is called a **morphism of unital associative algebra** if $f(\mu_A(x \otimes y)) = \mu_{A'}(f(x) \otimes f(y))$ for all $x, y \in A$ and $f(1_A) = 1_{A'}$. The two latter conditions can be rewritten as

$$f \circ \mu_A = \mu_{A'} \circ (f \otimes f) \text{ and } f \circ \eta_A = \eta_{A'}$$

There exists a **tensor product** of unital associative algebra $A \otimes A'$ whose vector space is the tensor product of vector spaces of A and A' and whose multiplication is defined by

$$\mu_{A \otimes A'} = (\mu_A \otimes \mu_{A'}) \circ (id_A \otimes \tau_{A' \otimes A} \otimes id_{A'}) .$$

$\tau_{A' \otimes A} : A' \otimes A \longrightarrow A \otimes A'$; $\tau(x' \otimes x) = x \otimes x'$, is the linear 'flip' map, and the unit is defined by

$$\eta_{A \otimes A'} = (\eta_A \otimes \eta_{A'}) .$$

For each unital associative algebra A one can define the **opposite** algebra A^{op}. This is an unital associative algebra with the same underlying vector space as A, but with the new multiplication $\mu_{A^{op}} = \mu_A \circ \tau_{A \otimes A}$, and the unit η_A. That is, we have $\mu_{A^{op}}(x \otimes y) = \mu_A(y \otimes x)$, where $\mu_{A^{op}}$ and μ_A denote the products of A^{op} and A, respectively.

The unital associative algebra A is said to be commutative if $\mu_{A^{op}} = \mu_A$.

We now dualize this definition by reversing all arrows and replacing all mappings by the corresponding dual ones. In doing so, the multiplication $\mu : A \otimes A \longrightarrow A$ is replaced by the comultiplication $\Delta : A \longrightarrow A \otimes A$, the unit $\eta : \Bbbk \longrightarrow A$ by the counit $\varepsilon : A \longrightarrow \Bbbk$, which is dual to that of an associative algebra over a field in the sense that if we reverse all the arrows in the defining diagrams of an associative algebra, we get the concept of a coalgebra.

Definition 1.1.4 *A **counital coassociative coalgebra** is a vector space C over $\Bbbk$, equipped with two linear mappings $\Delta : C \longrightarrow C \otimes C$, called the comultiplication or the coproduct,* *and* $\varepsilon : C \longrightarrow \Bbbk$, *called the counit, such that*

$$(\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta, \tag{1.3}$$

$$(\varepsilon \otimes id_C) \circ \Delta = (id_C \otimes \varepsilon) \circ \Delta = id_C, \tag{1.4}$$

given such a counital coassociative coalgebra $C = (C, \Delta, \varepsilon)$. *Equation (1.3) is referred to as the coassociativity of the comultiplication* Δ, *because it dualizes the associativity (1.1) of the multiplication* μ. *Equation (1.4) is referred to as the counital of the counit* ε.

The conditions (1.3) and (1.4) are respectively equivalent to the following commutative diagrams:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow id_C \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes id_C} & C \otimes C \end{array} \qquad \begin{array}{ccccc} \Bbbk \otimes C & \xleftarrow{\varepsilon \otimes id_C} & C \otimes C & \xrightarrow{id_C \otimes \varepsilon} & C \otimes \Bbbk \\ \cong & & \uparrow \Delta & & \cong \\ C & \xleftarrow{id_C} & C & \xrightarrow{id_C} & C \end{array}$$

The **coopposite coalgebra** C^{cop} is the counital coalgebra on the vector space C equipped with the new comultiplication $\Delta_{C^{cop}} = \tau_{C \otimes C} \circ \Delta_C$ and the counit ε_C.

The counital coalgebra C is said to be cocommutative if $\Delta_{C^{cop}} = \Delta_C$.

Example 1.1.5 1. *The ground field* $\Bbbk$ *is a coalgebra by defining* $\Delta(1) = 1 \otimes 1$ *and* $\varepsilon(1) = 1$ *and extended linearly to all of* $\Bbbk$.

2. *Let* G *be a group,, and define* $\Bbbk G$ *to be the* $\Bbbk$*-vector space with the canonical basis* G. *Then by defining* $\Delta(g) = g \otimes g$ *and* $\varepsilon(g) = 1$, *for all* $g \in G$, *we have a cocommutative coalgebra* $(\Bbbk G, \Delta, \varepsilon)$ *by extending* Δ *and* ε *linearly to all of* $\Bbbk G$.

Next we describe the tensor product coalgebra construction, which is similar to the tensor product algebra structure of two algebras over the same field.

Example 1.1.6 *Let* $(C, \Delta_C, \varepsilon_C)$ *and* $(C', \Delta_{C'}, \varepsilon_{C'})$ *be two counital coassociative coalgebras over the field* $\Bbbk$.*Then we can construct a counital coassociative coalgebra on the tensor product vector space* $C \otimes C'$, *called the* ***tensor product of coalgebra****, by defining with comultiplication*

$$\Delta_{C \otimes C'} = (id_C \otimes \tau_{C' \otimes C} \otimes id_{C'}) \circ (\Delta_C \otimes \Delta_{C'}) : C \otimes C' \longrightarrow \left(C \otimes C'\right) \otimes \left(C \otimes C'\right)$$

and counit

$$\varepsilon_{C\otimes C'} = \varepsilon_C \otimes \varepsilon_{C'} : C \otimes C' \longrightarrow \Bbbk.$$

Definition 1.1.7 *Let C and C' be counital coassociative coalgebras. A $\Bbbk$-linear mapping $f : C \longrightarrow C'$ is said to be a* ***morphism of counital coalgebra*** *(or counital coalgebra morphism) if*

$$\Delta_{C'} \circ f = (f \otimes f) \circ \Delta_C \text{ and } \varepsilon_C = \varepsilon_{C'} \circ f.$$

Proposition 1.1.8 *Let $f : C \longrightarrow C'$ and $g : D \longrightarrow D'$ be a morphisms of coalgebras. The tensor product of f and g yield a coalgebra morphism*

$$f \otimes g : C \otimes D \longrightarrow C' \otimes D'$$

Proof. The fact that f and g are coalgebra morphisms implies commutativity of the top square in the diagram

$$\begin{array}{ccc} C \otimes D & \xrightarrow{f\otimes g} & C' \otimes D' \\ {\scriptstyle \Delta_C \otimes \Delta_D} \downarrow & & \downarrow {\scriptstyle \Delta_{C'} \otimes \Delta_{D'}} \\ (C \otimes C) \otimes (D \otimes D) & \xrightarrow{f\otimes f\otimes g\otimes g} & (C' \otimes C') \otimes (D' \otimes D') \\ {\scriptstyle id_C \otimes \tau_{C\otimes D} \otimes id_D} \downarrow & & \downarrow {\scriptstyle id_{C'} \otimes \tau_{C'\otimes D'} \otimes id_{D'}} \\ (C \otimes D) \otimes (C \otimes D) & \xrightarrow{f\otimes g\otimes f\otimes g} & (C' \otimes D') \otimes (C' \otimes D), \end{array}$$

while the bottom square obviously is commutative by the definitions. Commutativity of the outer rectangle means that $f \otimes g$ is a coalgebra morphism. ■

We will use notation attributed to Sweedler for the image of an element under the comultiplication of a coassociative coalgebra C. If x is an element of a coassociative coalgebra (C, Δ, ε), the element $\Delta(x) \in C \otimes C$ is a finite sumwhere the right hand side is a formal sum denoting an element of $C \otimes C$.

$$\Delta(x) = \sum_i x_{1i} \otimes x_{2i}$$

It denotes how Δ shares out x into linear combinations of a part $_{(1)}$ in the first factor of $C \otimes C$ and a part $_{(2)}$ in the second factor. For brevity, we simply write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}. \tag{1.5}$$

With this notation, the coassociativity of Δ is expressed by

$$(\Delta \otimes id_C) \circ \Delta(x) = \sum \Delta\left(x_{(1)}\right) \otimes x_{(2)} = \sum x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)}$$

$$(id_C \otimes \Delta) \circ \Delta(x) = \sum x_{(1)} \otimes \Delta\left(x_{(2)}\right) = \sum x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)}$$

and, hence, it is possible and convenient to shorten the notation by writing

$$(\Delta \otimes id_C) \circ \Delta(x) = (id_C \otimes \Delta) \circ \Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$$

$$(id_C \otimes id_C \otimes \Delta)(id_C \otimes \Delta) \circ \Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)}$$

$$(\Delta \otimes id_C \otimes id_C)(\Delta \otimes id_C) \circ \Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)}$$

and so on.

Let us define inductively mappings $\Delta^{(n)} : C \longrightarrow C^{\otimes n+1}$ by

$$\Delta^{(n)} = \left(id_C^{\otimes n-1} \otimes \Delta\right) \circ \Delta^{(n-1)} \quad n > 1 \quad and \quad \Delta^{(1)} = \Delta.$$

(From the coassociativity, it follows that $\Delta^{(n)}$ is in fact equal to $n-1$ compositions of Δ independently of their order, that is, $\Delta^{(2)} = (id_C \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta$, etc.)

For $n = 2$,

$$\Delta^{(2)}(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$$

For $n = 3$,

$$\Delta^{(3)}(x) = \left(id_C^{\otimes 2} \otimes \Delta\right) \circ \Delta^{(2)} = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)}$$

For $n = 4$,

$$\Delta^{(4)} = \left(id_C^{\otimes 3} \otimes \Delta\right) \circ \Delta^{(3)} = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)} \otimes x_{(5)}$$

Then the element $\Delta^{(n)}(x) \in C^{\otimes n+1}$ is denoted by

$$\Delta^{(n)} = x_{(1)} \otimes x_{(2)} \otimes ... \otimes x_{(n)} \otimes x_{(n+1)}$$

then $\Delta^{(n)}$ denotes the $n-$ary comultiplication.

The conditions for the counit are described by

$$\sum \varepsilon\left(x_{(1)}\right) x_{(2)} = \sum x_{(1)} \varepsilon\left(x_{(2)}\right) = x.$$

Then we denote the $n-$ary multiplication [57] by $\mu^{(n)} : A^{\otimes n+1} \longrightarrow A$

$$\mu^{(n)} = \mu^{(n-1)} \circ \left(id_A^{\otimes n-1} \otimes \mu\right) \quad n > 1 \quad and \quad \mu^{(1)} = \mu.$$

For $n = 2$

$$\mu^{(2)}\left(x_1 \otimes x_2 \otimes x_3\right) = \mu \circ \left(id_A \otimes \mu\right)\left(x_1 \otimes x_2 \otimes x_3\right) = \left(x_1 \cdot x_2 \cdot x_3\right)$$

Then the element $\mu^{(n)}\left(x_1 \otimes ... \otimes x_n \otimes x_{n+1}\right) \in A$ is denoted by

$$\mu^{(n)}\left(x_1 \otimes ... \otimes x_n \otimes x_{n+1}\right) = x_1 \cdot x_2 \cdot ... \cdot x_{n-1} \cdot x_n \cdot x_{n+1}.$$

1.2 Bialgebras and Hopf algebras

Historically, the concept of Hopf algebra originated in algebraic topology, where the term "Hopf algebra" was used for what we are calling a bialgebra. The term bialgebra was introduced later and is still rarely used in topology, the bialgebras that usually appear in algebraic topology automatically have antipodes, so that it is reasonable to ignore the distinction, and we do so where no confusion can arise. We have followed the algebraic literature in using the name antipode and distinguishing between bialgebras and Hopf algebras because of the more recent interest in Hopf algebras of a kind that do not seem to appear in algebraic topology, such as quantum groups.

Definition 1.2.1 *A* ***bialgebra*** $(B, \mu, \eta, \Delta, \varepsilon)$ *is an unital associative algebra* (B, μ, η) *with and a counital coassociative coalgebra* (B, Δ, ε) *such that the following diagrams are commutatives.*

$$\begin{tikzcd}
B \otimes B \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & B \arrow[r, "\Delta"] & B \otimes B \\
B \otimes B \otimes B \otimes B \arrow[rr, "id_B \otimes \tau_{B\otimes B} \otimes id_B"] & & B \otimes B \otimes B \otimes B \arrow[u, "\mu\otimes\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B & B \arrow[l, "\Delta"'] \\
\mathbb{k} \otimes \mathbb{k} \arrow[u, "\eta\otimes\eta"] & \mathbb{k} \arrow[l, "id_{\mathbb{k},\mathbb{k}^{\otimes 2}}"'] \arrow[u, "\eta"']
\end{tikzcd}$$

$$\begin{array}{cccc} & B\otimes B & \xrightarrow{\mu} & B \\ and & {}_{\varepsilon\otimes\varepsilon}\downarrow & & \downarrow_{\varepsilon} \\ & \Bbbk\otimes\Bbbk & \xrightarrow{id_{\Bbbk^{\otimes 2},\Bbbk}} & \Bbbk \end{array}$$

That is, μ and η are a morphisms of coassociative coalgebras or, equivalently, Δ and ε are a morphisms of associative algebras.

Example 1.2.2 *1. The field $\Bbbk$, with its algebra structure, and with the canonical coalgebra structure, is a bialgebra.*

2. Let G be a group, then $\Bbbk G$ endowed with a coalgebra structure as in Example 1.1.5 (in which $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, for all $g \in G$ is a bialgebra.

3. If B is a bialgebra, then B^{op}, B^{cop} and $B^{op,cop}$ are bialgebras, where B^{op} has an algebra structure opposite to the one of B, and the same coalgebra structure as B, B^{cop} has the same algebra structure as B and the coalgebra structure co-opposite to the one of B, and $B^{op,cop}$ has the algebra structure opposite to the one of B, and the coalgebra structure co-opposite to the one of B.

Also, if B is a bialgebra, then $B \otimes B$ is a bialgebra with the tensor product algebra structure and the tensor product coalgebra structure.

Definition 1.2.3 *Let B and B' be two bialgebras. A linear map $f : B \longrightarrow B'$ is called a **morphism of bialgebras** if it is a morphism of algebras and a morphism of coalgebras between the underlying algebras, respectively coalgebras of the two bialgebras.*

Definition 1.2.4 *Let H be a bialgebra. A linear map $S : H \longrightarrow H$ is called an **antipode** of the bialgebra H if S is the inverse of the identity map $id_H : H \longrightarrow H$ with respect to the convolution product $*$ defined by*

$$id_H * S = S * id_H = \eta \circ \varepsilon.$$

$$\mu \circ (id_V \otimes S) \circ \Delta = \mu \circ (S \otimes id_V) \circ \Delta = \eta \circ \varepsilon.$$

Definition 1.2.5 *A bialgebra H having an antipode is called a **Hopf algebra**.*

Remark 1.2.6 *In a Hopf algebra, the antipode is unique.*

Let H, H' be two Hopf algebras. A map $f : H \longrightarrow H'$ is called a **morphism of Hopf algebras** if it is a morphism of bialgebras. It is natural to ask whether a morphism of Hopf algebras should preserve antipode. The following result shows that this is indeed the case.

Proposition 1.2.7 *Let H, H' be two Hopf algebras with antipodes S_H and $S_{H'}$. If $f : H \longrightarrow H'$ is a morphism of Hopf algebras, then $S_{H'} \circ f = f \circ S_H$.*

Remark 1.2.8 *Let H be a Hopf algebra with antipode S. Then the bialgebra $H^{op,cop}$ is a Hopf algebra with the same antipode S. If moreover S is bijective, then the bialgebras H^{op} and H^{cop} are Hopf algebras with antipode S^{-1}.*

If necessary, we will denote a Hopf algebra $H = (H, \mu, \eta, \Delta, \varepsilon, S)$ where S is the antipode. We next describe some basic properties of the antipode of a Hopf algebra, this result shows that S is an algebra anti-homomorphism and a coalgebra anti-homomorphism.

Proposition 1.2.9 *Let $H = (V, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebras with antipode S.*

i $S \circ \mu = \mu \circ (S \otimes S) \circ \tau$;

ii $S (\eta (1_{\Bbbk})) = \eta (1_{\Bbbk})$;

iii $\Delta \circ S = (S \otimes S) \circ \tau \circ \Delta$;

iv $\varepsilon \circ S = \varepsilon$.

Example 1.2.10 *Let G be a group. Then the group algebra $\Bbbk G$ is a Hopf algebra with canonical basis G. The antipode map is induced by the group inverse, so that*

$$S (g) = g^{-1}$$

for all $g \in G$, and extended linearly to all of $\Bbbk G$. The fact that S is the antipode of $\Bbbk G$ follows from the identity

$$\mu (S \otimes id) \Delta (g) = \mu (S (g) \otimes g) = g^{-1} g = 1_G$$

$$\mu (id \otimes S) \Delta (g) = \mu (g \otimes S (g)) = g g^{-1} = 1_G$$

for all $g \in G$, since $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. Since G is a basis for $\Bbbk G$, we have that S is the antipode of $\Bbbk G$.

Let H and H' be Hopf algebras. Then the tensor product $H \otimes H'$ is a Hopf algebra with the tensor product algebra and tensor product coalgebra structures.

The antipode of $H \otimes H'$ is given by $S_H \otimes S_{H'}$, where S_H and $S_{H'}$ are the antipodes of H and H', respectively.

1.3 Classification in low dimensions

In this section, we show that for a fixed dimension n, the set of bialgebrasis endowed with a structure of algebraic variety and a natural structure transport action which describes the set of isomorphic algebras. Solving such systems of polynomial equations leads to classifications of such structure. We aim at classifying bialgebras of dimension 2 and 3.

Let V be an n-dimensional vector space over $\Bbbk$. Setting a basis $\{e_i\}_{i\in\{0,1,2,\ldots n\}}$ of V, a multiplication μ (resp. a comultiplication Δ) is identified with its n^3 structure constants $C_{i,j}^k \in \Bbbk$ (resp. D_i^{jk}), where $\mu(e_i \otimes e_j) = \sum_{k=1}^{n} C_{i,j}^k e_k$ and $\Delta(e_i) = \sum_{j,k=1}^{n} D_i^{j,k} e_j \otimes e_k$. The counit ε is identified to its n structure constants ξ_i. We assume that e_1 is the unit.

A collection $\left\{\left(C_{i,j}^k, D_i^{j,k}, \xi_i\right)_{i,j,k\in\{1,\ldots,n\}}\right\}$ represents a bialgebra if the underlying multiplications, comultiplication, and the counit satisfy the appropriate conditions which translate to following polynomial equations.

$$\left\{\begin{array}{l} \sum\limits_{l=1}^{n}\left(C_{ij}^l C_{lk}^s - C_{jk}^l C_{il}^s\right) = 0 \\ C_{1i}^j = C_{i1}^j = \delta_{ij} \end{array}\right. \quad \forall i,j,k,s \in \{1,\ldots,n\},$$

$$\left\{\begin{array}{l} \sum\limits_{l=1}^{n}\left(D_s^{lk} D_l^{ij} - D_s^{il} D_l^{jk}\right) = 0 \\ \sum\limits_{l=1}^{n} D_i^{jl}\zeta_l = \sum\limits_{l=1}^{n} D_i^{lj}\zeta_l = \delta_{ij} \end{array}\right. \quad \forall i,j,k,s \in \{1,\ldots,n\},$$

$$\left\{\begin{array}{ll} \sum\limits_{l=1}^{n} C_{ij}^l D_l^{ks} - \sum\limits_{r,t,p,q=1}^{n} D_i^{rt} D_j^{pq} C_{rp}^k C_{tq}^s = 0 & \\ D_1^{11} = 1, D_1^{ij} = 0 & (i,j) \neq (1,1) \\ \zeta_1 = 1, \sum\limits_{l=1}^{n} C_{ij}^l \zeta_l = \zeta_i\zeta_j & \end{array}\right. \forall i,j,k,s \in \{1,\ldots,n\}.$$

Then, the set of n-dimensional bialgebras, which we denote by B_n, carries a structure of algebraic variety imbedded in $\mathbb{k}^{2n^3+n}$ with its natural structure of algebraic variety.

The "structure transport" action is defined by the action of $GL_n(V)$ on B_n. It corresponds to the change of basis.

Let $B = (V, \mu, \eta, \Delta, \varepsilon)$ be a bialgebras and $f : V \longrightarrow V$ be an invertible endomorphism, then the action of f on B transports the bialgebra structure into a bialgebra $B = (V, \mu', \eta', \Delta', \varepsilon')$ defined by

$$\mu' = f \circ \mu \circ \left(f^{-1} \otimes f^{-1}\right) \text{ and } \eta' = f \circ \eta$$

$$\Delta' = (f \otimes f) \circ \Delta \circ f \text{ and } \varepsilon' = \varepsilon \circ f^{-1}.$$

1.3.1 Classifications in Dimension 2

The set of 2-dimensional unital associative algebras yields two non-isomorphic algebras (see [15]). Let $\{e_1, e_2\}$ be a basis of $\mathbb{k}^2$, then the algebras are given by the following non-trivial products.

$$\bullet \mu_1^2(e_1, e_i) = \mu_1^2(e_i, e_1) = e_i, \; i = 1, 2, \; \mu_1^2(e_2, e_2) = e_2,$$

$$\bullet \mu_2^2(e_1, e_i) = \mu_2^2(e_i, e_1) = e_i, \; i = 1, 2, \; \mu_1^2(e_2, e_2) = 0.$$

In the sequel we consider that all the algebras are unital and the unit η corresponds to e_1.

In the following, we list the coalgebras which, combined with μ_1, give bialgebra structures (up to isomorphism).

$\bullet \Delta_{1,1}^2(e_1) = e_1 \otimes e_1$; $\Delta_{1,1}^2(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - 2e_2 \otimes e_2$;

$\varepsilon_{1,1}^2(e_1) = 1$; $\varepsilon_{1,1}^2(e_2) = 0$;

$\bullet \Delta_{1,2}^2(e_1) = e_1 \otimes e_1$; $\Delta_{1,2}^2(e_2) = e_2 \otimes e_2$;

$\varepsilon_{1,2}^2(e_1) = 1$; $\varepsilon_{1,2}^2(e_2) = 1$;

$\bullet \Delta_{1,3}^2(e_1) = e_1 \otimes e_1$; $\Delta_{1,3}^2(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$;

$\varepsilon_{1,3}^2(e_1) = 1$; $\varepsilon_{1,3}^2(e_2) = 0$;

1.3.2 Classifications in Dimension 3

First, we recall the classification of 3-dimensional unital associative algebras (see [15]). Let $\{e_1, e_2, e_3\}$, be a basis of $\mathbb{k}^3$, then the algebras are given by the following non-trivial products.

$\bullet\mu_1^3(e_1, e_i) = \mu_1^3(e_i, e_1) = e_i,\ i = 1, 2, 3;\ \mu_1^3(e_j, e_2) = \mu_1^3(e_2, e_j) = e_j,\ j = 2.3,\ \mu_1^3(e_3, e_3) = e_3,$

$\bullet\mu_2^3(e_1, e_i) = \mu_2^3(e_i, e_1) = e_i,\ i = 1, 2, 3;\ \mu_2^3(e_j, e_2) = \mu_2^3(e_2, e_j) = e_j,\ j = 2.3,\ \mu_2^3(e_3, e_3) = 0,$

$\bullet\mu_3^3(e_1, e_i) = \mu_3^3(e_i, e_1) = e_i,\ i = 1, 2, 3;\ \mu_3^3(e_2, e_2) = e_2,$

$\bullet\mu_4^3(e_1, e_i) = \mu_4^3(e_i, e_1) = e_i,\ i = 1, 2, 3,$

$\bullet\mu_5^3(e_1, e_i) = \mu_5^3(e_i, e_1) = e_i,\ i = 1, 2, 3;\ \mu_5^3(e_2, e_j) = e_j,\ j = 2, 3.$

Thanks to computer algebra, we obtain the following coalgebras associated to the previous algebras in order to obtain a bialgebra structures. We denote the comultiplications by $\Delta_{i,j}^3$ and the counits by $\varepsilon_{i,j}^3$, where i indicates the item of the multiplication and j the item of the comultiplication which combined with the multiplication i determine a bialgebra.

For the multiplication μ_1^3, we have:

$\bullet\Delta_{1,1}^3(e_1) = e_1 \otimes e_1$; $\Delta_{1,1}^3(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta_{1,1}^3(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 - 2e_3 \otimes e_3$; $\varepsilon_{1,1}^3(e_1) = 1$; $\varepsilon_{1,1}^3(e_2) = 0$; $\varepsilon_{1,1}^3(e_3) = 0$.

$\bullet\Delta_{1,2}^3(e_1) = e_1 \otimes e_1$; $\Delta_{1,2}^3(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta_{1,2}^3(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_3$; $\varepsilon_{1,2}^3(e_1) = 1$; $\varepsilon_{1,2}^3(e_2) = 0$; $\varepsilon_{1,2}^3(e_3) = 0$.

$\bullet\Delta_{1,3}^3(e_1) = e_1 \otimes e_1$; $\Delta_{1,3}^3(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta_{1,3}^3(e_3) = e_1 \otimes e_3 - e_2 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_2 - e_3 \otimes e_3$; $\varepsilon_{1,3}^3(e_1) = 1$; $\varepsilon_{1,3}^3(e_2) = 0$; $\varepsilon_{1,3}^3(e_3) = 0$.

$\bullet\Delta_{1,4}^3(e_1) = e_1 \otimes e_1$; $\Delta_{1,4}^3(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta_{1,4}^3(e_3) = e_1 \otimes e_3 - e_2 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_2$; $\varepsilon_{1,4}^3(e_1) = 1$; $\varepsilon_{1,4}^3(e_2) = 0$; $\varepsilon_{1,4}^3(e_3) = 0$.

$\bullet\Delta_{1,5}^3(e_1) = e_1 \otimes e_1$; $\Delta_{1,5}^3(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta_{1,5}^3(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 - e_2 \otimes e_3$; $\varepsilon_{1,5}^3(e_1) = 1$; $\varepsilon_{1,5}^3(e_2) = 0$; $\varepsilon_{1,5}^3(e_3) = 0$.

$\bullet\Delta_{1,6}^3(e_1) = e_1 \otimes e_1$; $\Delta_{1,6}^3(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta_{1,6}^3(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_2$; $\varepsilon_{1,6}^3(e_1) = 1$; $\varepsilon_{1,6}^3(e_2) = 0$; $\varepsilon_{1,6}^3(e_3) = 0$.

$\bullet\Delta^3_{1,7}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,7}(e_2) = e_2 \otimes e_2$; $\Delta^3_{1,7}(e_3) = e_2 \otimes e_3 + e_3 \otimes e_2 - 2e_3 \otimes e_3$; $\varepsilon^3_{1,7}(e_1) = 1$; $\varepsilon^3_{1,7}(e_2) = 1$; $\varepsilon^3_{1,7}(e_3) = 0$.

$\bullet\Delta^3_{1,8}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,8}(e_2) = e_2 \otimes e_2$; $\Delta^3_{1,8}(e_3) = e_2 \otimes e_3 + e_3 \otimes e_2 - e_3 \otimes e_3$; $\varepsilon^3_{1,8}(e_1) = 1$; $\varepsilon^3_{1,8}(e_2) = 1$; $\varepsilon^3_{1,8}(e_3) = 0$.

$\Delta^3_{1,9}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,9}(e_2) = e_1 \otimes e_3 + e_2 \otimes e_2 - e_2 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_2$; $\Delta^3_{1,9}(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_3$; $\varepsilon^3_{1,9}(e_1) = 1$; $\varepsilon^3_{1,9}(e_2) = 1$; $\varepsilon^3_{1,9}(e_3) = 0$.

$\bullet\Delta^3_{1,10}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,10}(e_2) = e_1 \otimes e_3 + e_2 \otimes e_2 - e_2 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_2 + e_3 \otimes e_3$; $\Delta^3_{1,10}(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 - 2e_3 \otimes e_3$; $\varepsilon^3_{1,10}(e_1) = 1$; $\varepsilon^3_{1,10}(e_2) = 1$; $\varepsilon^3_{1,10}(e_3) = 0$.

$\bullet\Delta^3_{1,11}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,11}(e_2) = e_2 \otimes e_2 + e_3 \otimes e_1 - e_3 \otimes e_2$; $\Delta^3_{1,11}(e_3) = e_2 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_3$; $\varepsilon^3_{1,11}(e_1) = 1$; $\varepsilon^3_{1,11}(e_2) = 1$; $\varepsilon^3_{1,11}(e_3) = 0$.

$\bullet\Delta^3_{1,12}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,12}(e_2) = e_1 \otimes e_3 + e_2 \otimes e_2 - e_2 \otimes e_3$; $\Delta^3_{1,12}(e_3) = e_1 \otimes e_3 + e_3 \otimes e_2 - e_3 \otimes e_3$; $\varepsilon^3_{1,12}(e_1) = 1$; $\varepsilon^3_{1,12}(e_2) = 1$; $\varepsilon^3_{1,12}(e_3) = 0$.

$\bullet\Delta^3_{1,13}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,13}(e_2) = e_1 \otimes e_2 - e_1 \otimes e_3 + e_2 \otimes e_1 - 2e_2 \otimes e_2 + 2e_2 \otimes e_3 - e_3 \otimes e_1 + 2e_3 \otimes e_2 - e_3 \otimes e_3$; $\Delta^3_{1,13}(e_3) = e_2 \otimes e_3 + e_3 \otimes e_2 - 2e_3 \otimes e_3$; $\varepsilon^3_{1,13}(e_1) = 1$; $\varepsilon^3_{1,13}(e_2) = 1$; $\varepsilon^3_{1,13}(e_3) = 1$.

$\bullet\Delta^3_{1,14}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,14}(e_2) = e_1 \otimes e_2 - e_1 \otimes e_3 + e_2 \otimes e_1 - e_2 \otimes e_2 + e_2 \otimes e_3 - e_3 \otimes e_1 + e_3 \otimes e_2$; $\Delta^3_{1,14}(e_3) = e_2 \otimes e_3 + e_3 \otimes e_2 - e_3 \otimes e_3$; $\varepsilon^3_{1,14}(e_1) = 1$; $\varepsilon^3_{1,14}(e_2) = 1$; $\varepsilon^3_{1,14}(e_3) = 1$.

$\bullet\Delta^3_{1,15}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,15}(e_2) = e_2 \otimes e_2$; $\Delta^3_{1,15}(e_3) = e_3 \otimes e_3$; $\varepsilon^3_{1,15}(e_1) = 1$; $\varepsilon^3_{1,15}(e_2) = 1$; $\varepsilon^3_{1,15}(e_3) = 1$.

$\bullet\Delta^3_{1,16}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,16}(e_2) = e_2 \otimes e_2$; $\Delta^3_{1,16}(e_3) = e_2 \otimes e_2 - e_2 \otimes e_3 - e_3 \otimes e_2 + 2e_3 \otimes e_3$; $\varepsilon^3_{1,16}(e_1) = 1$; $\varepsilon^3_{1,16}(e_2) = 1$; $\varepsilon^3_{1,16}(e_3) = 1$.

$\bullet\Delta^3_{1,17}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,17}(e_2) = e_2 \otimes e_3 + e_3 \otimes e_2 - e_3 \otimes e_3$; $\Delta^3_{1,17}(e_3) = e_3 \otimes e_3$; $\varepsilon^3_{1,17}(e_1) = 1$; $\varepsilon^3_{1,17}(e_2) = 1$; $\varepsilon^3_{1,17}(e_3) = 1$.

$\bullet\Delta^3_{1,18}(e_1) = e_1 \otimes e_1$; $\Delta^3_{1,18}(e_2) = e_2 \otimes e_1 - e_3 \otimes e_1 + e_3 \otimes e_2$; $\Delta^3_{1,18}(e_3) = e_3 \otimes e_3$; $\varepsilon^3_{1,18}(e_1) = 1$; $\varepsilon^3_{1,18}(e_2) = 1$; $\varepsilon^3_{1,18}(e_3) = 1$.

For the multiplication μ^3_2, we have

$\bullet\Delta^3_{2,1}(e_1) = e_1 \otimes e_1$; $\Delta^3_{2,1}(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta^3_{2,1}(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_2$; $\varepsilon^3_{2,1}(e_1) = 1$; $\varepsilon^3_{2,1}(e_2) = 0$; $\varepsilon^3_{2,1}(e_3) = 0$.

$\Delta^3_{2,2}(e_1) = e_1 \otimes e_1$; $\Delta^3_{2,2}(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta^3_{2,2}(e_3) = e_1 \otimes e_3 + e_2 \otimes e_3 + e_3 \otimes e_1$; $\varepsilon^3_{2,2}(e_1) = 1$; $\varepsilon^3_{2,2}(e_2) = 0$; $\varepsilon^3_{2,2}(e_3) = 0$.

•$\Delta^3_{2,3}(e_1) = e_1 \otimes e_1$; $\Delta^3_{2,3}(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$; $\Delta^3_{2,3}(e_3) = e_1 \otimes e_3 - e_2 \otimes e_3 + e_3 \otimes e_1 - e_3 \otimes e_2 + \lambda e_3 \otimes e_3$; $\varepsilon^3_{2,3}(e_1) = 1$; $\varepsilon^3_{2,3}(e_2) = 0$; $\varepsilon^3_{2,3}(e_3) = 0$.

For the multiplication μ^3_3, we have

•$\Delta^3_{3,1}(e_1) = e_1 \otimes e_1$; $\Delta^3_{3,1}(e_2) = e_2 \otimes e_2$; $\Delta^3_{3,1}(e_3) = e_2 \otimes e_3 + e_3 \otimes e_2$; $\varepsilon^3_{3,1}(e_1) = 1$; $\varepsilon^3_{3,1}(e_2) = 1$; $\varepsilon^3_{3,1}(e_3) = 0$.

•$\Delta^3_{3,2}(e_1) = e_1 \otimes e_1$; $\Delta^3_{3,2}(e_2) = e_2 \otimes e_2$; $\Delta^3_{3,2}(e_3) = e_1 \otimes e_3 + e_3 \otimes e_2$; $\varepsilon^3_{3,2}(e_1) = 1$; $\varepsilon^3_{3,2}(e_2) = 1$; $\varepsilon^3_{3,2}(e_3) = 0$.

•$\Delta^3_{3,3}(e_1) = e_1 \otimes e_1$; $\Delta^3_{3,3}(e_2) = e_2 \otimes e_2$; $\Delta^3_{3,3}(e_3) = e_2 \otimes e_3 + e_3 \otimes e_1$; $\varepsilon^3_{3,3}(e_1) = 1$; $\varepsilon^3_{3,3}(e_2) = 1$; $\varepsilon^3_{3,3}(e_3) = 0$.

For the multiplication μ^3_4, there does not exist any bialgebras.

For the multiplication μ^3_5, we have

•$\Delta^3_{5,1}(e_1) = e_1 \otimes e_1$; $\Delta^3_{5,1}(e_2) = e_2 \otimes e_2$; $\Delta^3_{5,1}(e_3) = e_2 \otimes e_3 + e_3 \otimes e_2$; $\varepsilon^3_{5,1}(e_1) = 1$; $\varepsilon^3_{5,1}(e_2) = 1$; $\varepsilon^3_{5,1}(e_3) = 0$.

In the sequel we consider that all the algebras are unital and the unit η corresponds to e_1.

Chapter 2

Hom-bialgebras and Hom-Hopf algebras

In this chapter, we first recall basics on unital Hom-associative algebras, counital Hom-coalgebras and Hom-bialgebras, and describe some properties of those structures extending the classical structures of algebras, coalgebra and bialgebra [37],[36].

All vector spaces (Hom-algebras, Hom-coalgebras, Hom-bialgebras) will be over a ground field $\mathbb{k}$. In the classification of Andruskiewitsch and Schneider, however we do not require this for the general theory.

2.1 Unital Hom-associative algebras

Definition 2.1.1 *([34], [38]) A* ***Hom-associative algebra*** *is a triple $A = (A, \mu, \alpha)$ consisting of a $\mathbb{k}$-vector space A, a linear map $\mu : A \otimes A \longrightarrow A$ (multiplication), and a homomorphism $\alpha : A \longrightarrow A$ satisfying the Hom-associativity condition*

$$\mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha) . \tag{2.1}$$

We assume moreover in this paper that $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$.

To generalize quantum groups to the Hom setting, we need a suitably weakened notion of a multiplicative identity for Hom-associative algebras.

Definition 2.1.2 *A Hom-associative algebra A is called* ***unital*** *if there exists a linear map $\eta : \mathbb{k} \longrightarrow A$ such that $\alpha \circ \eta = \eta$ and*

$$\mu \circ (\eta \otimes id_A) = \mu \circ (id_A \otimes \eta) = \alpha \tag{2.2}$$

The unit element is $1_A = \eta(1_{\Bbbk})$, we refer a unital Hom-associative algebra by (A, μ, η, α)

Hom-associativity and unitality conditions (2.1) and (2.2) may be expressed by the followings commutative diagrams.

$$
\begin{array}{ccccccccc}
A \otimes A \otimes A & \xrightarrow{\mu\otimes\alpha} & A \otimes A & & \Bbbk \otimes A & \xrightarrow{\eta\otimes id_A} & A \otimes A & \xleftarrow{id_A\otimes\eta} & A \otimes \Bbbk \\
{\scriptstyle\alpha\otimes\mu}\downarrow & & \downarrow{\scriptstyle\mu} & & \cong & & \downarrow\mu & & \cong \\
A \otimes A & \xrightarrow{\mu} & A & & A & \xrightarrow{\alpha} & A & \xleftarrow{\alpha} & A.
\end{array}
$$

where we have identified

$$\Bbbk \otimes A \cong A \cong A \otimes \Bbbk$$

Remark 2.1.3 .

1. *We recover the classical associative algebra when the twisting map α is the identity map.*

2. *We have $\alpha \circ \eta(1_{\Bbbk}) = \eta(1_{\Bbbk})$ then $\alpha(1_A) = 1_A$ and $\mu(1_A \otimes 1_A) = 1_A$.*

3. *We call Hom-associator the linear map $\mathfrak{as}_A$ defined on $A^{\otimes 3}$ by $\mu \circ (\alpha \otimes \mu - \mu \otimes \alpha)$.*

Example 2.1.4 *Here are some examples of Hom-associative algebras :*

1. *Let $\{e_1, e_2, e_3\}$ be a basis of a 3-dimensional linear space A over $\Bbbk$. The following multiplication μ and linear map α on A define a Hom-associative algebra over $\Bbbk^3$:*

$$
\begin{aligned}
&\mu(e_1 \otimes e_1) = ae_1, \quad \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = ae_2, \\
&\mu(e_2 \otimes e_2) = ae_2, \quad \mu(e_1 \otimes e_3) = \mu(e_3 \otimes e_1) = be_3, \\
&\mu(e_2 \otimes e_3) = be_3, \quad \mu(e_3 \otimes e_2) = \mu(e_3 \otimes e_3) = 0,
\end{aligned}
$$

$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = ae_2, \quad \alpha(e_3) = be_3,$$

where a, b are parameters in $\Bbbk$.

The algebras are not associative when $a \neq b$ and $b \neq 0$, since

$$\mu(\mu(e_1 \otimes e_1) \otimes e_3) - \mu(e_1 \otimes \mu(e_1 \otimes e_3)) = (a - b)\,be_3.$$

2. *Let* $\Bbbk G$ *be the group-algebra over the group* G. *As a vector space,* $\Bbbk G$ *is generated by* $\{e_g,\ g \in G\}$. *If* $\alpha : G \longrightarrow G$ *is a group homomorphism, then it can be extended to an algebra endomorphism of* $\Bbbk G$ *by setting*

$$\alpha\left(\sum_{g\in G} a_g e_g\right) = \sum_{g\in G} a_g \alpha\left(e_g\right) = \sum_{g\in G} a_g e_{\alpha(g)}$$

Consider the usual bialgebra structure on $\Bbbk G$ *and* α *a bialgebra morphism. Then, we define a generalized Hom-bialgebra* $(\Bbbk G, \mu, \Delta, \alpha)$ *over* $\Bbbk G$ *by setting:*

$$\mu\left(e_g \otimes e_{g'}\right) = \alpha\left(e_{g\cdot g'}\right),\ \Delta\left(e_g\right) = \alpha\left(e_g\right) \otimes \alpha\left(e_g\right).$$

3. *Let* $A = (A, \mu, \alpha)$ *be a Hom-associative algebra. Then* $(\mathcal{M}_n(A), \mu', \alpha')$, *where* $\mathcal{M}_n(A)$ *is the vector space of* $n \times n$ *matrix with entries in* A, *is also a Hom-associative algebra in which the multiplication* μ' *is given by matrix multiplication* μ *and* α' *is given by* α *in each entry.*

Example 2.1.5 **1.** *The tensor product of two unital Hom-associative algebras* $(A_1, \mu_1, \eta_1, \alpha_1)$ *and* $(A_2, \mu_2, \eta_2, \alpha_2)$ *is defined by* $(A_1 \otimes A_2, \tilde{\mu}, \tilde{\eta}, \tilde{\alpha})$ *such that* $\tilde{\mu} = (\mu_1 \otimes \mu_2) \circ \tau_{2,3}$, $\tilde{\eta} = \eta_1 \otimes \eta_2$, *and* $\tilde{\alpha} = \alpha_1 \otimes \alpha_2$, *where* $\tau_{23} = id_{A_1} \otimes \tau_{A_2 \otimes A_1} \otimes id_{A_2}$ *and* $\tau_{A_2\otimes A_1} : A_2 \otimes A_1 \longrightarrow A_1 \otimes A_2$; $\tau_{A_2\otimes A_1}(x_2 \otimes x_1) = x_1 \otimes x_2$, *is the linear 'flip' map.*

2. *Given a Hom-associative algebra* $A = (A, \mu, \alpha)$, *we define the opposite Hom-associative algebra* $A^{op} = (A, \mu^{op}, \alpha)$ *as the Hom-associative algebra with the same underlying vector space* A, *but with a multiplication defined by*

$$\mu^{op} = \mu \circ \tau_{A\otimes A},\ \mu^{op}(x \otimes y) = \mu(y \otimes x)$$

A Hom-associative algebra (A, μ, α) *is commutative if and only if* $\mu^{op} = \mu$.

Definition 2.1.6 *Let* (A, μ, α) *and* (A', μ', α') *be two Hom-associative algebras. A linear map* $f : A \to A'$ *is said to be a* ***Hom-associative algebras morphism*** *if*

$$\mu' \circ f^{\otimes 2} = f \circ \mu,\ \text{and } f \circ \alpha = \alpha' \circ f. \tag{2.3}$$

It is said to be a **weak morphism** *if holds only the first condition. If further more, the Hom-associative algebras are unital* with respect to η and η', then $f \circ \eta = \eta'$.

If $A = A'$, then the Hom-associative algebras (resp. unital Hom-associative algebras) are **isomorphic** if there exists a bijective linear map $f : A \to A$ such that

$$\mu = f^{-1} \circ \mu' \circ f^{\otimes 2},\ \alpha = f^{-1} \circ \alpha' \circ f, \tag{2.4}$$

$$(\text{resp. } \mu = f^{-1} \circ \mu' \circ f^{\otimes 2},\ \alpha = f^{-1} \circ \alpha' \circ f \textit{ and } \eta = f^{-1} \circ \eta'). \tag{2.5}$$

Proposition 2.1.7 *Let (A_1, μ, η, α), $(A_2, \mu', \eta', \alpha')$ be two unital Hom-associative algebras.*

The maps i_1 and i_2

$$\begin{array}{llll} i_1: & A_1 & \to & A_1 \otimes A_2 \\ & x & \to & i_1(x) = \alpha(x) \otimes 1_{A_2} \end{array} \qquad \begin{array}{llll} i_2: & A_2 & \to & A_1 \otimes A_2 \\ & y & \to & i_2(y) = 1_{A_1} \otimes \alpha'(y) \end{array}$$

are the morphisms of unital Hom-associative algebras.

Proof. First we check condition (2.3) for the map i_1.

It holds if and only if

$$\tilde{\mu} \circ (i_1 \otimes i_1) = i_1 \circ \mu,\ i_1 \circ \alpha = \left(\alpha \otimes \alpha'\right) \circ i_1,\ \textit{and } i_1 \circ \eta = \eta \otimes \eta'. \tag{2.6}$$

where $\tilde{\mu}$ is defined as in Example 2.1.5. For all $x_1, y_1 \in V_1$, we have

$$\begin{aligned} \tilde{\mu} \circ (i_1 \otimes i_1)(x_1 \otimes y_1) &= \left(\mu \otimes \mu'\right)(id_{A_1} \otimes \tau_{A_2 \otimes A_1} \otimes id_{A_2})(i_1(x_1) \otimes i_1(y_1)) \\ &= \left(\mu \otimes \mu'\right)(id_{A_1} \otimes \tau_{A_2 \otimes A_1} \otimes id_{A_2})(\alpha(x_1) \otimes 1_{A_2} \otimes \alpha(y_1) \otimes 1_{A_2}) \\ &= \mu(\alpha(x_1) \otimes \alpha(y_1)) \otimes \mu'(1_{A_2} \otimes 1_{A_2}) \\ &= \alpha(\mu(x_1 \otimes y_1)) \otimes 1_{A_2} \\ &= i_1 \circ \mu(x_1 \otimes y_1). \end{aligned}$$

So the first condition is satisfied. For all $x \in A_1$. we have

$$\begin{aligned} \tilde{\alpha} \circ i_1(x) &= (\alpha \otimes \alpha')(\alpha(x) \otimes 1_{A_2}) \\ &= \alpha(\alpha(x)) \otimes 1_{A_2} \\ &= i_1 \circ \alpha(x). \end{aligned}$$

So the scond condition is satisfied.

Finally

$$i_1 \circ \eta\left(1_{\Bbbk}\right) = \alpha\left(\eta\left(1_{\Bbbk}\right)\right) \otimes 1_{A_2} = \eta\left(1_{\Bbbk}\right) \otimes \eta'\left(1_{\Bbbk}\right).$$

Which shows that i_1 is a unital Hom-associative algebras morphism. Proof for i_2 is similar. ■

Proposition 2.1.8 *([49]) Let (A, μ, η, α) be a unital Hom-associative algebra and $\beta : A \longrightarrow A$ be a weak morphism of Hom-associative algebra, i.e. $\beta \circ \mu = \mu \circ \beta^{\otimes 2}$, $\beta \circ \alpha = \alpha \circ \beta$, and $\beta \circ \eta = \eta$. Then $A_\beta = (A, \mu_\beta = \beta \circ \mu, \eta_\beta = \beta \circ \eta, \alpha_\beta = \beta \circ \alpha)$ is a unital Hom-associative algebra.*

Hence, we denote by β^n the n-fold composition of n copies of β, with $\beta^0 = id_A$, $\beta^n \circ \mu = \mu \circ \left(\beta^{\otimes 2}\right)^n$, then $A_{\beta^n} = (A, \mu_{\beta^n} = \beta^n \circ \mu, \eta_{\beta^n} = \beta^n \circ \eta, \alpha_{\beta^n} = \beta^n \circ \alpha)$ is a unital Hom-associative algebra.

Proof. We have

$$\begin{aligned}
\mu_\beta \circ \alpha_\beta^{\otimes 2} &= (\beta \circ \mu) \circ [(\beta \circ \alpha) \otimes (\beta \circ \alpha)] = (\beta \circ \mu) \circ \left(\beta^{\otimes 2} \circ \alpha^{\otimes 2}\right) \\
&= \beta \circ \beta \circ \left(\mu \circ \alpha^{\otimes 2}\right) = \beta \circ (\beta \circ \alpha) \circ \mu = (\beta \circ \alpha) \circ (\beta \circ \mu) \\
&= \alpha_\beta \circ \mu_\beta.
\end{aligned}$$

Since β is a weak morphism of Hom-associative algebra, so α_β is a weak morphism of Hom-associative algebra.

We show that $(A, \mu_\beta, \eta_\beta, \alpha_\beta)$ satisfies the Hom-associativity. Indeed

$$\begin{aligned}
\mu_\beta\left(\mu_\beta \otimes \alpha_\beta\right) &= (\beta \circ \mu) \circ (\beta \circ \mu \otimes \beta \circ \alpha) \\
&= \beta \circ (\beta \circ (\mu \circ (\mu \otimes \alpha))) \\
&\overset{(2.1)}{=} \beta \circ (\beta \circ (\mu \circ (\alpha \otimes \mu))) \\
&= \beta \circ (\mu \circ (\beta \circ \alpha \otimes \beta \circ \mu)) \\
&= (\beta \circ \mu) \circ (\beta \circ \alpha \otimes \beta \circ \mu) \\
&= \mu_\beta\left(\alpha_\beta \otimes \mu_\beta\right).
\end{aligned}$$

The second assertion is proved similarly, so $(A, \mu_\beta, \eta_\beta, \alpha_\beta)$ is a unital Hom-associative algebra. ■

Remark 2.1.9 *In particular, if $\alpha = id_A$, one can construct a Hom-associative algebra starting from an associative algebra and an algebra endomorphism.*

2.2 Counital Hom-coassociative coalgebras

We define first the fundamental notion of a Hom-coassociative coalgebra, which is dual to that of a Hom-associative algebra, in the sense that if we reverse all the arrows in the defining diagrams of a Hom-associative algebra, we get the concept of a Hom-coassociative coalgebra.

Definition 2.2.1 *[29] A **Hom-coassociative coalgebra** is a triple (C, Δ, β) where C is a $\Bbbk$-vector space, $\Delta : C \longrightarrow C \otimes C$, is a linear map, and $\beta : C \longrightarrow C$ is a homomorphism satisfying the Hom-coassociativity condition,*

$$(\Delta \otimes \beta) \circ \Delta = (\beta \otimes \Delta) \circ \Delta. \tag{2.7}$$

We assume moreover that $\Delta \circ \beta = \beta^{\otimes 2} \circ \Delta$.

A Hom-coassociative coalgebra is said to be **counital** if there exists a linear map $\varepsilon : C \longrightarrow \Bbbk$ such that $\varepsilon \circ \beta = \varepsilon$ and

$$(\varepsilon \otimes id_C) \circ \Delta = (id_C \otimes \varepsilon) \circ \Delta = \beta. \tag{2.8}$$

Conditions (2.7) and (2.8) are respectively equivalent to the following commutative diagrams:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \beta \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes \beta} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} \Bbbk \otimes C & \xleftarrow{\varepsilon \otimes id_C} & C \otimes C & \xrightarrow{id_C \otimes \varepsilon} & C \otimes \Bbbk \\ \cong & & \uparrow \Delta & & \cong \\ C & \xleftarrow{\beta} & C & \xrightarrow{\beta} & C \end{array}$$

Example 2.2.2 *The ground field* $\mathbb{k}$ *is a Hom-coalgebra by defining*

$$\Delta(1\otimes 1)=1,\ \varepsilon(1)=1,\ \textit{and}\ \beta(1)=1$$

and extended linearly to all of $\mathbb{k}$.

Remark 2.2.3 **1.** *We recover the classical coassociative coalgebra when the twisting map* β *is the identity map.*

2. *Given a Hom-coassociative coalgebra* $C=(C,\Delta,\beta)$, *we define the* ***coopposite*** *Hom-coassociative coalgebra* $C^{cop}=(C,\Delta^{cop},\beta)$ *to be the Hom-coassociative coalgebra with the same underlying vector space as* C *and with comultiplication defined by* $\Delta^{cop}=\tau_{C\otimes C}\circ\Delta$.

3. *A Hom-coassociative coalgebra* (C,Δ,β) *is* ***cocommutative*** *if and only if* $\Delta^{cop}=\Delta$.

Definition 2.2.4 *Let* (C,Δ,β) *and* (C',Δ',β') *be two Hom-coassociative coalgebras. A linear map* $f:C\to C'$ *is a* ***Hom-coassociative coalgebras morphism*** *if*

$$f^{\otimes 2}\circ\Delta=\Delta'\circ f\ ,\ \textit{and}\ f\circ\beta=\beta'\circ f. \tag{2.9}$$

It is said to be a ***weak morphism*** *if holds only the first condition. If furthermore the Hom-coassociative coalgebras admit counits* ε *and* ε', *we have moreover* $\varepsilon=\varepsilon'\circ f$.

That is, $f:C\to C'$ *is a weak morphism of Hom-coalgebras if the diagrams*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ f\downarrow & & \downarrow f\otimes f \\ C' & \xrightarrow{\Delta'} & C'\otimes C' \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & C' \\ \varepsilon\downarrow & \swarrow\varepsilon' & \\ \mathbb{k} & & \end{array}$$

commute, and $f\circ\beta=\beta'\circ f$.

We say that a Hom-coassociative coalgebra (C,Δ,β) is **isomorphic** to a Hom-coassociative coalgebra (C',Δ',β') if there exists a bijective Hom-coassociative coalgebra morphism $f:C\longrightarrow C'$, and we denote this by $C\cong C'$ when the context is clear, shach that

$$\Delta'=f^{\otimes 2}\circ\Delta\circ f^{-1},\ \varepsilon'=\varepsilon\circ f^{-1}\ \textit{and}\ \beta'=\beta\circ f^{-1},$$

Definition 2.2.5 *If (C,Δ,β) is a Hom-coassociative coalgebra and D is a vector subspace of C, then we say that D is a **Hom-subcoalgebra** if $\Delta(D)\subseteq D\otimes D$. In this case, $(D,\Delta|_D,\beta|_D)$ is a Hom-coassociative coalgebra contained in the Hom-coassociative coalgebra (C,Δ,β).*

The Hom-coassociativity of Δ is expressed by

$$(\Delta\otimes\beta)\circ\Delta(x)=\sum\Delta\left(x_{(1)}\right)\otimes\beta\left(x_{(2)}\right)=\sum x_{(1)(1)}\otimes x_{(1)(2)}\otimes\beta\left(x_{(2)}\right)$$
$$(\beta\otimes\Delta)\circ\Delta(x)=\sum\beta\left(x_{(1)}\right)\otimes\Delta\left(x_{(2)}\right)=\sum\beta\left(x_{(1)}\right)\otimes x_{(2)(1)}\otimes x_{(2)(2)}$$

Let us define inductively mappings $\Delta^{(n)}:C\longrightarrow C^{\otimes n+1}$ by

$$\Delta^{(n)}=\left(\beta^{\otimes n-1}\otimes\Delta\right)\circ\Delta^{(n-1)}\quad n>1\quad \textit{and}\quad \Delta^{(1)}=\Delta,$$

then the element $\Delta^{(n)}(x)\in C^{\otimes n+1}$ is denoted by

$$\Delta^{(n)}=\beta^{n-1}\left(x_{(1)}\right)\otimes\beta^{n-2}\left(x_{(2)}\right)\otimes...\otimes\beta^{2}\left(x_{(n-2)}\right)\otimes\beta\left(x_{(n-1)}\right)\otimes x_{(n)}\otimes x_{(n+1)}$$

or

$$\Delta^{(n)}=x_{(1)}\otimes x_{(2)}\otimes\beta\left(x_3\right)\otimes\beta^{2}\left(x_4\right)\otimes...\otimes\beta^{n-2}\left(x_{(n)}\right)\otimes\beta^{n-1}\left(x_{(n+1)}\right)$$

then $\Delta^{(n)}$ denotes the $n-$ary Hom-comultiplication, then we denote the $n-$ary Hom-multiplication [57] by $\mu^{(n)}:A^{\otimes n+1}\longrightarrow A$

$$\mu^{(n)}=\mu^{(n-1)}\circ\left(\alpha^{\otimes n-1}\otimes\mu\right)\quad n>1\quad \textit{and}\quad \mu^{(1)}=\mu.$$

Then the element $\mu^{(n)}\left(x_1\otimes...\otimes x_n\otimes x_{n+1}\right)\in A$ is denoted by

$$\mu^{(n)}\left(x_1\otimes...\otimes x_n\otimes x_{n+1}\right)=\alpha^{n-1}\left(x_1\right)\cdot\alpha^{n-2}\left(x_2\right)\cdot...\cdot\alpha\left(x_{n-1}\right)\cdot x_n\cdot x_{n+1}$$

or

$$\mu^{(n)}\left(x_1\otimes...\otimes x_n\otimes x_{n+1}\right)=x_1\cdot x_2\cdot\alpha\left(x_3\right)\cdot...\cdot\alpha^{n-2}\left(x_n\right)\cdot\alpha^{n-1}\left(x_{n+1}\right)$$

The counit is described by

$$\sum\varepsilon\left(x_{(1)}\right)x_{(2)}=\sum x_{(1)}\varepsilon\left(x_{(2)}\right)=\beta\left(x\right)$$

Next we describe the tensor product Hom-coassociative coalgebra construction, which is similar to the tensor product Hom-associative algebra structure of two Hom-associative algebras over the same eld.

Proposition 2.2.6 *Let $(C_1, \Delta_1, \varepsilon_1, \beta_1)$ and $(C_2, \Delta_2, \varepsilon_2, \beta_2)$ be two counital Hom-coassociative coalgebras. Then the composite map*

$$C_1 \otimes C_2 \overset{\Delta_1 \otimes \Delta_2}{\longrightarrow} (C_1 \otimes C_1) \otimes (C_2 \otimes C_2) \overset{id_{C_1} \otimes \tau_{C_2 \otimes C_1} \otimes id_{C_2}}{\longrightarrow} (C_1 \otimes C_2) \otimes (C_1 \otimes C_2)$$

where : $\tau_{C_2 \otimes C_1} : C_2 \otimes C_1 \longrightarrow C_1 \otimes C_2$ is the linear 'flip' map, defines a Hom-coassociative comultiplication $\tilde{\Delta} = (id_{C_1} \otimes \tau_{C_2 \otimes C_1} \otimes id_{C_2}) \circ \Delta_1 \otimes \Delta_2$ on $C_1 \otimes C_2$, and with the counit ε_1 of C_1 and ε_2 of C_2 the map $\varepsilon_1 \otimes \varepsilon_2 : C_1 \otimes C_2 \longrightarrow \mathbb{k}$ is a counit of $C_1 \otimes C_2$.

Definition 2.2.7 ***Tensor product*** *$C_1 \otimes C_2$ of two counital Hom-coassociative coalgebras $(C_1, \Delta_1, \varepsilon_1, \beta_1)$ and $(C_2, \Delta_2, \varepsilon_2, \beta_2)$ is defined by $(C_1 \otimes C_2, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{\beta})$ such that*

$$\tilde{\Delta} = (id_{C_1} \otimes \tau_{C_2 \otimes C_1} \otimes id_{C_2}) \circ \Delta \otimes \Delta',\ \tilde{\varepsilon} = \varepsilon_1 \otimes \varepsilon_2, \text{and } \tilde{\beta} = \beta_1 \otimes \beta_2 \tag{2.10}$$

Dual to the notion of an ideal of a Hom-associative algebra is that of a coideal of a Hom-coassociative coalgebra. With coideals, we will be able to construct quotient Hom-coassociative coalgebras on the corresponding quotient vector spaces.

Definition 2.2.8 *Let $(C, \Delta, \varepsilon, \beta)$ be a Hom-coassociative coalgebra and I a subspace of C. Then I is a **left coideal** of C if $\Delta(I) \subseteq C \otimes I$. Similarly, I is called a **right coideal** of C if $\Delta(I) \subseteq I \otimes C$. We say that I is a **coideal** of C if $\Delta(I) \subseteq C \otimes I + I \otimes C$, $\varepsilon(I) = I$ and $\alpha(I) = I$.*

Remark 2.2.9 *Let (A, μ, η, α), and $(C, \Delta, \varepsilon, \beta)$ tow unital Hom-associative algebras, we have*

1. $\alpha^n \circ \mu = \mu \circ (\alpha^n \otimes \alpha^n)\ \forall n \geq 1$.

2. $(\beta^n \otimes \beta^n) \circ \Delta = \Delta \circ \beta^n \quad \forall n \geq 1$.

Proposition 2.2.10 *Let $C=(C,\Delta,\varepsilon,\alpha)$ be a counital Hom-coassociative coalgebra and $\beta:C\longrightarrow C$ be a weak morphism of Hom-coassociative coalgebra, i. e. $\Delta\circ\beta=\beta^{\otimes 2}\circ\Delta$, $\beta\circ\alpha=\alpha\circ\beta$, and $\varepsilon\circ\beta=\varepsilon$ then $C_\beta=(C,\Delta_\beta=\Delta\circ\beta,\varepsilon_\beta=\varepsilon\circ\beta,\alpha_\beta=\alpha\circ\beta)$ is a counital Hom-coassociative coalgebra.*

Hence, we denote by β^n the n-fold composition of n copies of β, with $\beta^0=id_V$, $\Delta\circ\beta^n=\left(\beta^{\otimes 2}\right)^n\circ\Delta$, then $C_{\beta^n}=(C,\Delta_{\beta^n}=\Delta\circ\beta^n,\varepsilon_{\beta^n}=\varepsilon\circ\beta^n,\alpha_{\beta^n}=\alpha\circ\beta^n)$ is a counital Hom-coassociative coalgebra.

Proof. We have

$$\begin{aligned}\alpha_\beta^{\otimes 2}\circ\Delta_\beta &= \left(\beta^{\otimes 2}\circ\alpha^{\otimes 2}\right)\circ(\Delta\circ\beta)\\ &=\beta^{\otimes 2}\circ\left(\alpha^{\otimes 2}\circ\Delta\right)\circ\beta=\beta^{\otimes 2}\circ\Delta\circ\alpha\circ\beta=\Delta\circ\beta\circ\alpha\circ\beta\\ &=\Delta_\beta\circ\alpha_\beta.\end{aligned}$$

Since β is a weak morphism of Hom-coassociative coalgebra, so α_β is a morphism of Hom-coassociative coalgebra.

We show that $(C,\Delta_\beta,\varepsilon_\beta,\alpha_\beta)$ satisfies the Hom-coassociativity. Indeed

$$\begin{aligned}(\Delta_\beta\otimes\alpha_\beta)\circ\Delta_\beta &= (\Delta\circ\beta\otimes\alpha\circ\beta)\circ(\Delta\circ\beta)\\ &=(\Delta\otimes\alpha)\circ\left(\beta^{\otimes 2}\circ\Delta\right)\circ\beta\\ &=((\Delta\otimes\alpha)\circ\Delta)\circ\beta\circ\beta\\ &\overset{(2.7)}{=}(\alpha\otimes\Delta)\circ\Delta\circ\beta\circ\beta\\ &=(\alpha\circ\beta\otimes\Delta\circ\beta)\circ\Delta\circ\beta\\ &=(\alpha_\beta\otimes\Delta_\beta)\circ\Delta_\beta.\end{aligned}$$

The second assertion is proved similarly, so $(C,\Delta_\beta,\varepsilon_\beta,\alpha_\beta)$ is a Hom-counital coassociative coalgebra. ■

If $\alpha=id_C$, this proposition shows how to construct a Hom-coassociative coalgebra starting from a coalgebra and a coalgebra morphism ([35]). It is a Hom-coalgebra version of the Proposition 2.1.8. We need only the coassociative comultiplication of the coalgebra [see [55]].

2.3 Duality between Hom-associative algebras and Hom-coassociative coalgebras

We will often use the following simple fact: if V and W are $\Bbbk$-vector spaces, and x is an element of $V \otimes W$, then x can be represented as $x = \sum_{i=1}^{n} x_i \otimes y_i$ for some positive integer n, some linearly independent $(x_i)_{i=1,\dots,n}$ in V, and some $(y_i)_{i=1,\dots,n}$ in W. Similarly, it can be written as a sum of tensor monomials with the elements appearing on the second tensor position being linearly independent.

The following lemma is well known from linear algebra.

Lemma 2.3.1 *Let M, N, and V are three $\Bbbk$-vector spaces, and linear maps $\phi : M^* \otimes V \longrightarrow Hom\,(M,V)$, $\phi' : Hom\,(M,N^*) \longrightarrow (M \otimes N)^*$, $\rho : M^* \otimes N^* \longrightarrow (M \otimes N)^*$ defined by*

$$\phi\,(f \otimes v)\,(m) = f\,(m)\,v \;\; for \;\; f \in M^*, \;\; v \in V, \;\; m \in M$$

$$\phi'\,(g)\,(m \otimes n) = g\,(m)\,(n) \quad for\; g \in Hom\,(M,N^*)\,, \;\; m \in M, \;\; n \in N,$$

$$\rho\,(f \otimes g)\,(m \otimes n) = f\,(m)\,g\,(n) \;\; for \;\; f \in M^*, \;\; g \in N^*, \;\; m \in M, \;\; n \in N$$

where $Hom\,(M,V) = \{f : M \longrightarrow V, \;\; f \text{ is a linear map}\}$ and $M^* = Hom\,(M,\Bbbk)$.

Then

i) *The map ϕ is injective. If moreover V is finite dimensional, then ϕ is an isomorphism.*

ii) *The map ϕ' is an isomorphism.*

iii) *The map ρ is injective. If moreover N is finite dimensional, then ρ is an isomorphism. ρ is commutative.*

Corollary 2.3.2 *For any $\Bbbk$-vector spaces $M_1, \dots, M_n$ the map*

$$\theta : M_1^* \otimes \dots \otimes M_n^* \longrightarrow (M_1 \otimes \dots \otimes M_n)^*$$

defined by

$$\theta\,(f_1 \otimes \dots \otimes f_n)\,(m_1 \otimes \dots \otimes m_n) = f_1\,(m_1) \dots f_n\,(m_n)$$

is injective. Moreover, if all spaces M_i are finite dimensional, then θ is an isomorphism.

Proof. The assertion follows immediatly by induction from asertion iii) of the Lemma 2.3.1. ■

If V, W are $\mathbb{k}$-vector spaces and $v : V \longrightarrow W$ is a $\mathbb{k}$-linear map, we will denote by $v^* : W^* \longrightarrow V^*$ the map define by

$$v^*(f^*) = f^* v \text{ } for \text{ } any \text{ } f^* \in W^*. \tag{2.11}$$

We made all the necessary preparations for constructing the dual Hom-associative algebra of a Hom-coassociative coalgebra.

Theorem 2.3.3 *Let $(C, \Delta, \varepsilon, \beta)$ be a counital Hom-coassociative coalgebra and C^* be the linear dual of C. We define the maps $\mu : C^* \otimes C^* \xrightarrow{\rho} (C \otimes C)^* \xrightarrow{\Delta^*} C^*$, $\mu = \Delta^* \rho$, where ρ is define as in Lamma 2.3.1, and $\eta : \mathbb{k} \xrightarrow{\phi} \mathbb{k}^* \xrightarrow{\varepsilon^*} C^*$, $\eta = \varepsilon^* \phi$ where $\phi : \mathbb{k} \longrightarrow \mathbb{k}^*$ is the canonical isomorphism, and $\eta(1_{\mathbb{k}}) = 1_{C^*}$ where $1_{C^*}(x) = \varepsilon(x)$, and the homomorphism $\alpha : C^* \longrightarrow C^*$, $\alpha(h) = h \circ \beta$.*

Then (C^*, μ, η, α) is an unital Hom-associative algebra.

This is checked in exactly the same way as for Hom-coassociative coalgebras, as was done in [35, Corollary 4.12].

Proof. The product $\mu = \Delta^* \rho$ is defined from $C^* \otimes C^*$ to C^* by

$$\mu(f^* \otimes g^*)(x) = \Delta^* \rho(f^* \otimes g^*)(x) = \rho(f^* \otimes g^*)(\Delta(x)) = \sum_{(x)} f^*\left(x_{(1)}\right) g^*\left(x_{(2)}\right) \tag{2.12}$$

for all $x \in C$ and $f^*, g^* \in C^*$

From this it follows that for $f^*, g^*, h^* \in C^*$, we have

$$\begin{aligned}
\mu(\mu(f^* \otimes g^*) \otimes \alpha(h^*)) &= \rho(\mu(f^* \otimes g^*) \otimes \alpha(h^*)) \Delta \\
&= \rho(\rho(f^* \otimes g^*) \Delta \otimes h^* \circ \beta) \Delta \\
&= \rho(\rho(f^* \otimes g^*) \otimes h^*)(\Delta \otimes \beta) \Delta \\
&\overset{(2.7)}{=} \rho(f^*(\beta) \otimes \rho(g^* \otimes h^*) \Delta) \Delta \\
&= \mu(\alpha(f^*) \otimes \mu(g^* \otimes h^*))
\end{aligned}$$

So the Hom-associativity $\mu \circ (\mu \otimes \alpha) = \mu \circ (\alpha \otimes \mu)$ follows from the Hom-coassociativity $(\Delta \otimes \beta) \circ \Delta = (\beta \otimes \Delta) \circ \Delta$.

Moreover, if C has a counit ε satisfying $(id_C \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes id_C) \circ \Delta = \beta$.

The second condition from the definition of a Hom-associative algebra is equivalent to the fact that η is an unital for the multiplication defined by μ,

We remark now that for $f^* \in C^*$ we have

$$\begin{aligned}
\mu\left(\eta \otimes id_{C^*}\right)\left(f^*\right) = \Delta^* \rho\left(\eta\left(1_{\Bbbk}\right) \otimes f^*\right) &\overset{(2.12)}{=} \rho\left(\eta\left(1_{\Bbbk}\right) \otimes f^*\right) \Delta \\
&= \rho\left(\varepsilon \otimes f^*\right) \Delta = \rho\left(id_{C^*} \otimes f^*\right)\left(\varepsilon \otimes id_C\right) \Delta \\
&\overset{(2.8)}{=} \rho\left(id_{C^*} \otimes f^*\right) \circ \beta \\
&= f^* \circ \beta = \alpha\left(f^*\right).
\end{aligned}$$

and

$$\begin{aligned}
\mu\left(id_{C^*} \otimes \eta\right)\left(f^*\right) &= \Delta^* \rho\left(f^* \otimes \eta\left(1_{\Bbbk}\right)\right) \\
&\overset{(2.12)}{=} \rho\left(f^* \otimes \eta\left(1_{\Bbbk}\right)\right) \Delta \\
&= \rho\left(f^* \otimes \varepsilon\right) \Delta \\
&= \rho\left(f^* \otimes id_{C^*}\right)\left(id_C \otimes \varepsilon\right) \Delta \\
&\overset{(2.8)}{=} \rho\left(f^* \otimes id_{C^*}\right) \circ \beta \\
&\overset{(2.12)}{=} f^* \circ \beta = \alpha\left(f^*\right).
\end{aligned}$$

which shows that η is the unit in C^*.

Conversely, does a unital Hom-associative algebra (A, μ, α) lead to a counital Hom-coassociative coalgebra on A^*? It turns out that it is not possible to perform a construction similar to the one of the dual unital Hom-associative algebra, due to the inexistence of a canonical morphism $(A \otimes A)^* \longrightarrow A^* \otimes A^*$. However, if A is finite-dimensional, the canonical morphism $\rho : (A \otimes A)^* \longrightarrow A^* \otimes A^*$ is bijective. ■

Theorem 2.3.4 *[35, Corollary 4.12] Let* (A, μ, η, α) *is a finite dimensionel unital Hom-associative algebra, and* A^* *be the linear dual of* A*. We define the comultiplication by the* *composition*

$$\Delta : A^* \xrightarrow{\mu^*} (A \otimes A)^* \xrightarrow{\rho^{-1}} A^* \otimes A^* \text{ by } \Delta = \rho^{-1}\mu^*,$$

and

$$\varepsilon : A^* \xrightarrow{\eta^*} \Bbbk^* \xrightarrow{\psi} \Bbbk,\ \ \varepsilon = \psi\eta^*$$

where ρ is define as in Lamma 2.3.1, and ψ is the canonical isomorphism, $\varepsilon(f) = f(1_A)$ for $f \in A^*$, where $1_A = \eta(1_\Bbbk)$ and the homomorphism

$$\beta : A^* \longrightarrow A^*,\ \ \beta(h) = h \circ \alpha.$$

Then $(A^*, \Delta, \varepsilon, \beta)$ is a counital Hom-coassociative coalgebra.

Such a construction could be extended to a so called finite dual.

Proof. First we establish the Hom-coassociativity of the comultiplication, which follows from Hom-associativity of the multiplication.

We can define $\Delta(f^*) = \sum_{(f)} f^*_{(1)} \otimes f^*_{(2)}$, with the property that

$$f^* \circ \mu(x \otimes y) = \sum_{(f)} f^*_{(1)}(x) \otimes f^*_{(2)}(y)$$

The comultiplication is defined by

$$\Delta(f^*)(x \otimes y) = \rho^{-1} \circ \mu^*(f^*)(x \otimes y) = f^* \circ \mu(x \otimes y) \quad x, y \in A$$

$$\Delta(f^*) = f^* \circ \mu$$

For $f\ ^* \in A^*$ we have

$$\begin{aligned}
(\Delta \otimes \beta) \circ \Delta(f^*) &\overset{(1.5)}{=} \sum_{(f)} (\Delta \otimes \beta) \circ \left(f^*_{(1)} \otimes f^*_{(2)}\right) \\
&= \sum_{(f)} \Delta\left(f^*_{(1)}\right) \otimes \beta\left(f^*_{(2)}\right) \\
&= \sum_{(f)} f^*_{(1)} \circ \mu \otimes f^*_{(2)} \circ \alpha = \sum_{(f)} \left(f^*_{(1)} \otimes f^*_{(2)}\right) \circ (\mu \otimes \alpha) \\
&= (f^* \circ \mu) \circ (\mu \otimes \alpha) \overset{(\mathbf{??})}{=} f^* \circ \mu \circ (\alpha \otimes \mu) = \Delta(f^*)(\alpha \otimes \mu) \\
&= \sum_{(f)} f^*_{(1)} \circ \alpha \otimes f^*_{(2)} \circ \mu = \sum_{(f)} \beta\left(f^*_{(1)}\right) \otimes \Delta\left(f^*_{(2)}\right) \\
&= (\beta \otimes \Delta) \circ \Delta(f^*).
\end{aligned}$$

which shows that Δ is Hom-coassociative.

Moreover, if A has an unit η satisfying $\mu \circ (id_A \otimes \eta) = \mu \circ (\eta \otimes id_A) = \alpha$ then for f $^* \in A^*$ we have

$$\begin{aligned}(\varepsilon \otimes id_{A^*}) \circ \Delta (f^*) &= \sum_{(f)} \left(\varepsilon \left(f^*_{(1)} \right) \otimes f^*_{(2)} \right) = \sum_{(f)} \left(f^*_{(1)} \circ \eta (1_{\Bbbk}) \otimes f^*_{(2)} \right) \\ &= \sum_{(f)} \left(f^*_{(1)} \otimes f^*_{(2)} \right) (\eta \otimes id_A) = f^* \circ \mu \circ (\eta \otimes id_A) \\ &\overset{(2.2)}{=} f^* \circ \alpha = \beta (f)\end{aligned}$$

and so $(\varepsilon \otimes id_{A^*}) \circ \Delta = \beta$. Similarly, $(id_{A^*} \otimes \varepsilon) \circ \Delta = \beta$, showing that is in fact the counit of A^*. Therefore $(A^*, \Delta, \varepsilon, \beta)$ is a counital Hom-coassociative coalgebra. ■

Proposition 2.3.5 *Let $(C, \Delta, \varepsilon, \beta)$ and $(D, \Delta', \varepsilon', \beta')$ be counital Hom-coassociative coalgebras, and let (A, μ, η, α) and $(B, \mu', \eta', \alpha')$ be finite-dimensional unital Hom-associative algebras.*

1) *If $f : C \longrightarrow D$ is a Hom-coassociative coalgebra morphism, then $f^* : D^* \longrightarrow C^*$ is a Hom-associative algebra morphism.*

2) *If $f : A \longrightarrow B$ is a Hom-associative algebra morphism, then $f^* : B^* \longrightarrow A^*$ is a Hom-coassociative coalgebra morphism.*

Proof. 1) We verify that f^* is the Hom-associative algebra morphism .

Let $d^*, e^* \in D^*$ and $c \in C$. where μ_{D^*}, η_{D^*}, and α_{D^*} are define as in Theorem 2.3.3. We have

$$\begin{aligned}f^* \circ \mu_{D^*} (d^* \otimes e^*) (c) &= \mu_{D^*} (d^* \otimes e^*) (f (c)) \\ &\overset{(2.12)}{=} \rho (d^* \otimes e^*) \circ \Delta_D \circ f (c) \\ &\overset{(2.9)}{=} \rho (d^* \otimes e^*) \circ (f \otimes f) \circ \Delta_C (c) \ \ (f \text{ is a Hom-coassociative coalgebra morphism}) \\ &= \rho (d^* (f) \otimes e^* (f)) \Delta_C (c) \\ &\overset{(2.11)}{=} \mu_{C^*} (f^* (d^*) \otimes f^* (e^*)) (c) \\ &= \mu_{C^*} ((f^* \otimes f^*) (d^* \otimes e^*)) (c)\end{aligned}$$

Furthermore

$$f^* \circ \eta_{D^*}(1_{\Bbbk}) = f^*(\varepsilon_D) = \varepsilon_D(f) = \varepsilon_C = \eta_{C^*}(1_{\Bbbk}),$$

For $d^* \in D^*$ we have

$$\begin{aligned} f^* \circ \alpha_{D^*}(d^*) &= \alpha_{D^*}(d^*)(f) = d^* \circ \beta_D(f) \\ &= d^* \circ f \circ \beta_C \text{ (f is a Hom coalgebra morphism)} \\ &= f^*(d^*) \circ \beta_C = \alpha_{C^*} \circ f^*(d^*) \end{aligned}$$

Then f^* is a Hom-associative algebra morphism.

2) We have to show that the following diagram is commutative

$$\begin{array}{ccc} B^* & \xrightarrow{f^*} & A^* \\ \Delta_{B^*} \downarrow & & \downarrow \Delta_{A^*} \\ B^* \otimes B^* & \xrightarrow{f^* \otimes f^*} & A^* \otimes A^* \end{array}$$

Let $b^* \in B^*$, where Δ_{A^*}, ε_{A^*}, and β_{A^*} are defined as in Theorem 2.3.4. We have

$$\begin{aligned} (\Delta_{A^*} \circ f^*)(b^*) &= \Delta_{A^*} \circ (b^*(f)) \\ &= b^* \circ f \circ \mu_A = b^* \circ \mu_B(f \otimes f) \text{ (f is a Hom-associtive algebra morphism)} \\ &= (f^* \otimes f^*)(b^* \circ \mu_B) = (f^* \otimes f^*)\Delta_{B^*}(b^*) \end{aligned}$$

whitch proves the commutativity of the diagram. Also

$$\begin{aligned} (\varepsilon_{A^*} \circ f^*)(b^*) &= \varepsilon_{A^*}(b^*(f)) = b^*(f)(1_A) = b^* \circ f(\eta_A(1_{\Bbbk})) \\ &= b^* \circ (\eta_B(1_{\Bbbk})) = b^*(1_B) = \varepsilon_{B^*}(b^*), \end{aligned}$$

and

$$\begin{aligned} f^* \circ \beta_{B^*}(b^*) &= \beta_{B^*}(b^*)(f) = b^* \circ \alpha_B(f) \\ &= b^* \circ f \circ \alpha_A \text{ (f is a Hom-coassocitive coalgebra morphism)} \\ &= f^*(b^*) \circ \alpha_A = \beta_{A^*} \circ f^*(b^*) \end{aligned}$$

so f^* is a Hom-coassocitive coalgebra morphism. ■

Proposition 2.3.6 *Let* (V,μ,μ,α) , $(V,\Delta,\varepsilon,\alpha)$ *be respectively unital Hom-associative algebra and counital Hom-coassocitive coalgebra. The following statements are equivalent*

1. *The maps* μ *and* η *are morphisms of counital Hom-coassocitive coalgebras.*

2. *The maps* Δ *and* ε *are morphisms of unital Hom-associative algebras.*

Proof. Let μ be a morphism of the Hom-coassociative coalgebra

$$\mu : (V\otimes V,\tilde{\Delta},\tilde{\varepsilon},\tilde{\beta}) \longrightarrow (V,\Delta,\varepsilon,\alpha)$$

such that: $\tilde{\Delta}=\tau_{2,3}\circ\Delta\otimes\Delta$, $\tilde{\varepsilon}=\varepsilon\otimes\varepsilon$, $\tilde{\beta}=\alpha\otimes\alpha$ then the morphism μ satisfies the axiom (2.9)

$$(\mu\otimes\mu)\circ\tau_{2,3}\circ\Delta\otimes\Delta=\Delta\circ\mu,\ \textit{and}\ \mu\circ(\alpha\otimes\alpha)=\alpha\circ\mu. \tag{2.13}$$

And η is a morphism of the Hom-coassociative coalgebra

$$\eta : \left(\mathbb{k}, id_{\mathbb{k},\mathbb{k}^{\otimes 2}}, id_{\mathbb{k},\mathbb{k}}, id_{\mathbb{k},\mathbb{k}}\right) \longrightarrow (V,\Delta,\varepsilon,\alpha)$$

then :

$$(\eta\otimes\eta)\circ id_{\mathbb{k},\mathbb{k}^{\otimes 2}}=\Delta\circ\eta \tag{2.14}$$

$$\eta\circ id_{\mathbb{k},\mathbb{k}}=\alpha\circ\eta \tag{2.15}$$

by relations (2.13), (2.14) and α is a homomorphism, the comultiplication

$$\Delta : (V,\mu,\eta,\alpha) \longrightarrow (V\otimes V,\tilde{\mu},\tilde{\eta},\tilde{\alpha})$$

is a week morphism of Hom-associative algebra, such that

$$\tilde{\mu}=(\mu\otimes\mu)\circ\tau_{2,3},\quad \tilde{\eta}=\eta\otimes\eta\ \textit{and}\ \tilde{\alpha}=\alpha\otimes\alpha.$$

By the relations (2.13), (2.15) and α is a homomorphism, the counit

$$\varepsilon : (V,\mu,\eta,\alpha) \longrightarrow \left(\mathbb{k}, id_{\mathbb{k}^{\otimes 2},\mathbb{k}}, id_{\mathbb{k},\mathbb{k}}, id_{\mathbb{k},\mathbb{k}}\right)$$

$$\varepsilon\circ\mu=id_{\mathbb{k}^{\otimes 2},\mathbb{k}}\circ(\varepsilon\otimes\varepsilon)$$

is a morphism of Hom-associative algebra. ■

Lemma 2.3.7 *Let $(C, \Delta, \varepsilon, \beta)$ be a counital Hom-coassociative coalgebra and $f : C \longrightarrow C$ be a linear map which commutes with β satisfies $\beta \circ f = f \circ \beta$. Then*

1. $(\beta \otimes (id_C \otimes f) \circ \Delta) \circ \Delta = (\Delta \otimes (\beta \circ f)) \circ \Delta$.

2. $(\beta \otimes (f \otimes id_C) \circ \Delta) \circ \Delta = (((id_C \otimes f) \circ \Delta) \otimes \beta) \circ \Delta$.

3. $(((f \otimes id_C) \circ \Delta) \otimes \beta) \circ \Delta = ((\beta \circ f) \otimes \Delta) \circ \Delta$.

Proof. The proof in checking axiom of Hom-coassociative, we compute as follows

1. The first equality

$$\begin{aligned}
(\beta \otimes (id_C \otimes f) \circ \Delta) \circ \Delta &= (id_C \otimes (id_C \otimes f) \circ (\beta \otimes \Delta) \circ \Delta \\
&\overset{(2.7)}{=} ((id_C \otimes id_C) \otimes f) \circ (\Delta \otimes \beta) \circ \Delta \\
&= ((id_C \otimes id_C) \circ \Delta \otimes f \circ \beta) \circ \Delta \\
&= (\Delta \otimes (\beta \circ f)) \circ \Delta.
\end{aligned}$$

2. The second equality

$$\begin{aligned}
(\beta \otimes (f \otimes id_C) \circ \Delta) \circ \Delta &= (id_C \otimes (f \otimes id_C)\,(\beta \otimes \Delta) \circ \Delta \\
&\overset{(2.7)}{=} ((id_C \otimes f) \otimes id_C)(\Delta \otimes \beta) \circ \Delta \\
&= ((id_C \otimes f) \circ \Delta \otimes \beta) \circ \Delta.
\end{aligned}$$

3. The third equality

$$\begin{aligned}
((f \otimes id_C) \circ \Delta \otimes \beta) \circ \Delta &= ((f \otimes id_C) \otimes id_C)(\Delta \otimes \beta) \circ \Delta \\
&\overset{(2.7)}{=} (f \otimes (id_C \otimes id_C))(\beta \otimes \Delta) \circ \Delta \\
&= (f \circ \beta \otimes (id_C \otimes id_C) \circ \Delta) \circ \Delta \\
&= (f \circ \beta \otimes \Delta) \circ \Delta \\
&= (\beta \circ f \otimes \Delta) \circ \Delta.
\end{aligned}$$

This finishes the proof. ■

Remark 2.3.8 *This Lemma can be used in the proof of Proposition 4.2.1.*

Lemma 2.3.9 *Let $(C, \Delta, \varepsilon, \beta))$ be a counital Hom-coassociative coalgebra, and f de a linear map $f : C \to C^{\otimes m}$ satisfying $f \circ \beta = \beta^{\otimes m} \circ f$ then*

1. $((\Delta \otimes \beta^{\otimes m})(\Delta \otimes \beta^{\otimes(m-1)}) \circ f = \Big((\beta \otimes \Delta) \circ \Delta \otimes (\beta \circ \beta)^{\otimes(m-1)}\Big) \circ f.$
2. $\big(\beta^n \otimes \big(\beta^{n-1} \otimes f\big) \circ \Delta\big) \circ \Delta = (\Delta \otimes \beta^{\otimes m}) \big(\beta^{n-1} \otimes f\big) \circ \Delta.$
3. $\big(\big(\big(f \otimes \beta^{n-1}\big) \circ \Delta\big) \otimes \beta^n\big) \circ \Delta = \big((\beta^{\otimes m} \otimes \Delta) \big(f \otimes \beta^{n-1}\big)\big) \circ \Delta.$

Proof. This proof is completely analogous to that of Lemma 2.3.7.

1. The first equality

$$\begin{aligned}
((\Delta \otimes \beta^{\otimes m})(\Delta \otimes \beta^{\otimes(m-1)}) \circ f &= \Big(\Delta \otimes \beta \otimes \beta^{\otimes(m-1)}\Big) (\Delta \otimes \beta^{\otimes(m-1)}) \circ f \\
&\overset{(2.7)}{=} \Big((\Delta \otimes \beta) \, \Delta \otimes (\beta \circ \beta)^{\otimes(m-1)}\Big) \circ f \\
&= \Big((\beta \otimes \Delta) \, \Delta \otimes (\beta \circ \beta)^{\otimes(m-1)}\Big) \circ f.
\end{aligned}$$

2. The second equality

$$\begin{aligned}
\big(\beta^n \otimes \big(\beta^{n-1} \otimes f\big) \circ \Delta\big) \circ \Delta &= \big(\big(\beta^{n-1} \circ \beta\big) \otimes \big(\beta^{n-1} \otimes f\big) \circ \Delta\big) \circ \Delta \\
&\overset{(2.7)}{=} \big(\beta^{n-1} \otimes (\beta^{n-1} \otimes f)\big) \circ \big(\beta \otimes \Delta\big) \circ \Delta \\
&= \big(\beta^{n-1} \otimes (\beta^{n-1} \otimes f)\big) \circ \big(\Delta \otimes \beta\big) \circ \Delta \\
&= ((\beta^{n-1} \otimes \beta^{n-1}) \circ \Delta \otimes f \circ \beta) \circ \Delta \\
&= (\Delta \circ \beta^{n-1} \otimes \beta^{\otimes m} \circ f) \circ \Delta \\
&= (\Delta \otimes \beta^{\otimes m}) \circ (\beta^{n-1} \otimes f) \circ \Delta.
\end{aligned}$$

3. The third equality

$$
\begin{aligned}
\left(\left(\left(f\otimes\beta^{n-1}\right)\circ\Delta\right)\otimes\beta^{n}\right)\circ\Delta &= \left(\left(\left(f\otimes\beta^{n-1}\right)\circ\Delta\right)\otimes\beta^{n-1}\left(\beta\right)\right)\circ\Delta\\
&\overset{(2.7)}{=} \left(\left(f\otimes\beta^{n-1}\right)\otimes\beta^{n-1}\right)\left(\Delta\otimes\beta\right)\circ\Delta\\
&= \left(f\otimes\left(\beta^{n-1}\otimes\beta^{n-1}\right)\right)\left(\beta\otimes\Delta\right)\circ\Delta\\
&= \left(f\left(\beta\right)\otimes\left(\beta^{n-1}\right)^{\otimes 2}\Delta\right)\circ\Delta\\
&= \left(\beta^{\otimes m}\left(f\right)\otimes\Delta\left(\beta^{n-1}\right)\right)\circ\Delta\\
&= \left(\left(\beta^{\otimes m}\otimes\Delta\right)\left(f\otimes\beta^{n-1}\right)\right)\circ\Delta.
\end{aligned}
$$

This finishes the proof. ∎

This Lemma can be used in the proof of Propositions 4.2.1 and 4.2.3.

Remark 2.3.10 *The following result is an immediate consequence of Lemma 2.3.9*

$$\left(\left(\left(\beta^{n-1}\otimes f\right)\circ\Delta\right)\otimes\beta^{n}\right)\circ\Delta = \left(\left(\beta^{n}\otimes\left(f\otimes\beta^{n-1}\right)\right)\circ\Delta\right)\circ\Delta.$$

2.4 Hom-Hopf algebras

In this section, we introduce a generalization of Hopf algebras and show some relevant properties of the new structure.

2.4.1 Hom-bialgebras

A Hom-bialgebra is a vector space which is an unital Hom-associative algebra and a counital Hom-coassociative coalgebra such that the conditions in Proposition 2.3.6 hold,[34, 35], see also [51].

Definition 2.4.1 *A **Hom-bialgebra** is a tuple $B=(B,\mu,\eta,\alpha,\Delta,\varepsilon,\beta)$ in which (B,μ,η,α) is an unital Hom-associative algebra, $(B,\Delta,\varepsilon,\beta)$ is a counital Hom-coassociative coalgebra and the linear maps Δ and ε are morphisms of Hom-associative algebras, that is*

$$\Delta\circ\mu=\mu^{\otimes 2}\circ\tau_{2,3}\circ\Delta^{\otimes 2} \text{ and } \varepsilon\otimes\varepsilon=\varepsilon\circ\mu. \tag{2.16}$$

Remark 2.4.2 *1. ([34]) In terms of elements, condition (2.16) could be expressed by the following identities :*

$$\begin{cases} \Delta(1_B) = 1_B \otimes 1_B,\ \alpha(1_B) = 1_B,\ \text{and } \beta(1_B) = 1_B,\ \text{where } 1_B = \eta(1_{\Bbbk}) \\ \Delta(\mu(x \otimes y)) = \Delta(x) \cdot \Delta(y) = \sum_{(x)(y)} \mu\left(x_{(1)} \otimes y_{(1)}\right) \otimes \mu\left(x_{(2)} \otimes y_{(2)}\right) \\ \qquad \varepsilon(1_B) = 1_{\Bbbk},\ \varepsilon(\mu(x \otimes y)) = \varepsilon(x)\varepsilon(y),\ \text{and } \varepsilon \circ \alpha(x) = \varepsilon(x). \end{cases}$$

where using the Sweedler's notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ *and the dot "·" denotes the multiplication on tensor product.*

2. *If* $\alpha = \beta$ *the Hom-bialgebra is denoted* $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$.

3. *Observe that a Hom-bialgebra is neither associative nor coassociative, unless of course* $\alpha = \beta = id_V$, *in which case we have a bialgebra.*

Definition 2.4.3 *A* ***morphism of Hom-bialgebra*** *(resp. weak morphism of Hom-bialgebra) which is either a morphisms (resp. weak morphism) of Hom-associative algebra and Hom-coassociative coalgebra.*

Example 2.4.4 *Let* G *be a group and* $\Bbbk G$ *the corresponding group algebra over* $\Bbbk$. *As a vector space,* $\Bbbk G$ *is generated by* $\{e_g : g \in G\}$. *If* $\alpha : G \longrightarrow G$ *is a group homomorphism, then it can be extended to an algebra endomorphism of* $\Bbbk G$ *by setting*

$$\alpha\left(\sum_{g\in G} a_g e_g\right) = \sum_{g\in G} a_g \alpha(e_g) = \sum_{g\in G} a_g e_{\alpha(g)}$$

Consider the usual bialgebra structure on $\Bbbk G$ *and a bialgebra morphism. Then, we define a Hom-bialgebra* $(\Bbbk G, \mu, \Delta, \alpha)$ *over* $\Bbbk G$ *by setting:*

$$\mu\left(e_g \otimes e_{g'}\right) = \alpha\left(e_{gg'}\right),\ \Delta(e_g) = \alpha(e_g) \otimes \alpha(e_g).$$

Example 2.4.5 *Let* X *be a set and consider the set of non-commutative polynomials* $A = \Bbbk\langle X_i \rangle$. *It carries a bialgebra structure with a comultiplication defined for* $x \in X$ *by*

$$\Delta(x) = 1_A \otimes x + x \otimes 1_A \text{ and } \Delta(1) = 1_A \otimes 1_A.$$

Let α be a bialgebra morphism. We define a Hom-bialgebra $(A, \mu', \Delta', \alpha')$ by

$$\mu'(f \otimes g) = f(\alpha(X))\, g(\alpha(X)), \Delta'(x) = \alpha(1_A) \otimes \alpha(x) + \alpha(x) \otimes \alpha(1_A)$$

$$\text{and } \Delta(1) = \alpha(1_A) \otimes \alpha(1_A).$$

Combining Propositions 2.1.8 and 2.2.10, we obtain the following proposition:

Proposition 2.4.6 *Let $B = (B, \mu, \eta, \Delta, \varepsilon, \alpha)$be a Hom-bialgebra and $\beta : B \longrightarrow B$ be a Hom-bialgebra morphism, then $B_\beta = (B, \mu_\beta, \eta_\beta, \Delta_\beta, \varepsilon_\beta, \alpha_\beta)$ is a Hom-bialgebra.*

Hence, $(B, \mu_{\beta^n}, \eta_{\beta^n}, \Delta_{\beta^n}, \varepsilon_{\beta^n}, \alpha_{\beta^n})$ is a Hom-bialgebra.

Proof. According to Propositions 2.1.8 and 2.2.10, $(B, \mu_\beta, \eta_\beta, \alpha_\beta)$ is a unital Hom-associative algebra, and $(B, \Delta_\beta, \varepsilon_\beta, \alpha_\beta)$ is a counital Hom-coassociative coalgebra. It remains to establish condition (2.16) in B_β. Using $\mu_\beta = \beta \circ \mu = \mu \circ \beta^{\otimes 2}$, $\Delta_\beta = \Delta \circ \beta = \Delta \circ \beta^{\otimes 2}$, $\tau_{B\otimes B} \circ \beta^{\otimes 2} = \beta^{\otimes 2} \circ \tau_{B\otimes B}$, and the condition (2.16) in the Hom-bialgebra B, we compute as follows:

$$\begin{aligned}
\Delta_\beta \circ \mu_\beta &= \Delta \circ \beta \circ \beta \circ \mu = \beta^{\otimes 2} \circ \beta^{\otimes 2} \circ \Delta \circ \mu \overset{(2.16)}{=} \beta^{\otimes 2} \circ \beta^{\otimes 2} \circ \mu^{\otimes 2} \circ \tau_{2,3} \circ \Delta^{\otimes 2} \\
&= \beta^{\otimes 2} \circ (\beta \circ \mu)^{\otimes 2} \circ \tau_{2,3} \circ \Delta^{\otimes 2} = \left(\beta \circ \mu \circ \beta^{\otimes 2}\right)^{\otimes 2} \circ \tau_{2,3} \circ \Delta^{\otimes 2} \\
&= \mu_\beta^{\otimes 2} \circ \beta^{\otimes 4} \circ \tau_{2,3} \circ \Delta^{\otimes 2} = \mu_\beta^{\otimes 2} \circ \tau_{2,3} \circ \beta^{\otimes 4} \circ \Delta^{\otimes 2} \\
&= \mu_\beta^{\otimes 2} \circ \tau_{2,3} \circ \Delta_\beta^{\otimes 2}.
\end{aligned}$$

We have shown that B_β is a Hom-bialgebra. ■

This construction method of Hom-bialgebra, starting with a given Hom-bialgebra or a bialgebra and a morphism, is called twisting principle.

Notice that the category of Hom-bialgebra is not closed under weak Hom-bialgebra morphisms.

Example 2.4.7 ***(Hom-Type Taft-Sweedler bialgebra)*** *We consider T_2, the 4-dimensional unital Taft-Sweedler algebra generated by g, x and the relations $(g^2 = 1,\ x^2 = 0,\ xg = -gx)$. The comultiplication is defined by $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$, the counit is given by $\varepsilon(g) = 1,\ \varepsilon(x) = 0$. Set $\{e_1 = 1,\ e_2 = g,\ e_3 = x,\ e_4 = gx\}$ be a basis.*

Pick any $\lambda \in \Bbbk$, the map $\alpha : T_2 \to T_2$ defined by $\alpha(e_1) = e_1,\ \alpha(e_2) = e_2,\ \alpha(e_3) = \lambda e_3$, $\alpha(e_4) = \lambda e_4$ is a bialgebra morphism. Therefore, we obtain a Hom-bialgebra $(T_2)_\lambda$ which is defined by the following table that describes multiplying the ith row elements by the jth column elements.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	λe_3	λe_4
e_2	e_2	e_1	λe_4	λe_3
e_3	λe_3	$-\lambda e_4$	0	0
e_4	λe_4	$-\lambda e_3$	0	0

and

$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = e_2 \otimes e_2,\ \Delta(e_3) = \lambda(e_3 \otimes e_1 + e_2 \otimes e_3),$$
$$\Delta(e_4) = \lambda(e_4 \otimes e_2 + e_1 \otimes e_4).$$

$$\varepsilon(e_1) = \varepsilon(e_2) = 1, \quad \varepsilon(e_3) = \varepsilon(e_4) = 0.$$

Example 2.4.8 *1. A unital Hom-associative algebra (A, μ, η, α) becomes a Hom-bialgebra when equipped with the trivial comultiplication $\Delta = 0$. Likewise, a counital Hom-coassociative coalgebra $(C, \Delta, \varepsilon, \beta)$ becomes a Hom-bialgebra when equipped with the trivial multiplication $\mu = 0$.*

2. Let $(B, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra. Then so are $(B, -\mu, \eta, -\Delta, \varepsilon, \alpha)$, and $(B, \mu^{op}, \eta, \Delta^{cop}, \varepsilon, \alpha)$ where $\mu^{op} = \mu \circ \tau_{B\otimes B}$ and $\Delta^{cop} = \tau_{B\otimes B} \circ \Delta$.

Proposition 2.4.9 *Let $B = (B, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a finite dimensional Hom-bialgebra. Then $B^* = (B^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*, \alpha^*)$ is a Hom-bialgebra, together with the Hom-associative algebra structure which is dual to the Hom-coassociative coalgebra structure of B, and with the Hom-coassociative coalgebra structure which is a dual to the Hom-associative algebra structure of B, is a Hom-bialgebra called dual Hom-bialgebra of B.*

Proposition 2.4.10 *If B is a finite dimensional Hom-bialgebra, then B is cocommutative if and only if B^* is commutative, and B is commutative if and only if B^* is cocommutative.*

2.4.2 Hom-Hopf algebras

A Hom-bialgebra $(B, \mu, \eta, \Delta, \varepsilon, \alpha)$ is a Hom-Hopf algebra if and only if the identity map of B is invertible in the Hom-algebra $Hom(B, B)$. Moreover, the preceding characterization implies that an antipode of a Hom-Hopf algebra is uniquely determined. To define this antipode, we will need the notion of a convolution product.

Proposition 2.4.11 *Let (A, μ, η, α) be a unital Hom-associative algebra and $(C, \Delta, \varepsilon, \beta)$ be a counital Hom-coassociative coalgebra. Then, the vector space $Hom(C, A)$ of $\Bbbk$-linear mappings of C to A equipped with the* ***convolution product*** *defined by*

$$(f * g)(x) = \mu \circ (f \otimes g) \circ \Delta(x) \quad x \in C$$

and the unit being $\eta \circ \varepsilon$ is a unital Hom-associative algebra with the homomorphism map defined by $\gamma(f) = \alpha \circ f \circ \beta$.

Proof. By the Hom-associativity of the multiplication in A and the Hom-coassociativity of the comultiplication in C, we obtain

$$\begin{aligned}
((f * g) * \gamma(h)) &= \mu \circ ((f * g) \otimes \gamma(h)) \circ \Delta \\
&= \mu \circ ((\mu \circ (f \otimes g) \circ \Delta) \otimes (\alpha \circ h \circ \beta)) \circ \Delta \\
&= \mu \circ (\mu \otimes \alpha)(f \otimes g \otimes h)(\Delta \otimes \beta) \circ \Delta \\
&= \mu \circ (\alpha \otimes \mu)(f \otimes g \otimes h)(\beta \otimes \Delta) \circ \Delta \\
&= \mu \circ (\alpha \circ f \circ \beta \otimes \mu \circ (g \otimes h) \circ \Delta) \circ \Delta \\
&= (\gamma(f) * (g * h))
\end{aligned}$$

that is, the convolution product is Hom-associative. The relation

$$\begin{aligned}
(\eta \circ \varepsilon) * f &= \mu \circ (\eta \circ \varepsilon \otimes f) \circ \Delta \\
&= \mu \circ (\eta \otimes id_V) \circ f \circ (\varepsilon \otimes id_V) \circ \Delta \\
&\overset{(2.2),(2.8)}{=} \alpha \circ f \circ \beta = \gamma(f)
\end{aligned}$$

shows that $\eta \circ \varepsilon$ is a left unit. It is similarly proved that $\eta \circ \varepsilon$ is a right unit.

The homomorphism γ satisfies

$$\begin{aligned}(\gamma\circ f)*(\gamma\circ g) &= \mu\circ(\alpha\circ f\circ\beta\otimes\alpha\circ g\circ\beta)\circ\Delta\\ &= \mu\circ(\alpha\otimes\alpha)\circ(f\otimes g)\circ(\beta\otimes\beta)\circ\Delta\\ &= \alpha\circ\mu\circ(f\otimes g)\circ\Delta\circ\beta\\ &= \alpha\circ(f*g)\circ\beta\\ &= \gamma\circ(f*g)\,.\end{aligned}$$

Then $(Hom\,(C,A)\,,*,\gamma)$ is a Hom-associative algebra. ■

Corollary 2.4.12 *Let $(C,\Delta,\varepsilon,\beta)$ be a counital Hom-coassociative coalgebra and (A,μ,η,α) a unital Hom-associative algebra. Then the vector spaces $Hom(C\otimes C,A)$ and $Hom(C,A\otimes A)$ are unital Hom-associative algebras.*

Proof. Consider $C\otimes C$ with the tensor product of Hom-coassociative coalgebras structure (resp. $A\otimes A$ with the tensor product of Hom-associative algebras structure) and A with the Hom-associative algebra structure (resp. C with the Hom-coassociative coalgebra structure). Then it makes sense to speak about the unital Hom-associative algebra $Hom(C\otimes C,A)$ (resp. unital Hom-associative algebra $Hom(C,A\otimes A)$), with the multiplication given by convolution, defined by

$$(f*g)\,(x\otimes y)=\mu\circ(f\otimes g)\circ\tilde{\Delta}\,(x\otimes y)\ \text{ where }\tilde{\Delta}=(id_C\otimes\tau_{C\otimes C}\otimes id_C)\circ\Delta\otimes\Delta,$$

$$(\text{resp.}\,(f*g)\,(x)=\tilde{\mu}\circ(f\otimes g)\circ\Delta\,(x)\ \text{ where }\tilde{\mu}=(\mu\otimes\mu)\,(id_A\otimes\tau_{A\otimes A}\otimes id_A)\,.$$

The identity element of the Hom-associative algebra $Hom(C\otimes C,A)$ is $\eta\circ(\varepsilon\otimes\varepsilon):C\otimes C\longrightarrow A$ (resp. the unit of Hom-associative algebra $Hom(C,A\otimes A)$ is $(\eta\otimes\eta)\circ\varepsilon:C\longrightarrow A\otimes A$). ■

Now let $(B,\mu,\eta,\Delta,\varepsilon,\alpha)$ be a Hom-bialgebra.

An endomorphism S is an **antipode** if it is the inverse of the identity over B for the Hom-algebra $Hom(B,B)$ with the multiplication given by the convolution product. The unit being $\eta\circ\varepsilon$, (recall that concatenation denotes composition of maps).

The conditions may be expressed by the identities :

$$\mu \circ (id_B \otimes S) \circ \Delta = \mu \circ (S \otimes id_B) \circ \Delta = \eta \circ \varepsilon. \tag{2.17}$$

Condition (2.17) means that S is the convolution inverse of the identity mapping, that is,

$$S * id_B = id_B * S = \eta \circ \varepsilon.$$

Definition 2.4.13 *A **Hom-Hopf algebra** is a Hom-bialgebra with an antipode. It is denoted by the tuple* $(H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$.

Remark 2.4.14 *If* $(H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$ *is a Hom-Hopf algebra, the antipode* S *satisfies*

$$\mu \circ (id_H \otimes S) \circ \Delta = \mu \circ (S \otimes id_H) \circ \Delta = \eta \circ \varepsilon$$

$$\alpha \circ \mu \circ (id_H \otimes S) \circ \Delta \circ \beta = \alpha \circ \mu \circ (S \otimes id_H) \circ \Delta \circ \beta = \alpha \circ \eta \circ \varepsilon \circ \beta$$

$$\mu_\alpha \circ (id_H \otimes S) \circ \Delta_\beta = \mu_\alpha \circ (S \otimes id_H) \circ \Delta_\beta = \eta \circ \varepsilon.$$

Then $(H, \mu_\alpha, \eta, \Delta_\beta, \varepsilon, \alpha, S)$ *is also a Hom-Hopf algebra.*

Hence, let $(H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$ be Hom-Hopf algebra and $\beta : H \longrightarrow H$ be a morphism of Hom-bialgebra, then $(H, \mu_{\beta^n}, \eta_{\beta^n}, \Delta_{\beta^n}, \varepsilon_{\beta^n}, \alpha_{\beta^n}, S)$ is a Hom-Hopf algebra.

Definition 2.4.15 *Let* H *and* H' *be two Hom-Hopf algebras. A map* $f : H \longrightarrow H'$ *is a called a **Hom-Hopf algebras morphism** if it is a Hom-bialgebras morphism.*

It is natural to ask whether a Hom-Hopf algebra morphism should preserve antipodes. The following result shows that this is indeed the case.

2.4.3 Antipode's properties.

Let $(H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$ be a Hom-Hopf algebra. For any element $x \in H$, using the counity and Sweedler notation, one may write

$$\alpha(x) = \sum_{(x)} x_{(1)} \otimes \varepsilon\left(x_{(2)}\right) = \sum_{(x)} \varepsilon\left(x_{(1)}\right) \otimes x_{(2)} \tag{2.18}$$

Then, for any $f \in End_{\Bbbk}(H)$, we have

$$f \circ \alpha(x) = \sum_{(x)} f\left(x_{(1)}\right) \varepsilon\left(x_{(2)}\right) = \sum_{(x)} \varepsilon\left(x_{(1)}\right) f\left(x_{(2)}\right) \tag{2.19}$$

The convolution product of $f, g \in End_{\Bbbk}(H)$. One may write

$$(f * g)(x) = \sum_{(x)} \mu\left(f\left(x_{(1)}\right) \otimes g\left(x_{(2)}\right)\right) \quad x \in H \tag{2.20}$$

Since the antipode S is the inverse of the identity for the convolution product, then S satisfies

$$\sum \mu\left(S\left(x_{(1)}\right) \otimes x_{(2)}\right) = \sum \mu\left(x_{(1)} \otimes S\left(x_{(2)}\right)\right) = \varepsilon(x)\eta(1_{\Bbbk}) \quad for\ any\ x \in H \tag{2.21}$$

Proposition 2.4.16 *Let $(H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$ and $(H', \mu', \eta', \Delta', \varepsilon', \alpha', S')$ be two Hom-Hopf algebras with antipodes S and S'. If $f : H \longrightarrow H'$ is a morphism of Hom bialgebra, then*

$$S' \circ f = f \circ S. \tag{2.22}$$

Proof. Consider the Hom-associative algebra $Hom(H, H')$ with the convolution product, and the elements $S' \circ f$ and $f \circ S$ from this Hom-associative algebra. We show that they are equal. Indeed

$$\begin{aligned}
\left(\left(S' \circ f\right) * f\right)(x) &= \mu' \circ \left(S' \circ f \otimes f\right) \circ \Delta(x) \\
&= \mu' \circ \left(S' \otimes id_{H'}\right)(f \otimes f) \circ \Delta(x) \\
&= \mu' \circ \left(S' \otimes id_{H'}\right) \circ \Delta' \circ f(x) \\
&= \eta' \circ \varepsilon' \circ f(x) = \varepsilon'(f(x))\eta'(1_{\Bbbk}) \\
&= \varepsilon(x)\eta'(1_{\Bbbk})
\end{aligned}$$

So $S' \circ f$ is a left inverse for f. Also

$$\begin{aligned}
(f * (f \circ S))(x) &= \mu' \circ (f \otimes f \circ S) \circ \Delta(x) \\
&= \mu' \circ (f \otimes f)(id_H \otimes S) \circ \Delta(x) \\
&= f \circ \mu \circ (id_H \otimes S) \circ \Delta(x) \\
&= f \circ \eta \circ \varepsilon(x) = \eta' \circ \varepsilon(x) \\
&= \varepsilon(x)\eta'(1_{\Bbbk}).
\end{aligned}$$

Hence $f \circ S$ is also a right inverse for f. It follows that f is (convolution) anvertible, and that the left and right inverses are equal. ■

Remark 2.4.17 *Since α is a Homomorphism of Hom-bialgebra, so*

$$S \circ \alpha = \alpha \circ S. \tag{2.23}$$

The next proposition gives some important properties of the antipode (see [6], [35]). We show that the antipode of a Hom-Hopf algebra is an anti-morphism of Hom-associative algebras and anti-morphism of Hom-coassociative coalgebras. This means that $S : H \longrightarrow H^{op}$ is a Hom-associative algebra morphism and $S : H \longrightarrow H^{cop}$ is a Hom-coassociative coalgebra morphism.

Proposition 2.4.18 *Let $(H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$ be a Hom-Hopf algebra. The antipode S is unique.*

Proof. We have $S * id_H = id_H * S = \eta \circ \varepsilon$. Thus, $(S * id_H) * S = S * (id_H * S) = S$. If S' is another antipode of H then

$$S' = S' * id_H * S' = S' * id_H * S = S * id_H * S = S.$$

Therefore the antipode when it exists is unique. ■

The next proposition gives some important properties of the antipode.

Proposition 2.4.19 *Let $(H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$ be a Hom-Hopf algebras with antipode S. Were we denote the multiplication by a point, $\mu (x \otimes y) = x \cdot y$ Then:*

i $S (x \cdot y) = S (y) \cdot S (x)$ *or* $S \circ \mu = \mu \circ (S \otimes S) \circ \tau$;

ii $S (\eta (1_{\Bbbk})) = \eta (1_{\Bbbk})$;

iii $\Delta (S (x)) = S \left(x_{(2)}\right) \otimes S \left(x_{(1)}\right)$ *or* $\Delta \circ S = (S \otimes S) \circ \tau \circ \Delta$;

iv $\varepsilon \circ S = \varepsilon$.

The multiplication is denoted by a dot for simplicity.

Proof. i) Consider $H \otimes H$ with the tensor product of Hom-coassociative coalgebra structure $\tilde{\Delta}_\alpha$, H with the Hom-associative algebra structure μ. and a Hom-associative algebra $Hom(H \otimes H, H)$ with the multiplication given by convolution, (see Corollary 2.4.12)

$$f * g(x \otimes y) = \mu \circ (f \otimes g) \circ \tilde{\Delta}_\alpha (x \otimes y)$$

where $\tilde{\Delta}_\alpha = \tilde{\Delta} \circ (\alpha \otimes \alpha)$ is defined in Definition 2.2.7.

Consider the maps $F, G : H \otimes H \longrightarrow H$ defined by

$$F(x \otimes y) = S(y) \cdot S(x), \; G(x \otimes y) = S(x \cdot y)$$

for any x, $y \in H$. We show that μ is a left inverse (with respect to convolution) for F, and a right inverse for G. Indeed, for x, $y \in H$ we have

$$\begin{aligned}
\mu * F(x \otimes y) &= \mu \circ (\mu \otimes F) \circ \tilde{\Delta}_\alpha (x \otimes y) \\
&= \sum_{(x),(y)} \mu\left((\alpha \otimes \alpha)(x \otimes y)_{(1)}\right) \cdot F\left((\alpha \otimes \alpha)(x \otimes y)_{(2)}\right) \\
&= \sum_{(x),(y)} \mu\left(\alpha\left(x_{(1)}\right) \otimes \alpha\left(y_{(1)}\right)\right) \cdot F\left(\alpha\left(x_{(2)}\right) \otimes \alpha\left(y_{(2)}\right)\right) \\
&\overset{(2.23)}{=} \sum_{(x),(y)} \left(\left(\alpha\left(x_{(1)}\right) \cdot \alpha\left(y_{(1)}\right)\right) \cdot \alpha \circ \left(S\left(y_{(2)}\right) \cdot S\left(x_{(2)}\right)\right)\right) \\
&\overset{(2.1)}{=} \sum_{(x),(y)} \alpha^2 \circ \left(x_{(1)}\right) \cdot \left(\alpha\left(y_{(1)}\right) \cdot \left(S\left(y_{(2)}\right) \cdot S\left(x_{(2)}\right)\right)\right) \\
&= \sum_{(x),(y)} \alpha^2\left(x_{(1)}\right) \cdot \left(\left(y_{(1)} \cdot S\left(y_{(2)}\right)\right) \cdot \alpha\left(S\left(x_{(2)}\right)\right)\right) \\
&= \sum_{(x)} \alpha^2 \circ \left(x_{(1)}\right) \cdot \left((\eta \circ \varepsilon(y)) \cdot \alpha\left(S\left(x_{(2)}\right)\right)\right) \\
&= \sum_{(x)} \alpha^2 \circ \left(x_{(1)}\right) \cdot \left(\eta(1_{\Bbbk}) \cdot \alpha\left(S\left(x_{(2)}\right)\right)\right) \varepsilon(y) \\
&= \sum_{(x)} \left(\alpha^2 \circ \left(x_{(1)}\right) \cdot \alpha^2 \circ \left(S\left(x_{(2)}\right)\right)\right) \varepsilon(y) \\
&= \alpha^2 \circ \eta \circ \varepsilon(x) \varepsilon(y) \\
&= \eta_H \circ \varepsilon_{H \otimes H}(x \otimes y).
\end{aligned}$$

It shows that $\mu * F = \eta_H \circ \varepsilon_{H\otimes H}$

$$\begin{aligned}
G * \mu\,(x\otimes y) &= \mu \circ (G \otimes \alpha \circ \mu) \circ \tilde{\Delta}_\alpha\,(x\otimes y) \\
&= \sum_{x\otimes y} G\left((\alpha\otimes\alpha)\,(x\otimes y)_{(1)}\right) \cdot \mu\left((\alpha\otimes\alpha)\,(x\otimes y)_{(2)}\right) \\
&= \sum_{(x),(y)} G\left(\alpha\left(x_{(1)}\right)\otimes\alpha\left(y_{(1)}\right)\right) \cdot \mu\left(\alpha\left(x_{(2)}\right)\otimes\alpha\left(y_{(2)}\right)\right) \\
&= \sum_{(x),(y)} S\left(\alpha\left(x_{(1)}\right)\cdot\alpha\left(y_{(1)}\right)\right) \cdot \left(\alpha\left(x_{(2)}\right)\cdot\alpha\left(y_{(2)}\right)\right) \\
&= \alpha \circ \left(\sum_{(x),(y)} S\left(x_{(1)}\cdot y_{(1)}\right)\cdot\left(x_{(2)}\cdot y_{(2)}\right)\right) \\
&= \alpha \circ \left(\sum_{(x),(y)} S\left((x\cdot y)_{(1)}\right)\cdot\left((x\cdot y)_{(2)}\right)\right) \\
&= \alpha \circ \left(\sum_{xy} S\left((x\cdot y)_{(1)}\right)\cdot\left((x\cdot y)_{(2)}\right)\right) \\
&= \alpha \circ \eta_H \circ \varepsilon_{H\otimes H}\,(x\otimes y) = \eta_H \circ \varepsilon_{H\otimes H}\,(x\otimes y)
\end{aligned}$$

and $G * \mu = \eta_H \circ \varepsilon_{H\otimes H}$. Hence μ is a left inverse for F and a right inverse for G in a Hom-associative algebra, and therefore $F = G$. this means that i) holds.

ii) We apply the definition of the antipode for the element 1_H. Setting $1_H = \eta(1_{\Bbbk})$ and since $\Delta(1_H) = 1_H \otimes 1_H$ and $\varepsilon(1_H) = 1_{\Bbbk}$ one has

$$\begin{aligned}
(S * id_H)\,(1_H) &= \mu\,(S(1_H)\otimes 1_H) = \alpha \circ S(1_H) \\
&= S\circ\alpha(1_H) = S\circ\alpha\circ\eta(1_{\Bbbk}) = S\circ\eta(1_{\Bbbk}) \\
(S * id_H)\,(1_H) &= \eta\circ\varepsilon(1_H) = \eta(1_{\Bbbk}).
\end{aligned}$$

Using the antipode property we get $S\circ\eta(1_{\Bbbk}) = \eta(1_{\Bbbk})$.

iii)We use the same technique that we applied in part (i). We consider the linear maps

$$Q : H \longrightarrow H\otimes H \quad \text{and } R : H \longrightarrow H\otimes H$$

in the convolution Hom-associative algebra $Hom\,(H, H\otimes H)$ (see Corollary 2.4.12) where $\tilde{\mu}_\alpha = \alpha\circ\tilde{\mu}$ is defined in Example 2.1.5.

$$f * g\,(x) = \tilde{\mu}_\alpha \circ (f\otimes g)\circ\Delta\,(x)$$

given by

$$Q(x) = S(x)_1 \otimes S(x)_2 \text{ and } R(x) = S(x_2) \otimes S(x_1)$$

for all $x \in H$. We will again prove that $Q = R$ by showing that Q and R are both the convolution inverse of $\Delta : H \longrightarrow H \otimes H$.

iv) We apply ε to the relation $\mu \circ (id_H \otimes S) \circ \Delta(x) = \varepsilon(x)\eta(1_{\Bbbk})$. Since ε and S are linear maps, we obtain

$$\begin{aligned}
\varepsilon \circ \mu \circ (id_H \otimes S) \circ \Delta(x) &= \varepsilon(\varepsilon(x)\eta(1_{\Bbbk})) \\
(\varepsilon \otimes \varepsilon \circ S) \circ \Delta(x) &= \varepsilon(x)\varepsilon(1_H) = \varepsilon(x) \\
\varepsilon \circ S \circ (\varepsilon \otimes id_H) \circ \Delta(x) &\overset{(2.8)}{=} \varepsilon(x) \\
\varepsilon \circ S \circ \alpha(x) &\overset{(2.23)}{=} \varepsilon(x) \\
\varepsilon \circ \alpha \circ S(x) &= \varepsilon \circ S(x) = \varepsilon(x).
\end{aligned}$$

Thus $\varepsilon \circ S = \varepsilon$. ■

Proposition 2.4.20 *Let $(H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$ be a Hom-Hopf algebra with antipode S. Then the following assertions are equivalent:*

1. $\sum_{(x)} S(x_{(2)})\, x_{(1)} = \varepsilon(x) 1_H$ *for any $x \in H$. Or* $\mu \circ (S \otimes id_H) \circ \tau \circ \Delta = \varepsilon \circ \eta$
2. $\sum_{(x)} x_{(2)} S(x_{(1)}) = \varepsilon(x) 1_H$ *for any $x \in H$. Or* $\mu \circ (id_H \otimes S) \circ \tau \circ \Delta = \varepsilon \circ \eta$
3. $S^2 = id_H$ *(by S^2 we mean the composition of S with itself)*

Proof. (1)$\Longrightarrow$(3) We know that id_H is inverse of S with respect to convolution. We show that S^2 is a right convolution inverse of S, and by the uniqueness of the inverse it will follow that $S^2 = id_H$. We have

$$\begin{aligned}
(S * S^2)(x) = \mu \circ (S \otimes S^2) \circ \Delta(x) &= \sum_{(x)} S(x_{(1)}) \cdot S^2(x_{(2)}) \\
&= \sum_{(x)} S\left(S(x_{(2)}) \cdot (x_{(1)})\right) \text{ (S is an anti-morphism of Hom-associative algebras)} \\
&= S(\varepsilon(x) 1_H) = \varepsilon(x)\eta(1_{\Bbbk}) = \eta \circ \varepsilon(x)
\end{aligned}$$

This shows that indeed $S * S^2 = \eta \circ \varepsilon$.

(3)$\Longrightarrow$(2) We know that $\sum_{(x)} x_{(1)} \cdot S\left(x_{(2)}\right) = \varepsilon(x) 1_H$ Applying the anti-morphism of Hom-associative algebra S we obtain $\sum_{(x)} S^2\left(x_{(2)}\right) \cdot S\left(x_{(1)}\right) = \varepsilon(x) 1_H$. Since $S^2 = id_H$, this becomes $\sum_{(x)} x_{(2)} \cdot S\left(x_{(1)}\right) = \varepsilon(x) 1_H$.

(2)$\Longrightarrow$(3) We proceed as in (1)$\Longrightarrow$(3), and we show that $S^2 = id_H$ is a left convolution inverse for S. Indeed,

$$\begin{aligned}\left(S^2 * S\right)(x) = \mu \circ \left(S^2 \otimes S\right) \circ \Delta(x) &= \sum_{(x)} S^2\left(x_{(1)}\right) \cdot S\left(x_{(2)}\right) \\ &= S\left(\sum_{(x)} x_{(2)} \cdot S\left(x_{(1)}\right)\right) = S\left(\varepsilon(x) 1_H\right) = \eta \circ \varepsilon(x).\end{aligned}$$

(3)$\Longrightarrow$(1) We apply S to the equation $\sum_{(x)} S\left(x_{(1)}\right) \cdot x_{(2)} = \varepsilon(x) 1_H$, and using $S^2 = id_H$ we obtain $\sum_{(x)} S\left(x_{(2)}\right) \cdot x_{(1)} = \varepsilon(x) 1_H$. ∎

Corollary 2.4.21 *Let H be a commutative or cocommutative Hom-Hopf algebra then $S^2 = id_H$.*

Proof. If H is commutative ($\mu(x \otimes y) = y \otimes x$), then by $\sum_{(x)} S\left(x_{(1)}\right) \cdot x_{(2)} = \varepsilon(x) 1_H$ it follows that $\sum_{(x)} x_{(2)} \cdot S\left(x_{(1)}\right) = \varepsilon(x) 1_H$, i.e. (2) from the proceding proposition.

If H is cocommutative, then

$$\sum_{(x)} x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)}$$

and then by $\sum_{(x)} S\left(x_{(1)}\right) \cdot x_{(2)} = \varepsilon(x) 1_H$ it follows that $\sum_{(x)} S\left(x_{(2)}\right) \cdot x_{(1)} = \varepsilon(x) 1_H$, i.e. (1) from the proceding proposition. ∎

Remark 2.4.22 *Let $(H, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-Hopf algebra with antipode S. Then the Hom-bialgebra $H^{op,cop} = (H, \mu^{op}, \eta, \Delta^{cop}, \varepsilon, \alpha)$ is a Hom-Hopf algebra with the same antipode S.*

In Proposition 2.4.9 we saw that if H is a finite dimensional Hom-bialgebra, then its dual is a Hom-bialgebra. The following result shows that if H is a Hom-Hopf algebra, then its dual also has a Hom-Hopf algebra structure.

Proposition 2.4.23 *Let H be a finite dimensional Hom-Hopf algebra, with antipode S. Then the Hom-bialgebra H^* is a Hom-Hopf algebra, with antipode S^*.*

Proof. We know already that H^* is a Hom-bialgebra. We therefore need only show that S^* is the antipode of H^*. To this end, we have that,

$$\begin{aligned}\mu_{H^*} \circ (S^* \otimes id_{H^*}) \circ \Delta_{H^*}(f^*) &= (\rho \circ (S^* \otimes id_{H^*}) \circ \Delta_H)(f^* \circ \mu_H) \\ &= (S \otimes id_H)^* \Delta_H (f^* \circ \mu_H) \\ &= (f^* \circ \mu_H)(S \otimes id_H) \circ \Delta_H \\ &= (\rho \circ f^* \circ \eta_H \circ \varepsilon_H) = \eta_{H^*} \circ \varepsilon_{H^*}(f^*).\end{aligned}$$

using $\rho^{-1}(S \otimes id_H)^* \rho = (S^* \otimes id_{H^*})$

Similarly,

$$\mu_{H^*} \circ (id_{H^*} \otimes S^*) \circ \Delta_{H^*}(f^*) = \eta_{H^*} \circ \varepsilon_{H^*}(f^*).$$

for all $f^* \in H^*$, since $\varepsilon_{H^*}(f^*) = f^*(1_H) = f^* \circ \eta_H(1_{\Bbbk})$, and $\eta_{H^*}(1_{\Bbbk}) = 1_{H^*} = \varepsilon$, we have that S^* is the convolution inverse of id_{H^*} and therefore S^* is the antipode of H^*. ■

Chapter 3

Modules and comodules of Hom-Hopf algebras

In this chapter, we recall the definitions of modules and comodules over Hom-associative algebras and study their tensor products. The definitions of action and coactions are simply a polarisation of those of Hom-algebras and Hom-coalgebras, so we include them now among the basic definitions.

A Hom-module is a pair (M, α) [56] in which M is a vector space and $\alpha : M \longrightarrow M$ is a linear map. A morphism $(M, \alpha_M) \longrightarrow (N, \alpha_N)$ of Hom-modules is a linear map $f : M \longrightarrow N$ such that $\alpha_N \circ f = f \circ \alpha_M$. We will often abbreviate a Hom-module (M, α) to M. The tensor product of the Hom-modules (M, α_M) and (N, α_N) consists of the vector space $M \otimes N$ and the linear self-map $\alpha_M \otimes \alpha_N$.

3.1 Modules over Hom-associative algebras.

In this section, we define the notion of a module acting on a Hom-associative algebra. Let $A = (A, \mu_A, \eta_A, \alpha_A)$ be a unital Hom-associative algebra and (M, α_M) be a Hom-module.

Definition 3.1.1 *The vector space M is called a* ***left A-module*** *(or left module over Hom-algebra A) if there exists a morphism $\lambda_l : A \otimes M \longrightarrow M$ of Hom-modules, written as $\lambda_l (a \otimes m) = a \triangleright m$, called the structure map, such that*

$$\lambda_l \circ (\alpha_A \otimes \lambda_l) = \lambda_l \circ (\mu_A \otimes \alpha_M) \quad \textit{and} \quad \lambda_l \circ (\eta_A \otimes id_M) = \alpha_M \tag{3.1}$$

or, equivalently, that for all $a, b \in A$ and $m \in M$

$$\alpha_A(a) \triangleright (b \triangleright m) = (ab) \triangleright \alpha_M(m) \quad \text{and} \quad 1_A \triangleright m = \alpha_M(m)$$

where the first identity of (3.1) acts on $A \otimes A \otimes M$ and the second of (3.1) on $\Bbbk \otimes M \simeq M$.

The axioms are depicted in the commutative diagrams:

$$\begin{array}{ccccccc} A \otimes A \otimes M & \overset{\alpha_A \otimes \lambda_l}{\longrightarrow} & A \otimes M & & \Bbbk \otimes M & \cong & M \\ {\scriptstyle \mu_A \otimes \alpha_M} \downarrow & & \downarrow {\scriptstyle \lambda_l} & & {\scriptstyle \eta \otimes id_M} \downarrow & & \downarrow {\scriptstyle \alpha_M} \\ A \otimes M & \overset{\lambda_l}{\longrightarrow} & M & & A \otimes M & \overset{\lambda_l}{\longrightarrow} & M \end{array}$$

Remark 3.1.2 *The map λ_l is then called a left action of A on M.*

If (M, α_M) and $(M', \alpha_{M'})$ are lefts A-modules, then a **morphism of left A-modules** $f : M \longrightarrow M'$ is a morphism of the underlying Hom-modules such that

$$f \circ \lambda_l = \lambda_l' \circ (id_A \otimes f) \qquad \text{or} \qquad f(a \triangleright m) = a \triangleright f(m). \tag{3.2}$$

If f is invertible, it is a left A-modules **isomorphism**.

Definition 3.1.3 *A **right A-module** (or right module over Hom-associative algebra A) is a vector space M with a morphism $\lambda_r : M \otimes A \longrightarrow M$ of Hom-modules, right action of A on M and written as $\lambda_r(m \otimes a) = m \triangleleft a$, such that*

$$\lambda_r \circ (\lambda_r \otimes \alpha_A) = \lambda_r \circ (\alpha_M \otimes \mu_A) \quad \text{and} \quad \lambda_r \circ (id_M \otimes \eta) = \alpha_M. \tag{3.3}$$

The two conditions on a right A-module (3.3) can be expressed as for all $a, b \in A$ and $m \in M$

$$(m \triangleleft a) \triangleleft \alpha_A(b) = \alpha_M(m) \triangleleft (ab) \quad \text{and} \quad m \triangleleft 1_A = \alpha_M(m)$$

or as the commutativity of the diagrams

$$\begin{array}{ccccccc} M \otimes A \otimes A & \overset{\alpha_M \otimes \mu_A}{\longrightarrow} & M \otimes A & & M \otimes \Bbbk & \cong & M \\ {\scriptstyle \lambda_r \otimes \alpha_A} \downarrow & & \downarrow {\scriptstyle \lambda_r} & & {\scriptstyle id_M \otimes \eta} \downarrow & & \downarrow {\scriptstyle \alpha_M} \\ M \otimes A & \overset{\lambda_r}{\longrightarrow} & M & & M \otimes A & \overset{\lambda_r}{\longrightarrow} & M \end{array}$$

Remark 3.1.4 *The map λ_r is then called a right action of A on M. We will often denote a left or right A-module by (M, λ, α_M) and refer to λ as the left or right action of A on M.*

Example 3.1.5 *If A is an unital Hom-associative algebra, then we may consider A as either a left or right A-module where the action is given by the multiplication μ.*

The Hom-associativity and unit properties give, respectively, the corresponding properties for a A-module.

If M and M' are rights A-modules, then a **morphism of right** A-module $f : M \longrightarrow M'$ if

$$f \circ \lambda_r = \lambda'_r \circ (f \otimes id_A) \qquad or \qquad f (m \lhd a) = f (m) \lhd a \tag{3.4}$$

If f is invertible, it is an **isomorphism of right A-modules**.

Proposition 3.1.6 *A right A-module is nothing else than a left module over the opposite unital Hom-associative algebra A^{op}. Therfore we need only consider left modules which shall for simplicity be called Hom-module in the sequel.*

Proof. Indeed,

$$\begin{aligned}
\lambda_r^{op} \circ (\alpha_A \otimes \lambda_r^{op}) (x \otimes y \otimes m) &= \lambda_r^{op} \circ (\alpha_A (x) \otimes \lambda_r^{op} (m \otimes y)) \\
&= \lambda_r \circ (\lambda_r (m \otimes y) \otimes \alpha_A (x)) \\
&= \lambda_r \circ (\alpha_M (m) \otimes \mu (y \otimes x)) \\
&= \lambda_r^{op} \circ (\mu^{op} (x \otimes y) \otimes \alpha_M (m)) \\
&= \lambda_r^{op} \circ (\mu^{op} \otimes \alpha_M) (x \otimes y \otimes m) .
\end{aligned}$$

We have shown that λ_r^{op} is the structure map of a left A-module. ■

Theorem 3.1.7 *Let (A, μ, η, α_A) be an unital Hom-associative algebra and (M, λ_l, α_M) be a left A-module Then $\left(M, \alpha_M^n \circ \lambda_l, \alpha_M^{n+1}\right)$ is a another left A_{α^n}-module over unital Hom-associative algebra $A_{\alpha^n} = (A, \mu_{\alpha^n}, \eta_{\alpha^n}, \alpha_A^{n+1})$ defined in Proposition 2.1.8.*

Theorem 3.1.8 *Let $(A, \mu_A, \eta_A, \alpha_A)$ be an unital Hom-associative algebra and (M, λ_l, α_M) be a left A-module with structure map $\lambda_l : A \otimes M \longrightarrow M$. Define the map*

$$\tilde{\lambda}_l^{2,0} = \lambda_l \circ \left(\alpha_A^2 \otimes id_M\right) : A \otimes M \longrightarrow M. \tag{3.5}$$

Then $\tilde{\lambda}_l^{2,0}$ is the structure map of another left A-module structure on (M, α_M).

Hence, for any $n \in \mathbb{N}$, $\tilde{\lambda}_l^{n,0}$ is the structure map of another left A-Hom-module structure on M, where

$$\tilde{\lambda}_l^{n,0} = \lambda_l \circ (\alpha_A^n \otimes id_M) : A \otimes M \longrightarrow M.$$

Proof. The fact that $\lambda : A \otimes M \longrightarrow M$ is a morphism of Hom-modules means that

$$\alpha_M \circ \lambda = \lambda \circ \alpha_{A \otimes M} \tag{3.6}$$

To see that $\tilde{\lambda}_l^{n,0}$ is a morphism of Hom-modules, we compute as follows

$$\begin{aligned}
\alpha_M \circ \tilde{\lambda}_l^{n,0} &= \alpha_M \circ \lambda_l \circ (\alpha_A^n \otimes id_M) \\
&\overset{(3.6)}{=} \lambda_l \circ (\alpha_A \otimes \alpha_M) \circ (\alpha_A^n \otimes id_M) \\
&= \lambda_l \circ (\alpha_A^n \otimes id_M) \circ (\alpha_A \otimes \alpha_M) \\
&= \tilde{\lambda}_l^{n,0} \circ (\alpha_A \otimes \alpha_M)
\end{aligned}$$

To see that $\tilde{\lambda}_l^{2,0}$ satisfies (3.1) (with $\tilde{\lambda}_l^{2,0}$ in place of λ_l), we compute as follows:

$$\begin{aligned}
\tilde{\lambda}_l^{n,0} \circ \left(\alpha_A \otimes \tilde{\lambda}_l^{n,0}\right) &= \lambda_l \circ (\alpha_A^n \otimes id_M) \circ (\alpha_A \otimes (\lambda_l \circ (\alpha_A^n \otimes id_M))) \\
&= \lambda_l \circ (\alpha_A \otimes \lambda_l) \circ (\alpha_A^n \otimes \alpha_A^n \otimes id_M) \\
&\overset{(3.1)}{=} \lambda_l \circ (\mu_A \otimes \alpha_M) \circ (\alpha_A^n \otimes \alpha_A^n \otimes id_M) \\
&= \lambda_l \circ (\alpha_A^n \circ \mu_A) \otimes \alpha_M \text{ by multiplicativity of } \alpha_A \\
&= \lambda_l \circ (\alpha_A^n \otimes id_M) \circ (\mu_A \otimes \alpha_M) \\
&= \tilde{\lambda}_l^{2,0} \circ (\mu_A \otimes \alpha_M)
\end{aligned}$$

and

$$\tilde{\lambda}_l^{n,0} \circ (\eta_A \otimes id_M) = \lambda_l \circ (\alpha_A^n \otimes id_M) \circ (\eta_A \otimes id_M) = \lambda_l \circ (\eta_A \otimes id_M) = \alpha_M$$

$\lambda_l \circ (\eta_A \otimes id_M) = \alpha_M$. We have shown that $\tilde{\lambda}_l^{n,0}$ is the structure map of a left A-module structure on (M, α_M). ■

Theorem 3.1.9 *Let $(A, \mu_A, \eta_A, \alpha_A)$ be an unital Hom-associative algebra and (M, λ_r, α_M) be a right A-module with structure map $\lambda_r : M \otimes A \longrightarrow M$. Define the map*

$$\tilde{\lambda}_r^{2,0} = \lambda_r \circ \left(id_M \otimes \alpha_A^2\right) : M \otimes A \longrightarrow M.$$

Then $\tilde{\lambda}_r^{2,0}$ is the structure map of another right A-module structure on (M, α_M).

Hence, $\forall n \in \mathbb{N}$, if (M, λ_r, α_M) is a right A-module. Then $\left(M, \tilde{\lambda}_r^{n,0}, \alpha_M\right)$ is the another right A-module, where

$$\tilde{\lambda}_r^{n,0} = \lambda_r \circ (id_M \otimes \alpha_A^n) : M \otimes A \longrightarrow M.$$

Proof. This proof is completely analogous to that of Theorem 3.1.8 ■

Theorem 3.1.10 *Let (A, μ, η, α_A) be an unital Hom-associative algebra and (M, α_M) be a left A-module with structure map : $\lambda_l : A \otimes M \longrightarrow M$. For each integer $k \geq 0$, define the map*

$$\tilde{\lambda}_l^{0,k} = \alpha_M^k \circ \lambda_l : A \otimes M \longrightarrow M.$$

Then each the Hom-module $\left(M, \tilde{\lambda}_l^{0,k}, \alpha_M^{k+1}\right)$ gives the structure of a left module over unital Hom-associative algebra $(A, \mu_{\alpha^k}, \eta_{\alpha^k}, \alpha_{\alpha^k})$ in Proposition 2.1.8, for any $m \in \mathbb{N}$.

Proof. Since $\tilde{\lambda}_l^{0,k} = \alpha_M^k \circ \lambda_l$, it suffices to prove that $\tilde{\lambda}_l^{0,k}$ gives $\left(M, \alpha_M^{k+1}\right)$ the structure of an A_{α^k}-module, where $A_{\alpha^k} = (A, \mu_{\alpha^k}, \eta_{\alpha^k}, \alpha_{\alpha^k})$.

First, the morphism of Hom-modules in this case says

$$\begin{aligned} \alpha_M^{k+1} \circ \tilde{\lambda}_l^{0,k} &= \alpha_M^{2k+1} \circ \lambda_l = \alpha_M^{2k} \circ \lambda_l \circ (\alpha_A \otimes \alpha_M) \\ &= \alpha_M^k \circ \lambda_l \circ \left(\alpha_A^{k+1} \otimes \alpha_M^{k+1}\right) = \tilde{\lambda}_l^{0,k} \circ \left(\alpha_A^{k+1} \otimes \alpha_M^{k+1}\right) \end{aligned}$$

by the multiplicativity of α_A and α_M with respect to λ_l.

Next, the condition (3.1) (with $\tilde{\lambda}_l^{0,k}$ in place of λ_l) in this case says

$$\begin{aligned}\tilde{\lambda}_l^{0,k} \circ \left(\alpha_A^{k+1} \otimes \tilde{\lambda}_l^{0,k}\right) &= \alpha_M^k \circ \lambda_l \circ \left(\alpha_A^{k+1} \otimes \alpha_M^k \circ \lambda_l\right) = \alpha_M^k \circ \lambda_l \circ (\alpha_A \otimes \lambda_l) \circ \left(\alpha_A^k \otimes \alpha_A^k \otimes \alpha_M^k\right) \\ &= \alpha_M^k \circ \lambda_l \circ (\mu_A \otimes \alpha_M) \circ \left(\alpha_A^k \otimes \alpha_A^k \otimes \alpha_M^k\right) \\ &= \alpha_M^k \circ \lambda_l \circ \left(\alpha_A^k \circ \mu_A \otimes \alpha_M^{k+1}\right) \\ &= \tilde{\lambda}_l^{0,k} \circ \left(\mu_{\alpha^k} \otimes \alpha_M^{k+1}\right)\end{aligned}$$

and

$$\begin{aligned}\tilde{\lambda}_l^{0,k} \circ (\eta_{\alpha^k} \otimes id_M) = \alpha_M^k \circ \lambda_l \circ (\eta_{\alpha^k} \otimes id_M) &= \lambda_l \circ \left(\alpha_A^k \otimes \alpha_M^k\right) \circ (\eta_{\alpha^k} \otimes id_M) \\ &= \lambda_l \circ (\eta_{\alpha^k} \otimes id_M) \circ \alpha_M^k = \alpha_M^{k+1} = \tilde{\lambda}_l^{0,k} \circ (id_M \otimes \eta_{\alpha^k})\end{aligned}$$

Then $\left(M, \tilde{\lambda}_l^{0,k}, \alpha_M^{k+1}\right)$ gives the structure of a left module. ■

Corollary 3.1.11 *Let (A, μ, η, α_A) be an unital Hom-associative algebra and (M, α_M) be a left A-module with structure map $\lambda_l : A \otimes M \longrightarrow M$. For any integers $n, k \geq 0$, define the map*

$$\tilde{\lambda}_l^{n,k} = \alpha_M^k \circ \lambda_l \circ (\alpha_A^n \otimes id_M) : A \otimes M \longrightarrow M.$$

Then the Hom-module $\left(M, \tilde{\lambda}_l^{n,k}, \alpha_M^{k+1}\right)$ gives the structure of a left module over the unital Hom-associative algebra $(A, \mu_{\alpha^k}, \eta_{\alpha^k}, \alpha_{\alpha^k})$.

Proof. Apply Theorem 3.1.10 to the A-module M with structure map $\tilde{\lambda}_l^{n,0} = \lambda_l \circ (\alpha_A^n \otimes id_M)$ in Theorem 3.1.8, and observe that $\tilde{\lambda}_l^{n,k} = \left(\tilde{\lambda}_l^{n,0}\right)^{0,k}$. We have

$$\begin{aligned}\tilde{\lambda}_l^{n,k} \circ \left(\alpha_A^{k+1} \otimes \tilde{\lambda}_l^{n,k}\right) &= \alpha_M^k \circ \lambda_l \circ (\alpha_A^n \otimes id_M) \circ \left(\alpha_A^{k+1} \otimes \alpha_M^k \circ \lambda_l \circ (\alpha_A^n \otimes id_M)\right) \\ &= \alpha_M^k \circ \lambda_l^{n,0} \circ \left(\alpha_A^{k+1} \otimes \alpha_M^k \circ \lambda_l^{n,0}\right) = \alpha_M^{2k} \circ \lambda_l^{n,0} \circ \left(\alpha_A \otimes \lambda_l^{n,0}\right) \\ &= \alpha_M^{2k} \circ \lambda_l^{n,0} \circ (\mu_A \otimes \alpha_M) \text{ by Theorem3.1.8} \\ &= \alpha_M^{2k} \circ \lambda_l \circ (\alpha_A^n \otimes id_M) \circ (\mu_A \otimes \alpha_M) \\ &= \alpha_M^k \circ \lambda_l \circ (\alpha_A^n \otimes id_M) \circ \left(\alpha_M^k \circ \mu_A \otimes \alpha_M^{k+1}\right) \\ &= \tilde{\lambda}_l^{n,k} \circ \left(\alpha_M^k \circ \mu_A \otimes \alpha_M^{k+1}\right)\end{aligned}$$

The second assertion is proved similarly, so $\left(M, \tilde{\lambda}_l^{n,k}, \alpha_M^{k+1}\right)$ gives the structure of a left A_{α^k}-module. ■

Definition 3.1.12 *A vector space M is called an A-**bimodule** if M is both a left A-module with action $a \rhd m$ and a right A-module with action $m \lhd a$ satisfying the compatibility condition*

$$(a \rhd m) \lhd \alpha_A(b) = \alpha_A(a) \rhd (m \lhd b)$$

for $a, b \in A$ and $m \in M$. Or

$$\lambda_r \circ (\lambda_l \otimes \alpha_A) = \lambda_l \circ (\alpha_A \otimes \lambda_r) \tag{3.7}$$

Then, we refer to the tuple $(M, \lambda_l, \lambda_r, \alpha_M)$ for an A-bimodule M.

If M and M' are A-bimodules, a map $f : M \longrightarrow M'$ is a **morphism of A-bimodules** if it is a morphism of both left A-module and right A-module.

Every Hom-associative algebra (A, μ_A, α_A) is an A-bimodule with $\lambda_l = \lambda_r = \mu_A$.

Proposition 3.1.13 *Let $(A, \mu_A, \eta_A, \alpha_A)$ be an unital Hom-associative algebra and $(M, \lambda_l, \lambda_r, \alpha_M)$ be a A-bimodule. Define the maps*

$$\tilde{\lambda}_l^{2,0} = \lambda_l \circ \left(\alpha_A^2 \otimes id_M\right) \text{ and } \tilde{\lambda}_r^2 = \lambda_r \circ \left(id_M \otimes \alpha_A^2\right).$$

Then $\left(M, \tilde{\lambda}_l^{2,0}, \tilde{\lambda}_r^{2,0}, \alpha_M\right)$ is a A-bimodule.

Hence, $\forall n \in \mathbb{N}$, if $(M, \lambda_l, \lambda_r, \alpha_M)$ is a A-bimodule. Then $\left(M, \tilde{\lambda}_l^{n,0}, \tilde{\lambda}_r^{n,0}, \alpha_M\right)$ is the another A-bimodule,

Proof. We use Theorems 3.1.8 and 3.1.9, that (M, λ_l, α_M) is a left A-module, and (M, λ_r, α_M) is a right A-module. It remains to establish compatibility condition, we compute as follows

$$\begin{aligned}
\tilde{\lambda}_r^{n,0} \circ \left(\tilde{\lambda}_l^{n,0} \otimes \alpha_A\right) &= \lambda_r \circ (id_M \otimes \alpha_A^n) \circ (\lambda_l \circ (\alpha_A^n \otimes id_M) \otimes \alpha_A) \\
&= \lambda_r \circ (\lambda_l \otimes \alpha_A) \circ (\alpha_A^n \otimes id_M \otimes \alpha_A^n) \\
&\overset{(3.7)}{=} \lambda_l \circ (\alpha_A \otimes \lambda_r) \circ (\alpha_A^n \otimes id_M \otimes \alpha_A^n) \\
&= \lambda_l \circ (\alpha_A^n \otimes id_M) \circ (\alpha_A \otimes \lambda_r \circ (id_M \otimes \alpha_A^n)) \\
&= \tilde{\lambda}_l^{n,0} \circ \left(\alpha_A \otimes \tilde{\lambda}_r^2\right).
\end{aligned}$$

Then $\left(M, \tilde{\lambda}_l^{n,0}, \tilde{\lambda}_r^{n,0}, \alpha_M\right)$ is a A-bimodule. ■

Example 3.1.14 *Let $M' = V \otimes M \otimes V$, and consider structure maps $\lambda_l^\mu = \mu \otimes \alpha_M \otimes \alpha_V$ and $\lambda_r^\mu = \alpha_V \otimes \alpha_M \otimes \mu$. Then $\left(V \otimes M \otimes V, \lambda_l^\mu, \lambda_r^\mu\right)$ is an exterior A-bimodule; λ_l^μ and λ_r^μ called exterior bimodule structure maps. This is not to be confused with the notion of an exterior algebra.*

Proof. Indeed, the left A-module axioms for λ_l^μ now follows from that of λ_l and the identities

$$\begin{aligned}
\lambda_l^\mu \circ \left(\alpha_V \otimes \lambda_l^\mu\right) &= (\mu \otimes \alpha_M \otimes \alpha_V) \circ (\alpha_V \otimes \mu \otimes \alpha_M \otimes \alpha_V) \\
&= \mu \circ (\alpha_V \otimes \mu) \otimes \alpha_M^{\otimes 2} \otimes \alpha_V^{\otimes 2} \\
&\overset{(2.1)}{=} \mu \circ (\mu \otimes \alpha_V) \otimes \alpha_M^{\otimes 2} \otimes \alpha_V^{\otimes 2}
\end{aligned}$$

$$\begin{aligned}
\lambda_l^\mu \circ (\mu \otimes \alpha_{V \otimes M \otimes V}) &= (\mu \otimes \alpha_M \otimes \alpha_V) \circ (\mu \otimes \alpha_V \otimes \alpha_M \otimes \alpha_V) \\
&= \mu \circ (\mu \otimes \alpha_V) \otimes \alpha_M^{\otimes 2} \otimes \alpha_V^{\otimes 2}
\end{aligned}$$

and

$$\lambda_l^\mu \circ (\eta \otimes id_{V \otimes M \otimes V}) = (\mu \otimes \alpha_M \otimes \alpha_V) \circ (\eta \otimes id_{V \otimes M \otimes V})$$
$$\overset{(2.2)}{=} (\mu \circ (\eta \otimes id_V) \otimes \alpha_M \otimes \alpha_V) = \alpha_{V \otimes M \otimes V}$$

Likewise the right A-module axioms for λ_r^μ follow from that of λ_r and the identity $\lambda_r \circ (id_M \otimes \eta) = \alpha_M$.

$$\begin{aligned}
\lambda_r^\mu \circ (\lambda_r^\mu \otimes \alpha_V) &= (\alpha_V \otimes \alpha_M \otimes \mu) \circ (\alpha_V \otimes \alpha_M \otimes \mu \otimes \alpha_V) \\
&= \alpha_V^{\otimes 2} \otimes \alpha_M^{\otimes 2} \otimes \mu \circ (\mu \otimes \alpha_V) \\
&\overset{(2.1)}{=} \alpha_V^{\otimes 2} \otimes \alpha_M^{\otimes 2} \otimes \mu \circ (\alpha_V \otimes \mu) \\
\lambda_r^\mu \circ \left(\alpha_{V \otimes M \otimes V} \otimes \mu\right) &= (\alpha_V \otimes \alpha_M \otimes \mu) \circ (\alpha_V \otimes \alpha_M \otimes \alpha_V \otimes \mu) \\
&= \alpha_V^{\otimes 2} \otimes \alpha_M^{\otimes 2} \otimes \mu \circ (\alpha_V \otimes \mu)
\end{aligned}$$

and

$$\lambda_r^\mu \circ (id_{V\otimes M\otimes V} \otimes \eta) = (\alpha_V \otimes \alpha_M \otimes \mu) \circ (id_{V\otimes M\otimes V} \otimes \eta)$$
$$= \alpha_V \otimes \alpha_M \otimes \mu \circ (id_V \otimes \eta)$$
$$\overset{(2.2)}{=} \alpha_{V\otimes M\otimes V}.$$

Finally, compatibility conditionis (3.7) follows from the following calculation:

$$\lambda_r^\mu \circ \left(\lambda_l^\mu \otimes \alpha_V\right) = (\alpha_V \otimes \alpha_M \otimes \mu) \circ (\mu \otimes \alpha_M \otimes \alpha_V \otimes \alpha_V)$$
$$= \alpha_V \circ \mu \otimes \alpha_M^{\otimes 2} \otimes \mu \circ \alpha_V$$
$$= \mu \circ (\alpha_V \otimes \alpha_V) \otimes \alpha_M^{\otimes 2} \otimes (\alpha_V \otimes \alpha_V) \circ \mu$$
$$= \lambda_l^\mu \circ (\alpha_V \otimes \lambda_r^\mu).$$

We have shown that $\left(V \otimes M \otimes V, \lambda_l^\mu, \lambda_r^\mu\right)$ is an exterior A-bimodule. ■

3.2 Comodules over Hom-coassociative coalgebras.

Dualizing actions of Hom-associative algebras on Hom-modules leads to coactions of Hom-coassociative coalgebras on Hom-comodules. Let $(C, \Delta_C, \varepsilon_C, \beta_C)$ be a counital Hom-coassociative coalgebra and (M, β_M) be a Hom-module.

Definition 3.2.1 *A right coaction of $(C, \Delta_C, \varepsilon_C, \beta_c)$ on a vector space M, called then a* ***right C-comodule****, is a morphism $\rho_r : M \longrightarrow M \otimes C$ of Hom-modules, satisfying the identities*

$$(\rho_r \otimes \beta_C) \circ \rho_r = (\beta_M \otimes \Delta_C) \circ \rho_r \quad \textit{and} \quad (id_M \otimes \varepsilon_C) \circ \rho_r = \beta_M. \tag{3.8}$$

If ρ_r and ρ'_r are right comodules of C on Hom-module (M, β_M) and $(M', \beta_{M'})$, then a **morphism of right C-comodules** $f : M \longrightarrow M'$ is a morphism of the underlying Hom-comodules such that

$$\rho'_r \circ f = (id_C \otimes f) \circ \rho_r. \tag{3.9}$$

Definition 3.2.2 *A **left** C**-comodule** (or left comodule over Hom-coalgebra C) on (M, β_M) is a linear mapping $\rho_l : M \longrightarrow C \otimes M$ such that*

$$(\beta_C \otimes \rho_l) \circ \rho_l = (\Delta_C \otimes \beta_M) \circ \rho_l \quad \textit{and} \quad (\varepsilon_C \otimes id_M) \circ \rho_l = \beta_M. \tag{3.10}$$

The two conditions of (3.8) are equivalent to the requirement that the two diagrams

$$\begin{array}{ccccccc} M \otimes C \otimes C & \overset{\beta_M \otimes \Delta_C}{\longleftarrow} & M \otimes C & & M \otimes \Bbbk & \cong & M \\ {\scriptstyle \rho_r \otimes \beta_C} \uparrow & & \uparrow {\scriptstyle \rho_r} & & {\scriptstyle id_M \otimes \varepsilon} \uparrow & & \uparrow {\scriptstyle \beta_M} \\ M \otimes C & \overset{\rho_r}{\longleftarrow} & M & & M \otimes C & \overset{\rho_r}{\longleftarrow} & M \end{array}$$

are commutative. The diagrams of (3.10) for a right comodule are obtained from the diagrams of (3.3) for a right module by reversing arrows and replacing the multiplication μ by the comultiplication Δ and the unit η by the counit ε. Note also that the identities of (3.8) and (3.10) which characterize right and left comodules are just the "duals" of the equations of (3.1) and (3.3) which define right and left modules, respectively.

The map ρ_r (resp. ρ_l) is then called a right (resp. left) coaction of (M, β_M) on C.We will often denote a right or left comodule by (M, ρ, β_M) and refer to ρ as the left or right coaction of M on C.

Then, a *morphism of right C-comodule* is $f : M \to M'$ such that

$$\rho'_r \circ f = (f \otimes id_C) \circ \rho_r. \tag{3.11}$$

If f is invertible, it is an isomorphism of right C-modules.

Let all $m \in M$ we now adapt the Sweedler notation to coaction if (M, ρ_r, β_M) is a right C-comodule, then write

$$\rho_r(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$$

the elements on the first tensor position (the $m_{(0)}$'s) being in M, and elements on the second tensor position (the $m_{(1)}$'s) being in C. So equations (3.8) becomes

$$\sum \left(m_{(0)}\right)_{(0)} \otimes \left(m_{(0)}\right)_{(1)} \otimes \beta_C\left(m_{(1)}\right) = \sum \beta_M\left(m_{(0)}\right) \otimes \left(m_{(1)}\right)_{(1)} \otimes \left(m_{(0)}\right)_{(2)}$$

and

$$\sum \varepsilon_C\left(m_{(1)}\right) \otimes m_{(0)} = \beta_M(m)$$

Similarly, if (M, ρ_l, β_M) is a left C-comodule, we denote

$$\rho_l(m) = \sum_{(m)} m_{(-1)} \otimes m_{(0)}$$

for all $m \in M$, where $m_{(-1)} \in C$ and $m_{(0)} \in M$. In this case, equations (3.10) becomes

$$\sum \beta_C\left(m_{(-1)}\right) \otimes \left(m_{(0)}\right)_{(-1)} \otimes \left(m_{(0)}\right)_{(0)} = \sum \left(m_{(-1)}\right)_{(1)} \otimes \left(m_{(-1)}\right)_{(2)} \otimes \beta_M\left(m_{(0)}\right)$$

and

$$\sum \varepsilon_C\left(m_{(-1)}\right) \otimes m_{(0)} = \beta_M(m).$$

a slightly different adaptation of our original Sweedler notation.

The Hom-coassociativity and counit properties give, respectively, the corresponding properties for a C-comodule.

Proposition 3.2.3 *A right C-comodule is the same as a left comodule over the coopposite counital Hom-coassociative coalgebra C^{cop}.*

Remark 3.2.4 *The preceding proposition shows that any result that we obtain for right Hom-comodules has an analogue for left Hom-comodules. This is why we are going to work generally with right Hom-comodules, without explicitly mentioning the similar results for left C-comodules.*

For $m \in M$, In Sweedler notation for a right C-comodule, this can be written

$$\sum f\left(m_{(0)}\right) \otimes m_{(1)} = \sum f(m)_{(0)} \otimes f(m)_{(1)}$$

Theorems 3.1.8, 3.1.10 and Corollary 3.1.11 can be readily dualized by inverting the arrows and replacing μ by Δ in the various equalities in their proofs. Therefore, we omit the proofs of the following results, which are dual to Theorems 3.1.8, 3.1.10 and Corollary 3.1.11 respectively.

Theorem 3.2.5 *Let $(C, \Delta, \varepsilon, \beta_C)$ be a counital Hom-coassociative coalgebra and (M, β_M) be a right A-comodule with structure map $\rho_r : M \longrightarrow M \otimes C$. Define the map*

$$\tilde{\rho}_r^{2,0} = \left(id_M \otimes \beta_C^2\right) \circ \rho_r : M \longrightarrow M \otimes C. \tag{3.12}$$

Then $\tilde{\rho}_r^{2,0}$ *is the structure map of another right* C*-comodule structure on* M*.*

Hence, $\forall n \in \mathbb{N}$, $\left(M, \tilde{\rho}_r^{n,0}, \beta_M\right)$ *is the another right* A*-module, where*

$$\tilde{\rho}_r^{n,0} = (id_M \otimes \beta_C^n) \circ \rho_r : M \longrightarrow M \otimes C.$$

Proposition 3.2.6 *Let* $(C, \Delta, \varepsilon, \beta_C)$ *be a counital Hom-coassociative coalgebra and* (M, β_M) *be a left* C*-comodule with structure map* $\rho_l : M \longrightarrow C \otimes M$*. Define the map*

$$\tilde{\rho}_l^{2,0} = \left(\beta_C^2 \otimes id_M\right) \circ \rho_l : M \longrightarrow C \otimes M.$$

Then $\tilde{\rho}_l^2$ *is the structure map of another left* C*-comodule structure on* M*.*

Hence, $\forall n \in \mathbb{N}$, $\left(M, \tilde{\rho}_l^{n,0}, \beta_M\right)$ *is the another right* C*-comodule, where*

$$\tilde{\rho}_l^{n,0} = (\beta_C^n \otimes id_M) \circ \rho_l : M \longrightarrow C \otimes M.$$

Theorem 3.2.7 *Let* $(C, \Delta, \varepsilon, \beta_C)$ *be a counital Hom-coassociative coalgebra and* (M, β_M) *be a right* C*-comodule with structure map:* $\rho_r : M \longrightarrow M \otimes C$*. For each integer* $k \geq 0$*, define the map*

$$\tilde{\rho}_r^{0,k} = \rho_r \circ \beta_M^k : M \longrightarrow M \otimes C.$$

Then each the comodule $\left(M, \tilde{\rho}_r^{0,k}, \beta_M^{k+1}\right)$ *gives the structure of a right comodule over Hom-coassociative coalgebra* $C_{\beta^k} = (C, \Delta_{\beta^k}, \varepsilon_{\beta^k}, \beta_{\beta^k})$ *in Proposition 2.2.10.*

Corollary 3.2.8 *Let* $(C, \Delta, \varepsilon, \beta_C)$ *be a Hom-coassociative coalgebra and* (M, β_M) *be a right* C*-comodule with structure map* $\rho_r : M \longrightarrow M \otimes C$*. For any integers* $n, k \geq 0$*, define the map*

$$\tilde{\rho}_r^{n,k} = (id_M \otimes \beta_C^n) \circ \rho_r \circ \beta_M^k : M \longrightarrow M \otimes C.$$

Then the comodule $\left(M, \tilde{\rho}_r^{n,k}, \beta_M^{k+1}\right)$ gives the structure of a right comodule over the counital Hom-coassociative coalgebra $C_{\beta^k} = \left(C, \Delta_{\beta^k}, \varepsilon_{\beta^k}, \beta_{\beta^k}\right)$ in Proposition 2.2.10.

Definition 3.2.9 *A vector space* M *is called a* C***-bicomodule*** *(or bicomodule over counital Hom-coassociative coalgebra* C*) if* M *is both a left* C*-comodule and a right* C*-comodule satisfying the compatibility condition*

$$(\beta_C \otimes \rho_r) \circ \rho_l = (\rho_l \otimes \beta_C) \circ \rho_r \tag{3.13}$$

Then we call $(M, \rho_l, \rho_r, \alpha_M)$ is a A-bicomodule.

The compatibility may be written as

$$\sum \beta_C\left(m_{(-1)}\right) \otimes \left(m_{(0)}\right)_{(0)} \otimes \left(m_{(0)}\right)_{(1)} = \sum \left(m_{(0)}\right)_{(-1)} \otimes \left(m_{(0)}\right)_{(0)} \otimes \beta_C\left(m_{(1)}\right)$$

for any $m \in M$.

Remark 3.2.10 *Clearly, Δ is a right comodule and also a left comodule of C on itself. then Δ is a C-bicomodule Indeed, in the case $\rho_r = \rho_l = \Delta$ the equations (*3.8*), (*3.10*) and the equations (*3.13*) become the Hom-coalgebra axioms (*2.7*) and (*2.8*).*

If M and M' are C-bicomodules, a map $f : M \longrightarrow M'$ is a **morphism of C-Hom-bicomodules** if it is morphism of left C-comodule and it is morphism of right C-comodule.

Proposition 3.2.11 *Let $(C, \Delta_C, \varepsilon_C, \beta_C)$ be a counital Hom-coassociative coalgebra and $(M, \rho_l, \rho_r, \beta_M)$ be a C-bicomodule. Define the maps*

$$\tilde{\rho}_l^n = (\beta_C^n \otimes id_M) \circ \rho_l \text{ and } \tilde{\rho}_r^n = (id_M \otimes \beta_C^n) \circ \rho_r.$$

Then $(M, \tilde{\rho}_l^n, \tilde{\rho}_r^n, \beta_M)$ is a C-bimodule.

Example 3.2.12 *If $(C, \Delta_C, \varepsilon_C, \beta_C)$ is a counital Hom-coassociative coalgebra and V is a $\Bbbk$-vector space and β_V is a homomorphism of a vector space, then $V \otimes C$ becomes a right C-comodule with structure map*

$$\rho_r : V \otimes C \longrightarrow V \otimes C \otimes C$$

induced by Δ_C, hence $\rho_r = \beta_V \otimes \Delta_C$

Proof. We want ρ_r to satisfy the following equations:

$$(\rho_r \otimes \beta_C) \circ \rho_r = (\beta_{V\otimes C} \otimes \Delta_C) \circ \rho_r \quad \text{and} \quad (id_{V\otimes C} \otimes \varepsilon_C) \circ \rho_r = \beta_{V\otimes C}.$$

We show the first equation:

$$
\begin{aligned}
(\rho_r \otimes \beta_C) \circ \rho_r &= (\beta_V \otimes \Delta_C \otimes \beta_C) \circ \beta_V \otimes \Delta_C \\
&= (\beta_V \circ \beta_V \otimes (\Delta_C \otimes \beta_C) \, \Delta_C) \\
&= (\beta_V \circ \beta_V \otimes (\beta_C \otimes \Delta_C) \, \Delta_C) \\
&= (\beta_V \otimes \beta_C \otimes \Delta_C) \, (\beta_V \otimes \Delta_C) \\
&= (\beta_{V \otimes C} \otimes \Delta_C) \circ \rho_r
\end{aligned}
$$

The second equation follows:

$$
\begin{aligned}
(id_{V \otimes C} \otimes \varepsilon_C) \circ \rho_r &= (id_{V \otimes C} \otimes \varepsilon_C) \circ (\beta_V \otimes \Delta_C) \\
&= \beta_V \otimes (id_C \otimes \varepsilon_C) \, \Delta_C = \beta_V \otimes \beta_C = \beta_{V \otimes C}.
\end{aligned}
$$

Then ρ_r gives a right C-comodule on $V \otimes C$. ■

Example 3.2.13 *Let $N' = V \otimes N \otimes V$, and consider structure maps $\rho_l^{\Delta} = \Delta \otimes \beta_N \otimes \beta_V$ and $\rho_r = \beta_V \otimes \beta_N \otimes \Delta$. Then $\left(V \otimes N \otimes V, \rho_l^{\Delta}, \rho_r^{\Delta}\right)$ is an exterior C-bicomodule; ρ_l^{Δ} and ρ_r^{Δ} are called exterior bicomodule structure maps.*

Proof. This proof is completely analogous to that of Example 3.1.14. ■

3.3 Duality between modules and comodules of Hom-Hopf algebras

Let C be a counital Hom-coassociative coalgebra, and C^* the dual unital Hom-associative algebra. If M is a vector space, and $\omega : M \longrightarrow M \otimes C$ is a linear map, we dfine $\psi_\omega : C^* \otimes M \longrightarrow M$ by

$$\psi_\omega = \phi \, (\gamma \otimes id_M) \, (id_{C^*} \otimes \tau_{M \otimes C}) \, (id_{C^*} \otimes \omega)$$

where $\gamma : C^* \otimes C \longrightarrow \mathbb{k}$ is defined by $\gamma \, (x^* \otimes x) = x^* \, (x)$, and $\phi : \mathbb{k} \otimes M \longrightarrow M$ is the canonical isomorphism. If $\omega \, (m) = \sum_i m_i \otimes x_i$, then $\psi_\omega \, (x^* \otimes m) = \sum_i x^* \, (x_i) \, m_i$.

Proposition 3.3.1 *Let* $(C, \Delta, \varepsilon, \beta)$*be a counital Hom-coassociative coalgebra.* (M, ω, β_M) *is a right* C*-comodule if and only if* $(M, \psi_\omega, \alpha_M)$ *is a left* C^**-module where* $\alpha_M = \beta_M$*.*

Proof. From the previous Proposition we know that $(C^*, \mu_{C^*}, \eta_{C^*}, \alpha_{C^*})$ is a unital Hom-associative algebra when we define $\mu_{C^*}, \eta_{C^*}, \alpha_{C^*}$, as in Theorem 2.3.3.

Assume that (M, ω, β_M) is a right C-comodule. Denoting $\omega(m) = \sum\limits_{(m)} m_{(0)} \otimes x_{(1)}$, we have

$$\psi_\omega (x^* \otimes m) = \sum x^* \left(x_{(1)}\right) m_{(0)}.$$

First, we have that

$$\psi_\omega (1_{C^*} \otimes m) = \psi_\omega (\varepsilon_C \otimes m) = \phi (\gamma \otimes id_M) (id_{C^*} \otimes \tau_{M \otimes C}) (id_{C^*} \otimes \omega) (\varepsilon_C \otimes m)$$
$$\overset{(3.8)}{=} \sum \varepsilon_C \left(x_{(1)}\right) m_{(0)} = (id_M \otimes \varepsilon_C) \circ \omega (m) = \beta_M (m) = \alpha_M (m)$$

from the definition of a right C-comodule. Then, for x^*, $y^* \in C^*$, $m \in M$

$$\begin{aligned}
\psi_\omega \circ (\alpha_{C^*} \otimes \psi_\omega) (x^* \otimes y^* \otimes m) &= \psi_\omega \circ (\alpha_{C^*} (x^*) \otimes \psi_\omega (y^* \otimes m)) \\
&= \psi_\omega \circ \left(x^* (\beta_C) \otimes \sum y^* \left(x_{(1)}\right) m_{(0)}\right) \\
&= \sum y^* \left(x_{(1)}\right) \psi_\omega \left(x^* (\beta_C) \otimes m_{(0)}\right) \\
&= \sum y^* \left(x_{(1)}\right) x^* (\beta_C) \left(x_{(2)}\right) \left(m_{(0)}\right)_{(0)} \\
&= \sum x^* \circ \beta_C \left(x_{(2)}\right) y^* \left(x_{(1)}\right) \left(m_{(0)}\right) \\
&\overset{(3.8)}{=} \sum \rho (x^* \otimes y^*) \left(\beta_C \left(x_{(2)}\right) \otimes x_{(1)} \otimes m_{(0)}\right) \\
&= \sum \rho (x^* \otimes y^*) \left(x_{(2)} \otimes x_{(1)} \otimes \beta_M \left(m_{(0)}\right)\right) \\
&= \sum \Delta^* \rho (x^* \otimes y^*) x_{(1)} \otimes \beta_M \left(m_{(0)}\right) \\
&= \psi_\omega \circ (\mu_{C^*} \otimes \alpha_M) (x^* \otimes y^* \otimes m)
\end{aligned}$$

which shows that $(M^*, \psi_\omega, \alpha_M)$ is a left C^*-module. Assume now that $(M^*, \psi_\omega, \alpha_M)$ is a left C^*-module. From $\psi_\omega \circ (\eta_{C^*} \otimes id_{M^*}) = \alpha_{M^*}$, one obtains $\sum \varepsilon_C \left(x_{(1)}\right) m_{(0)} = \alpha_M (m)$.

It follows

$$(id_M \otimes \varepsilon_C) \circ \omega (m) = \sum \varepsilon_C \left(x_{(1)}\right) m_{(0)} = \alpha_M (m) = \beta_M (m),$$

Hence, the second condition from the definition of a right C-comodule is checked.

If x^*, $y^* \in C^*$, $m \in M$ then

$$\begin{aligned}
\psi_\omega \circ (\mu_{C^*} \otimes \alpha_M)(x^* \otimes y^* \otimes m) &= \psi_\omega \circ (\Delta^* \circ \rho(x^* \otimes y^*) \otimes \alpha_M(m)) \\
&= \sum \Delta_C^* \circ \rho(x^* \otimes y^*)\left(x_{(1)}\right) \alpha_M(m)_{(0)} \\
&= \sum \left(x^*\left(x_{(1)}\right)_{(1)} y^*\left(x_{(1)}\right)_{(2)}\right) \alpha_M(m)_{(0)} \\
&= \phi'(id_M \otimes y^* \otimes x^*)(\beta_M \otimes \Delta_C)\omega(m)
\end{aligned}$$

where $\phi' : M \otimes \mathbb{k} \otimes \mathbb{k} \longrightarrow M$ is the canonical isomorphism, and $\alpha_M = \beta_M$. Also

$$\begin{aligned}
\psi_\omega \circ (\alpha_{C^*} \otimes \psi_\omega)(x^* \otimes y^* \otimes m) &= \psi_\omega \circ (\alpha_{C^*}(x^*) \otimes \psi_\omega(y^* \otimes m)) \\
&= \psi_\omega \circ \left(\alpha_{C^*}(x^*) \otimes \sum y^*\left(x_{(1)}\right) m_{(0)}\right) \\
&= \sum y^*\left(x_{(1)}\right) \psi_\omega \circ \left(x^*(\beta_C) \otimes m_{(0)}\right) \\
&= \sum y^*\left(x_{(1)}\right) x^*\left(\beta_C\left(x_{(2)}\right)\right) m_{(0)} \\
&= \phi'(id_M \otimes y^* \otimes x^*)(\omega \otimes \beta_C)\omega(m)
\end{aligned}$$

Denoting

$$z = (\beta_M \otimes \Delta_C)\omega(m) - (\omega \otimes \beta_C)\omega(m) \in M \otimes C \otimes C$$

we have $(id_M \otimes y^* \otimes x^*)(z) = 0$ for any $y^*, x^* \in C^*$. This shows that $z = 0$.

Indeed, if we denote by $(e_i)_i$ a basis of C, we can write $y = \sum_{i,j} m_{ij} \otimes e_i \otimes e_j$ for some $m_{ij} \in M$. Fix i_0 and j_0 and consider the maps $e_i^* \in C^*$ defined by $e_i^*(e_j) = \delta_{i,j}$ for any j. Then $m_{i_0 j_0} = \left(id_M \otimes e_{i_0}^* \otimes e_{j_0}^*\right)(z) = 0$, and from this we get $z = 0$

Then $(\beta_M \otimes \Delta_C)\omega(m) = (\omega \otimes \beta_C)\omega(m)$. The proof is similar to the opposite case. ■

Proposition 3.3.2 *Let A be a finite-dimentional unital Hom-associative algebra. If M is a left A-module, then M is a right A^*-comodule.*

Theorem 3.3.3 *Let $(C, \Delta, \varepsilon, \beta)$ be a counital Hom-coassociative coalgebra. Then for any right C-comodule M, M^* is a left C^*module. Conversely, Let (A, μ, η, α) be a finite-dimensional unital Hom-associative algebra. If N is a left A-module then N^* is a right A^*-comodule.*

Proof. First let

$$M \xrightarrow{\rho_r} M \otimes C$$

be the right C-comodule structure on M . Define

$$\lambda_l' : C^* \otimes M^* \xrightarrow{\rho} (M \otimes C)^* \xrightarrow{\rho_r^*} M^*$$

We want $\lambda_l' = \rho_r^* \circ \rho$ and the map ρ, $\rho\left(x^* \otimes m^*\right)(m \otimes x) = x^*(x)\, m^*(m)$ to satisfy the following equations:

$$\lambda_l' \circ \left(\alpha_{C^*} \otimes \lambda_l'\right) = \lambda_l' \circ \left(\mu_{C^*} \otimes \alpha_{M^*}\right) \quad and \quad \lambda_l' \circ \left(\eta_{C^*} \otimes id_{M^*}\right) = \alpha_{M^*}$$

Transposing the equations (3.8) will give the desired result. We show the first equation:

$$\begin{aligned}
\lambda_l' \circ \left(\alpha_{C^*} \otimes \lambda_l'\right)\left(x^* \otimes y^* \otimes m^*\right) &= \rho_r^* \circ \rho \circ \left(\alpha_{C^*} \otimes \rho_r^* \circ \rho\right)\left(x^* \otimes y^* \otimes m^*\right) \\
&= \rho_r^* \circ \rho \circ \left(x^*\left(\beta_C\right) \otimes \rho\left(y^* \otimes m^*\right) \rho_r\right) \\
&= \rho \circ \left(x^* \otimes \rho\left(y^* \otimes m^*\right)\right)\left(\beta_C \otimes \rho_r\right) \rho_r \\
&= \rho \circ \left(\rho\left(x^* \otimes y^*\right) \otimes m^*\right)\left(\Delta_C \otimes \beta_M\right) \rho_r \\
&= \rho_r^* \circ \rho \circ \left(\Delta_C^* \circ \rho\left(x^* \otimes y^*\right) \otimes \alpha_{M^*}\left(m^*\right)\right) \\
&= \lambda_l' \circ \left(\mu_{C^*} \otimes \alpha_{M^*}\left(x^* \otimes y^* \otimes m^*\right)\right)
\end{aligned}$$

The second equation follows:

$$\begin{aligned}
\lambda_l' \circ \left(1_{C^*} \otimes id_{M^*}\right)\left(m^*\right) &= \rho_r^* \circ \rho \circ \left(\varepsilon_C \otimes m^*\right) = \rho \circ \left(\varepsilon_C \otimes m^*\right) \rho_r \\
&= m^*\left(id_M \otimes \varepsilon_C\right) \circ \rho_r = m^* \circ \beta_M = \alpha_{M^*}\left(m^*\right)
\end{aligned}$$

Then λ_l' gives a left A^*-module on M^*.

To go the other way, let

$$A \otimes N \xrightarrow{\lambda_l} N$$

be the left A-module structure on N. Define

$$\rho_r' : N^* \xrightarrow{\rho} (A \otimes N)^* \xrightarrow{\rho_r^*} N^* \otimes A^*$$

Then ρ_r' gives a A^*comodule on N^* . The proof is similar to the opposite case. ∎

3.4 Tensor product of bimodules and bicomodules

We will need to consider tensor products over $\Bbbk$ of bimodules. The tensor product of bimodules can be endowed with bimodule structures; if M and N are bimodules over A; we shall consider two bimodule structures on $M \otimes N$ (dual of each other), which we will denote by $M\overline{\otimes}N$ and $M\underline{\otimes}N$ (see for example [48] for details). These notations will also be used for the tensor product of bimodules or bicomodules, 'forgetting' some of the structures.

Proposition 3.4.1 *Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra and $(M, \lambda_l, \lambda_r)$, $(M', \lambda'_l, \lambda'_r)$ be H-bimodules.The internal (bimodule) tensor product of M with M' is the so called interior H-bimodule $M\overline{\otimes}M' = (M \otimes M', \lambda_l\overline{\otimes}\lambda'_l, \lambda_r\overline{\otimes}\lambda'_r)$ with*

$$\lambda_l^2: \quad H \otimes (M \otimes M') \quad \longrightarrow \quad M \otimes M'$$

$$\lambda_l^2 = \lambda_l\overline{\otimes}\lambda'_l = \left(\lambda_l \otimes \lambda'_l\right) \circ (id_H \otimes \tau_{H,M} \otimes id_{M'}) \circ (\Delta_H \otimes id_M \otimes id_{M'}) \tag{3.14}$$

and

$$\lambda_r^2: \quad (M \otimes M') \otimes H \quad \longrightarrow \quad M \otimes M'$$

$$\lambda_r^2 = \lambda_r\overline{\otimes}\lambda'_r = \left(\lambda_r \otimes \lambda'_r\right) \circ \left(id_M \otimes \tau_{M',H} \otimes id_H\right) \circ (id_M \otimes id_{M'} \otimes \Delta_H). \tag{3.15}$$

Proof. It is already shown in [51].

To see that λ_l^2 is a morphism of modules, we compute as follows:

$$\begin{aligned}
(\alpha_M \otimes \alpha_{M'}) \circ \lambda_l^2 &= (\alpha_M \otimes \alpha_{M'}) \circ \left(\lambda_l \otimes \lambda'_l\right) \circ (id_H \otimes \tau_{H,M} \otimes id_{M'}) \circ (\Delta_H \otimes id_M \otimes id_{M'}) \\
&\overset{(3.6)}{=} \left(\lambda_l \otimes \lambda'_l\right) (\alpha_H \otimes \alpha_M \otimes \alpha_H \otimes \alpha_{M'}) \circ (id_H \otimes \tau_{H,M} \otimes id_{M'}) \circ (\Delta_H \otimes id_M \otimes id_{M'}) \\
&= \left(\lambda_l \otimes \lambda'_l\right) \circ (id_H \otimes \tau_{H,M} \otimes id_{M'}) \circ (\Delta_H \otimes id_M \otimes id_{M'}) \circ (\alpha_H \otimes \alpha_M \otimes \alpha_{M'}) \\
&= \lambda_l^2 \circ (\alpha_H \otimes \alpha_M \otimes \alpha_{M'})
\end{aligned}$$

To see that λ_l^2 is a morphism of modules, we compute as follows, where some obvious subscripts have been left out:

$$\lambda_l^2 \circ \left(\alpha_H \otimes \lambda_l^2\right)$$

$$= \left(\lambda_l \otimes \lambda_l'\right)\left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right)\left(\Delta_H \otimes id_{M\otimes M'}\right)\left[\alpha_H \otimes \left(\lambda_l \otimes \lambda_l'\right)\left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right)\left(\Delta_H \otimes id_{M\otimes M'}\right)\right]$$

$$= \left(\lambda_l \otimes \lambda_l'\right)\left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right)\left(\alpha_H^{\otimes 2} \otimes \left(\lambda_l \otimes \lambda_l'\right)\right)\left[\Delta_H \otimes \left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right)\left(\Delta_H \otimes id_{M\otimes M'}\right)\right]$$

$$= \left(\lambda_l\left(\alpha_H \otimes \lambda_l\right) \otimes \lambda_l'\left(\alpha_H \otimes \lambda_l'\right)\right)\left[\Delta_H \otimes \left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right) \circ \left(\Delta_H \otimes id_{M\otimes M'}\right)\right]$$

$$\overset{(3.1)}{=} \left(\lambda_l\left(\mu_H \otimes \alpha_M\right) \otimes \lambda_l'\left(\mu_H \otimes \alpha_{M'}\right)\right)\left[\Delta_H \otimes \left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right) \circ \left(\Delta_H \otimes id_{M\otimes M'}\right)\right]$$

$$= \left(\lambda_l \otimes \lambda_l'\right)\left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right)\left(\mu_H^{\otimes 2} \otimes \alpha_{M\otimes M'}\right)\left[\Delta_H \otimes \left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right) \circ \left(\Delta_H \otimes id_{M\otimes M'}\right)\right]$$

$$= \left(\lambda_l \otimes \lambda_l'\right)\left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right)\left(\mu_H^{\otimes 2} \otimes id_{M\otimes M'}\right)\left(id_H \otimes \tau_{H,H} \otimes id_H \otimes id_{M\otimes M'}\right) \circ \left(\Delta_H^{\otimes 2} \otimes \alpha_{M\otimes M'}\right)$$

$$\overset{(2.16)}{=} \left(\lambda_l \otimes \lambda_l'\right)\left(id_H \otimes \tau_{H,M} \otimes id_{M'}\right) \circ \left(\Delta_H \otimes id_{M\otimes M'}\right) \circ \left(\mu_H \otimes \alpha_{M\otimes M'}\right)$$

$$= \lambda_l^2 \circ \left(\mu_H \otimes \alpha_{M\otimes M'}\right).$$

We have shown that λ_l^2 is the structure map of a left H-module structure on $M \otimes M'$.

■

Remark 3.4.2 *Since Δ_H is Hom-coassociative,*

$$\left(\lambda_l \overline{\otimes} \lambda_l'\right) \overline{\otimes} \alpha_H\left(\lambda_l''\right) = \alpha_H\left(\lambda_l\right) \overline{\otimes}\left(\lambda_l' \overline{\otimes} \lambda_l''\right)$$

and

$$\left(\lambda_r \overline{\otimes} \lambda_r'\right) \overline{\otimes} \alpha_H\left(\lambda_r''\right) = \alpha_H\left(\lambda_r\right) \overline{\otimes}\left(\lambda_r' \overline{\otimes}\left(\lambda_r''\right)\right)$$

Thus, the internal tensor product can be Hom-associatively applied to any finite family of H-bimodules.

Proof. We use the Hom-coassociativity axiom (2.7). We have

$$\left(\lambda_l\overline{\otimes}\lambda_l'\right)\overline{\otimes}\alpha_H\left(\lambda_l''\right)=\left(\left(\lambda_l\overline{\otimes}\lambda_l'\right)\otimes\alpha_H\left(\lambda_l''\right)\right)\circ\tau_{2,3}\circ\left(\Delta_H\otimes id_M^{\otimes 3}\right)$$
$$=\left[\left(\left(\lambda_l\otimes\lambda_l'\right)\circ\tau_{2,3}\circ\left(\Delta_H\otimes id_{M\otimes M'}\right)\right)\otimes\alpha_H\left(\lambda_l''\right)\right]\circ\tau_{2,3}\circ\left(\Delta_H\otimes id_{M\otimes M'\otimes M''}\right)$$
$$=\left[\left(\lambda_l\otimes\lambda_l'\right)\otimes\lambda_l''\right]\circ\left[\tau_{2,3}\circ\left(\Delta_H\otimes id_{M\otimes M'}\right)\otimes\alpha_{H\otimes M''}\right]\circ\tau_{2,3}\circ\left(\Delta_H\otimes id_{M\otimes M'\otimes M''}\right)$$
$$=\left[\lambda_l\otimes\left(\lambda_l'\otimes\lambda_l''\right)\right]\circ\left[\alpha_{H\otimes M}\otimes\tau_{2,3}\circ\left(\Delta_H\otimes id_{M'\otimes M''}\right)\right]\circ\tau_{2,3}\circ\left(\Delta_H\otimes id_{M\otimes M'\otimes M''}\right)$$
$$=\left(\alpha_H\left(\lambda_l\right)\otimes\left(\lambda_l'\overline{\otimes}\lambda_l''\right)\right)\circ\tau_{2,3}\circ\left(\Delta_H\otimes id_{M\otimes M'\otimes M''}\right)$$
$$=\alpha_H\left(\lambda_l\right)\overline{\otimes}\left(\lambda_l'\overline{\otimes}\lambda_l''\right).$$

The second assertion is proved similarly. ■

Corollary 3.4.3 *Let (M,λ_l,λ_r)be an H-bimodule. The structure maps $\overline{\lambda}_l^{\mu}=\mu\overline{\otimes}\lambda_l\overline{\otimes}\mu$ and $\overline{\lambda}_r^{\mu}=\mu\overline{\otimes}\lambda_r\overline{\otimes}\mu$ on the interior H-bimodule $H\overline{\otimes}M\overline{\otimes}H$ are called the two-sided interior extensions of λ_l and λ_r by μ respectively, by μ.*

Proposition 3.4.4 *Let (M,λ_l,λ_r)be an H-bimodule. The interior H-bimodule $M^{\overline{\otimes}n}=(M^{\otimes n},\lambda_l^n,\lambda_r^n)$, with*

$$\lambda_l^n=\left(\lambda_l\overline{\otimes}\lambda_l^{n-1}\right)=\left(\lambda_l\otimes\lambda_l^{n-1}\right)\circ\tau_{2,3}\circ\left(\Delta\otimes id_M^{\otimes n}\right)\tag{3.16}$$

and

$$\lambda_r^n=\left(\lambda_r^{n-1}\overline{\otimes}\lambda_r\right)=\left(\lambda_r^{n-1}\otimes\lambda_r\right)\circ\tau_{n,n+1}\circ\left(id_M^{\otimes n}\otimes\Delta\right)\tag{3.17}$$

is called the n-fold interior (bimodule) tensor power of M.

Proof. By induction. ■

Example 3.4.5 *Let $(H,\mu_H,\Delta_H,\alpha_H)$ be a Hom-bialgebra; for each $n\geq 1$, the n-fold interior (bimodule) tensor power of H is the interior H-bimodule $H^{\overline{\otimes}n}=(H^{\otimes n},\lambda_l^n,\lambda_r^n)$ with*

$$\lambda_l^n=\mu_H^{\otimes n}\circ(135...(2n-1)\,246...(2n))\circ\prod_{i=n}^{2n-2}\left(\Delta_H\otimes id_M^{\otimes(3n-i-2)}\right)\tag{3.18}$$

and

$$\lambda_r^n=\mu_H^{\otimes n}\circ(135...(2n-1)\,246...(2n))\circ\prod_{i=n}^{2n-2}\left(id_M^{\otimes(3n-i-2)}\otimes\Delta_H\right)\tag{3.19}$$

where $\lambda_l=\lambda_r=\mu_H$.

Proposition 3.4.6 *Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra and (N, ρ_l, ρ_r), (N', ρ'_l, ρ'_r) be H-bicomodules. The internal (bicomodule) tensor product of N with N' is the so-called interior H-bicomodule $N\underline{\otimes}N' = (N \otimes N', \rho_l\underline{\otimes}\rho'_l, \rho_r\underline{\otimes}\rho'_r)$ with*

$$\rho_l^2 = \rho_l\underline{\otimes}\rho'_l = (\mu_H \otimes id_{N\otimes N'}) \circ \tau_{2,3} \circ \left(\rho_l \otimes \rho'_l\right) \tag{3.20}$$

and

$$\rho_r^2 = \rho_r\underline{\otimes}\rho'_r = (id_{N\otimes N'} \otimes \mu_H) \circ \tau_{2,3} \circ \left(\rho_r \otimes \rho'_r\right). \tag{3.21}$$

Remark 3.4.7 *Since μ_H is Hom-associative, we have*

$$\left(\rho_l\underline{\otimes}\rho'_l\right)\underline{\otimes}\alpha\left(\rho''_l\right) = \alpha\left(\rho_l\right)\underline{\otimes}\left(\rho'_l\underline{\otimes}\rho''_l\right) \text{ and } \left(\rho_r\underline{\otimes}\rho'_r\right)\underline{\otimes}\alpha\left(\rho''_r\right) = \alpha\left(\rho_r\right)\underline{\otimes}\left(\rho'_r\underline{\otimes}\rho''_r\right).$$

Thus, the internal tensor product can be Hom-associatively applied to any finite family of H-bicomodules.

Corollary 3.4.8 *Let (N, ρ_l, ρ_r) be an H-bicomodule. The structure maps $\overline{\rho}_l^{\Delta} = \Delta\underline{\otimes}\rho_l\underline{\otimes}\Delta$ and $\overline{\rho}_r^{\Delta} = \Delta\underline{\otimes}\rho_r\underline{\otimes}\Delta$ on the interior H-bicomodule $H\underline{\otimes}N\underline{\otimes}H$ are called the two-sided interior extensions of ρ_l and ρ_r by Δ, respectively.*

Proposition 3.4.9 *Let (N, ρ_l, ρ_r)be an H-bicomodule. The interior H-bicomodule $N^{\underline{\otimes}n} = (N^{\otimes n}, \rho_l^n, \rho_r^n)$, with*

$$\rho_l^n = \rho_l\underline{\otimes}\rho_l^{n-1} = \left(\mu \otimes id_N^{\otimes n}\right) \circ \tau_{2,3} \circ \left(\rho_l \otimes \rho_l^{n-1}\right) \tag{3.22}$$

and

$$\rho_r^n = \rho_r^{n-1}\underline{\otimes}\rho_r = \left(id_N^{\otimes n} \otimes \mu\right) \circ \tau_{n,n+1} \circ \left(\rho_r^{n-1} \otimes \rho_r\right) \tag{3.23}$$

is called the n-fold internal (bicomodule) tensor power of N.

Proof. By induction. ■

Example 3.4.10 *Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra, consider the H-bicomodule (H, ρ_l, ρ_r) where $\rho_l = \rho_r = \Delta_H$. For each $n \geq 1$, the n-fold internal (bicomodule) tensor power of H is the interior H-bicomodule $H^{\underline{\otimes}n} = (H^{\otimes n}, \rho_l^n, \rho_r^n)$ with*

$$\rho_l^n = \prod_{i=n}^{2n-2} \left(\mu_H \otimes id_N^{\otimes i}\right) \circ (135...(2n-1)\,246...(2n))^{-1} \circ \Delta_H^{\otimes n}. \tag{3.24}$$

$$\rho_r^n = \prod_{i=n}^{2n-2} \left(id_N^{\otimes i} \otimes \mu_H\right) \circ \left(135...\left(2n-1\right)246...\left(2n\right)\right)^{-1} \circ \Delta_H^{\otimes n}. \tag{3.25}$$

Lemma 3.4.11 *Consider the interiors H-bicomodules, $H^{\overline{\otimes} n} = (H^{\otimes n}, \lambda_l^n, \lambda_r^n)$ and $H^{\underline{\otimes} n} = (H^{\otimes n}, \rho_l^n, \rho_r^n)$ with $\lambda_l^1 = \lambda_l^1 = \mu$, $\rho_l^1 = \rho_l^1 = \Delta$ and let $f : H^{\otimes p} \to H^{\otimes p}$ be a linear map which commutes with α and such that $\alpha^{\otimes p} \circ f = f \circ \alpha^{\otimes q}$, $p, q \in \mathbb{N}^*$ Then*

1. $\lambda_l^{p+1} \circ \left(\alpha^{p-1} \otimes \left(\alpha^{p-1} \otimes f\right) \circ \rho_l^p\right) = \left(\alpha^{p-1} \otimes \lambda_l^p \circ \left(\alpha^{p-1} \otimes f\right)\right) \circ \rho_l^{p+1}$
2. $\lambda_r^{p+1} \circ \left(\left(f \otimes \alpha^{p-1}\right) \circ \rho_r^p \otimes \alpha^{p-1}\right) = \left(\lambda_r^p \circ \left(f \otimes \alpha^{p-1}\right) \otimes \alpha^{p-1}\right) \circ \rho_r^{p+1}$.

This Lemma can be used in the proof of Proposition 4.2.1, and Theorem 4.2.5.

Chapter 4

Gerstenhaber-Schack Cohomology for Hom-bialgebras

Gerstenhaber-Schack cohomology of Hom-bialgebras is a twisted generalization of bialgebras cohomology, which was first discovered by Gerstenhaber and Schack [18, 19], extending associative algebras cohomology introduced by Hochschild in [24] to bialgebras. Deformation theories are intimately related to cohomology. It turns out that, we do not need a cohomology of Hom-Hopf algebras since it is enough to deform Hom-Hopf algebra as a Hom-bialgebra.

The cohomology of Hopf algebras was introduced in order to study deformations of Hopf algebras. In fact, the cohomology that we are going to study is adapted to deformations of Hom-Hopf algebras as Drinfel'd quasi-bialgebras; however, M. Gerstenhaber and S.D. Schack also defined a cohomology which studies the deformations of Hopf algebras as Hopf algebras. We refer to [18, 19] for the definition of bialgebra cohomology and its truncated version due to Gerstenhaber and Schack. We define the bicomplex extending Hochschild cohomology for horizontal faces and coalgebra Cartier cohomology for the vertical faces.

4.1 Hochschild Complexes

Definition 4.1.1 *A* ***chain complex*** $\mathcal{C}_-$ *is a sequence of abelian groups and homomorphisms*

$$\ldots \longrightarrow^{d_{n+1}} \mathcal{C}_n \longrightarrow^{d_n} \mathcal{C}_{n+1} \longrightarrow^{d_{n-1}} \mathcal{C}_{n+2} \ldots$$

with the property $d_{n+1} \circ d_n = 0$ *for all* n.

The homomorphisms d^n *are called coboundary operators or codifferentials.*

A cochain complex $\mathcal{C}^-$ *is a sequence of abelian groups and homomorphisms*

$$\ldots \longrightarrow^{d^{n-1}} \mathcal{C}^n \longrightarrow^{d^n} \mathcal{C}^{n+1} \longrightarrow^{d^{n+1}} \mathcal{C}^{n+2} \ldots$$

with the property $d^n \circ d^{n+1} = 0$ *for all* n.

A chain complex can be considered as a cochain complex by reversing the enumeration $\mathcal{C}^n = \mathcal{C}_{-n}$; $d^n = d_{-n}$.

A complex of A-Hom-modules is a complex for which $\mathcal{C}_n$ (respectively $\mathcal{C}^n$) are Hom-modules over a ring A and d_n (resp. d^n) are homomorphisms of Hom-modules.

Let $\operatorname{Im} d_{n+1}$ (resp. $\operatorname{Im} d^{n-1}$) be the image of $\mathcal{C}_{n+1}$ (resp. $\mathcal{C}^{n-1}$) by d_{n+1} (resp. d^{n-1}) and $\ker d_n$ (resp. $\ker d^n$) be the kernel of d_n (resp. d^n). Since $d_n \circ d_{n+1} = 0$ (resp. $d^n \circ d^{n-1} = 0$), we have $\operatorname{Im} d_{n+1} \subset \ker d_n$ (resp. $\operatorname{Im} d_n \subset \ker d_{n-1}$).

A homology of a chain complex $\mathcal{C}_n$ is the group $H_n(\mathcal{C}) = \ker d_n / \operatorname{Im} d_{n+1}$.

A cohomology of a chain complex $\mathcal{C}^n$ is the group $H^n(\mathcal{C}) = \ker d^n / \operatorname{Im} d^{n-1}$.

The elements of $\mathcal{C}_n$ are called n-dimensional chains, the elements of $\mathcal{C}^n$ are n-dimensional cochains, the elements of $Z_n := \ker d_n$ (resp. $Z^n := \ker d^n$) are n-dimensional cycles (resp. cocycles), the elements of $B_n := \operatorname{Im} d_{n+1}$ (resp. $B^n := \operatorname{Im} d^{n-1}$) are n-dimensional boundaries (resp. coboundaries).

If $\mathcal{C}^{\cdot}$ is a complex of A-Hom modules, its cohomology is an A-Hom-module. A complex is said to be acyclic (or an exact sequence) if $H^n(\mathcal{C}) = 0$ for all n.

A morphism $f : \mathcal{C}^{\cdot} \longrightarrow \mathcal{D}^{\cdot}$ is a family of group (Hom-module) homomorphisms $f^n : \mathcal{C}^n \longrightarrow \mathcal{D}^n$ commuting with differentials, that is $f^{n+1} \circ d^n_{\mathcal{C}} = d^n_{\mathcal{C}} \circ f^n$. A morphismn f induces a morphism of cohomology $H^-(f) = \{H^n(f) : H^n(\mathcal{C}^{\cdot}) \longrightarrow H^n(\mathcal{D}^{\cdot})\}$ by the formula $\{$the class of cocycle c $\}= \{$the class of cocycle $f(c)\}$.

4.2 Hochschild Cohomology for Hom-bialgebras

We recall the definition of Hom-bialgebra cohomology du to Gerstenhaber and Schack and its truncated version. For more details and greater generality we refer to ([18], [19]). Let $B = (B, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra on the $\Bbbk$-vector space B.

The **cochains** are given by the bicomplex.

We set for the cochains:

$$\mathcal{C}_{Hom}^{p,q} = Hom_{\Bbbk}\left(B^{\otimes q}, B^{\otimes p}\right), p, q \geq 1,$$

$$\mathcal{C}_{Hom}^{p,q} = \left\{f : B^{\otimes q} \longrightarrow B^{\otimes p}, f \textit{ is a linear map, } f \circ \alpha^{\otimes q} = \alpha^{\otimes p} \circ f\right\}.$$

We define the horizontal faces $\delta_{Hom,H}^{p,q} : \mathcal{C}_{Hom}^{p,q} \longrightarrow \mathcal{C}_{Hom}^{p,q+1}$ as

$$\delta_{Hom,H}^{p,q}(f) = \lambda_l^p \circ (\alpha^{q-1} \otimes f) + \sum_{i=1}^{q} (-1)^i f \circ \left(\alpha^{\otimes(i-1)} \otimes \mu \otimes \alpha^{\otimes(q-i)}\right) + (-1)^{q+1} \lambda_r^p \circ (f \otimes \alpha^{q-1}). \tag{4.1}$$

The vertical faces $\delta_{Hom,C}^{p,q} : \mathcal{C}_{Hom}^{p,q} \longrightarrow \mathcal{C}_{Hom}^{p+1,q}$ are defined as:

$$\delta_{Hom,C}^{p,q}(f) = (\alpha^{p-1} \otimes f) \circ \rho_l^q + \sum_{j=1}^{p} (-1)^j \left(\alpha^{\otimes(j-1)} \otimes \Delta \otimes \alpha^{\otimes(p-j)}\right) \circ f + (-1)^{p+1} (f \otimes \alpha^{p-1}) \circ \rho_r^q. \tag{4.2}$$

Proposition 4.2.1 *The composite*

$$\delta_{Hom,C}^{2,1} \circ \delta_{Hom,C}^{1,1} = 0, \quad \delta_{Hom,C}^{1,2} \circ \delta_{Hom,H}^{1,1} = \delta_{Hom,H}^{2,1} \circ \delta_{Hom,C}^{1,1}, \quad \delta_{Hom,H}^{1,2} \circ \delta_{Hom,H}^{1,1} = 0.$$

Proof. We prove the first identity. We have

$$\delta_{Hom,C}^{2,1}(f) = (\alpha \otimes f) \circ \rho_l^1 - (\Delta \otimes \alpha) \circ f + (\alpha \otimes \Delta) \circ f - (f \otimes \alpha) \circ \rho_r^1,$$

$\delta^{1,1}_{Hom,C}(f) = (id_B \otimes f) \circ \rho^1_l - \Delta \circ f + (f \otimes id_B) \circ \rho^1_r$, and $\rho^1_l = \rho^1_r = \Delta$

$$\begin{aligned}
\delta^{2,1}_{Hom,C} \circ \delta^{1,1}_{Hom,C}(f) &= (\alpha \otimes \delta^{1,1}_{Hom,C}(f)) \circ \Delta - (\Delta \otimes \alpha) \circ \delta^{1,1}_{Hom,C}(f) + (\alpha \otimes \Delta) \circ \delta^{1,1}_{Hom,C}(f) \\
&- (\delta^{1}_{Hom,C}(f) \otimes \alpha) \circ \Delta \\
&= (\alpha \otimes ((id_B \otimes f) \circ \Delta - \Delta \circ f + (f \otimes id_B) \circ \Delta)) \circ \Delta \\
&- (\Delta \otimes \alpha) \circ ((id_B \otimes f) \circ \Delta - \Delta \circ f + (f \otimes id_B) \circ \Delta) \\
&+ (\alpha \otimes \Delta) \circ ((id_B \otimes f) \circ \Delta - \Delta \circ f + (f \otimes id_B) \circ \Delta) \\
&- ((id_B \otimes f) \circ \Delta - \Delta \circ f + (f \otimes id_B) \circ \Delta) \otimes \alpha) \circ \Delta \\
&= (\alpha \otimes (id_B \otimes f) \circ \Delta) \circ \Delta - (\alpha \otimes (\Delta \circ f)) \circ \Delta + (\alpha \otimes (f \otimes id_B) \circ \Delta) \circ \Delta \\
&- (\Delta \otimes \alpha)(id_B \otimes f) \circ \Delta + (\Delta \otimes \alpha)(\Delta \circ f) - (\Delta \otimes \alpha)(f \otimes id_B) \circ \Delta \\
&+ (\alpha \otimes \Delta)(id_B \otimes f) \circ \Delta - (\alpha \otimes \Delta)(\Delta \circ f) + (\alpha \otimes \Delta)(f \otimes id_B) \circ \Delta \\
&- (((id_B \otimes f) \circ \Delta) \otimes \alpha) \circ \Delta + ((\Delta \circ f) \otimes \alpha) \circ \Delta - (((f \otimes id_B) \circ \Delta) \otimes \alpha) \circ \Delta \\
&\overset{(*)}{=} (\alpha \otimes (id_B \otimes f) \circ \Delta) \circ \Delta + (\alpha \otimes (f \otimes id_B) \circ \Delta) \circ \Delta \\
&- (\Delta \otimes \alpha)(id_B \otimes f) \circ \Delta + (\alpha \otimes \Delta)(f \otimes id_B) \circ \Delta \\
&- (((id_B \otimes f) \circ \Delta) \otimes \alpha) \circ \Delta - (((f \otimes id_B) \circ \Delta) \otimes \alpha) \circ \Delta,
\end{aligned}$$

where (*) was obtained by the Hom-coassociativity of Δ and Lemma 2.3.7.

From Lemma 2.3.9, we immediately obtain $\delta^{2,1}_{Hom,C} \circ \delta^{1,1}_{Hom,C}(f) = 0$.

For the second identity, we have

$\delta^{1,2}_{Hom,C}(f) = (id_B \otimes f) \circ \rho^2_l - \Delta \circ f + (f \otimes id_B) \circ \rho^2_r$,

$\delta^{1,1}_{Hom,C}(f) = (id_B \otimes f) \circ \rho^1_l - \Delta \circ f + (f \otimes id_B) \circ \rho^1_r$,

$\delta^{2,1}_{Hom,H}(f) = \lambda^2_l \circ (id_B \otimes f) - f \circ \mu + \lambda^2_r \circ (f \otimes id_B)$,

$$\delta^{1,1}_{Hom,H}(f) = \lambda^1_l \circ (id_B \otimes f) - f \circ \mu + \lambda^1_r \circ (f \otimes id_B), \text{ and } \rho^1_l = \rho^1_r = \Delta,\ \lambda^1_l = \lambda^1_r = \mu,$$

$$\begin{aligned}
\delta^{1,2}_{Hom,C} \circ \delta^{1,1}_{Hom,H}(f) &= (id_B \otimes (\mu \circ (id_B \otimes f) - f \circ \mu + \mu \circ (f \otimes id_B))) \circ \rho^2_l \\
&- \Delta \circ (\mu \circ (id_B \otimes f) - f \circ \mu + \mu \circ (f \otimes id_B)) \\
&+ ((\mu \circ (id_B \otimes f) - f \circ \mu + \mu \circ (f \otimes id_B)) \otimes id_B) \circ \rho^2_r \\
&= ((id_B \otimes \mu \circ (id_B \otimes f)) \circ \rho^2_l - (id_B \otimes f \circ \mu) \circ \rho^2_l + \\
&(id_B \otimes \mu \circ (f \otimes id_B)) \circ \rho^2_l - \Delta \circ \mu \circ (id_B \otimes f) - \Delta \circ f \circ \mu \\
&+ \Delta \circ \mu \circ (f \otimes id_B) + (\mu \circ (id_B \otimes f) \otimes id_B) \circ \rho^2_r \\
&- (f \circ \mu \otimes id_B) \circ \rho^2_r + (\mu \circ (f \otimes id_B) \otimes id_B) \circ \rho^2_r \\
&\overset{(**)}{=} \lambda^2_l \circ (id_B \otimes (id_B \otimes f) \circ \Delta) - (id_B \otimes f) \circ \Delta \circ \mu + \\
&\lambda^2_r \circ (((id_B \otimes f) \circ \Delta) \otimes id_B)) - \lambda^2_l \circ (id_B \otimes (\Delta \circ f)) \\
&- \Delta \circ f \circ \mu + \lambda^2_r \circ ((\Delta \circ f) \otimes id_B)) + \lambda^2_l \circ (id_B \otimes (f \otimes id_B) \circ \Delta) \\
&- (f \otimes id_B) \circ \Delta \circ \mu + \lambda^2_r \circ ((f \otimes id_B) \circ \Delta \otimes id_B) \\
&= \delta^{2,1}_{Hom,H} \circ \delta^{1,1}_{Hom,C}(f),
\end{aligned}$$

where $(**)$ is obtained by the compatibility condition and Lemma 3.4.11.

For the third identity see ([2]). ■

Proposition 4.2.2 *([2]) Let* $D^{p,q}_i : \mathcal{C}^{p,q}_{Hom}(B^{\otimes q}, B^{\otimes p}) \longrightarrow \mathcal{C}^{p,q+1}_{Hom}\left(B^{\otimes q+1}, B^{\otimes p}\right)$ *be linear operators defined for* $f \in \mathcal{C}^{p,q}_{Hom}(B^{\otimes q}, B^{\otimes p})$ *by:*

$$D^{p,q}_i(f) = \begin{cases} -\lambda^p_l \circ (\alpha^{q-1} \otimes f) + f \circ \left(\mu \otimes \alpha^{\otimes(q-1)}\right) & if \quad i = 0. \\ f \circ \left(\alpha^{\otimes i} \otimes \mu \otimes \alpha^{\otimes(q-i-1)}\right) & if \quad \forall 1 \leq i \leq q-2. \\ f \circ \left(\alpha^{\otimes q-1} \otimes \mu\right) - \lambda^p_r \circ (f \otimes \alpha^{q-1}) & if \quad i = q-1. \end{cases} \tag{4.3}$$

Then

$$D^{p,q+1}_i \circ D^{p;q}_j = D^{p,q+1}_{j+1} \circ D^{p,q}_i \quad 0 \leq i < j \leq q-1, \ \text{and} \ \ \delta^{p,q}_{Hom,H} = \sum_{i=0}^{q-1} (-1)^{i+1} D^{p,q}_i. \tag{4.4}$$

Proof. See ([2]) ■

Proposition 4.2.3 *Let $S_i^{p,q}:\mathcal{C}_{Hom}^{p,q}\left(B^{\otimes q},B^{\otimes p}\right)\longrightarrow\mathcal{C}_{Hom}^{p+1,q}\left(B^{\otimes q},B^{\otimes p+1}\right)$ be the linear operators defined for $g\in\mathcal{C}_{Hom}^{p,q}\left(B^{\otimes q},B^{\otimes p}\right)$ by:*

$$S_i^{p,q}(g)=\begin{cases}-(\alpha^{p-1}\otimes g)\circ\rho_l^q+\left(\Delta\otimes\alpha^{\otimes(p-1)}\right)\circ g & if \quad i=0.\\ \left(\alpha^{\otimes i}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\circ g \quad if \quad \forall 1\leq i\leq p-2. & \\ \left(\alpha^{\otimes p-1}\otimes\Delta\right)\circ g-(g\otimes\alpha^{p-1})\circ\rho_r^q \quad if \quad i=p-1. & \end{cases}\tag{4.5}$$

Then

$$S_j^{p+1,q}\circ S_i^{p,q}=S_{i+1}^{p+1,q}\circ S_j^{p,q}\quad 0\leq i<j\leq p-1,\ and\ \ \delta_{Hom,C}^{p,q}=\sum_{i=0}^{p-1}(-1)^{i+1}S_i^{p,q}.\tag{4.6}$$

Proof. Let $g\in\mathcal{C}_{Hom}^{p,q}\left(B^{\otimes q},B^{\otimes p}\right)$.

· When the $j=0$ and $1\leq i\leq p-2$, we have

$S_0^{p+1,q}=-\left(\alpha^p\otimes g\right)\circ\rho_l^q+\left(\Delta\otimes\alpha^{\otimes p}\right)\circ g,\ S_i^{p,q}(g)=\left(\alpha^{\otimes i}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\circ g.$

The left-hand side in 4.6 is:

$$\begin{aligned}\left(S_0^{p+1,q}\circ S_i^{p,q}\right)(g)&=-\left(\alpha^p\otimes S_i^{p,q}(g)\right)\circ\rho_l^q+\left(\Delta\otimes\alpha^{\otimes p}\right)\circ S_i^{p,q}(g)\\&=-\left(\alpha^p\otimes\left(\left(\alpha^{\otimes i}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\circ g\right)\right)\circ\rho_l^q\\&+\left(\Delta\otimes\alpha^{\otimes p}\right)\circ\left(\left(\alpha^{\otimes i}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\circ g\right)\\&=-\left(\alpha^p\otimes\left(\left(\alpha^{\otimes i}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\circ g\right)\right)\circ\rho_l^q\\&+\left(\Delta\circ\alpha\otimes(\alpha\circ\alpha)^{\otimes i-1}\otimes\alpha^{\otimes 2}\circ\Delta\otimes(\alpha\circ\alpha)^{\otimes(p-i-1)}\right)\circ g.\end{aligned}$$

On the other hand, we have

$S_{i+1}^{p+1,q}=\left(\alpha^{\otimes i+1}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\circ g,\ S_0^{p,q}(g)=-\left(\alpha^{p-1}\otimes g\right)\circ\rho_l^q+\left(\Delta\otimes\alpha^{\otimes(p-1)}\right)\circ g.$

$$\begin{aligned}\left(S_{i+1}^{p+1,q}\circ S_0^{p,q}\right)(g)&=\left(\alpha^{\otimes i+1}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\circ S_0^{p,q}(g)\\&=\left(\alpha^{\otimes i+1}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\left(-\left(\alpha^{p-1}\otimes g\right)\circ\rho_l^q+\left(\Delta\otimes\alpha^{\otimes(p-1)}\right)\circ g\right)\\&=-\left(\alpha^p\otimes\left(\left(\alpha^{\otimes i}\otimes\Delta\otimes\alpha^{\otimes(p-i-1)}\right)\circ g\right)\right)\circ\rho_l^q\\&+\left(\alpha^{\otimes 2}\circ\Delta\otimes(\alpha\circ\alpha)^{\otimes i-1}\otimes\Delta\circ\alpha\otimes(\alpha\circ\alpha)^{\otimes(p-i-1)}\right)\circ g.\end{aligned}$$

So $\left(S_0^{p+1,q}\circ S_i^{p,q}\right)(g)=\left(S_{i+1}^{p+1,q}\circ S_0^{p,q}\right)(g)$.

· The case $(j,i)=(0,p-1)$, the following prouves is an immediate of presidente computation, we have

$S_0^{p+1,q}(g) = -(\alpha^p \otimes g) \circ \rho_l^q - (\Delta \otimes \alpha^{\otimes p}) \circ g$, $S_{p-1}^{p,q}(g) = \left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g - \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q$,

The left-hand side in 4.6 is:

$$\begin{aligned}
S_0^{p+1,q} \circ S_{p-1}^{p,q}(g) &= -\left(\alpha^p \otimes S_{p-1}^{p,q}(g)\right) \circ \rho_l^q - \left(\Delta \otimes \alpha^{\otimes p}\right) \circ S_{p-1}^{p,q}(g) \\
&= -\left(\alpha^p \otimes \left(\left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g - \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q\right)\right) \circ \rho_l^q \\
&+ \left(\Delta \otimes \alpha^{\otimes p}\right) \circ \left(\left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g - \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q\right) \\
&= -\left(\alpha^p \otimes \left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g\right) \circ \rho_l^q + \left(\alpha^p \otimes \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q\right) \circ \rho_l^q \\
&+ \left(\left(\Delta \otimes \alpha^{\otimes p}\right) \circ \left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g\right) - \left(\Delta \otimes \alpha^{\otimes p}\right) \circ \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q \\
&= -\left(\alpha^p \otimes \left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g\right) \circ \rho_l^q + \left(\alpha^p \otimes \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q\right) \circ \rho_l^q \\
&+ \left(\left(\Delta(\alpha) \otimes (\alpha \circ \alpha)^{\otimes p-2} \otimes \alpha^{\otimes 2}(\Delta)\right) \circ g\right) - \left(\left(\left(\Delta \otimes \alpha^{\otimes p-1}\right) \circ g\right) \otimes \alpha\left(\alpha^{p-1}\right)\right) \circ \rho_r^q \\
&= -\left(\alpha^p \otimes \left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g\right) \circ \rho_l^q + \left(\alpha^p \otimes \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q\right) \circ \rho_l^q \\
&+ \left(\left(\Delta \circ \alpha \otimes (\alpha \circ \alpha)^{\otimes p-2} \otimes \Delta \circ \alpha\right) \circ g\right) - \left(\left(\left(\Delta \otimes \alpha^{\otimes p-1}\right) \circ g\right) \otimes \alpha^p\right) \circ \rho_r^q.
\end{aligned}$$

On the other hand, we have

$S_p^{p+1,q}(g) = (\alpha^{\otimes p} \otimes \Delta) \circ g - (g \otimes \alpha^p) \circ \rho_r^q$, $S_0^{p,q}(g) = -\left(\alpha^{p-1} \otimes g\right) \circ \rho_l^q + \left(\Delta \otimes \alpha^{\otimes p-1}\right) \circ g$.

So

$$\begin{aligned}
S_p^{p+1,q} \circ S_0^{p,q}(g) &= \left(\alpha^{\otimes p} \otimes \Delta\right) \circ S_0^{p,q}(g) - \left(S_0^{p,q}(g) \otimes \alpha^p\right) \circ \rho_r^q \\
&= \left(\alpha^{\otimes p} \otimes \Delta\right) \circ \left(-\left(\alpha^{p-1} \otimes g\right) \circ \rho_l^q + \left(\Delta \otimes \alpha^{\otimes p-1}\right) \circ g\right) \\
&- \left(\left(-\left(\alpha^{p-1} \otimes g\right) \circ \rho_l^q + \left(\Delta \otimes \alpha^{\otimes p-1}\right) \circ g\right) \otimes \alpha^p\right) \circ \rho_r^q \\
&= -\left(\left(\alpha^{\otimes p} \otimes \Delta\right) \circ \left(\alpha^{p-1} \otimes g\right) \circ \rho_l^q\right) + \left(\alpha^{\otimes p} \otimes \Delta\right) \circ \left(\left(\Delta \otimes \alpha^{\otimes p-1}\right) \circ g\right) \\
&+ \left(\left(\left(\alpha^{p-1} \otimes g\right) \circ \rho_l^q\right) \otimes \alpha^p\right) \circ \rho_r^q - \left(\left(\left(\Delta \otimes \alpha^{\otimes p-1}\right) \circ g\right) \otimes \alpha^p\right) \circ \rho_r^q \\
&\overset{(***)}{=} -\left(\alpha^p \otimes \left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g\right) \circ \rho_l^q + \left(\Delta \circ \alpha \otimes (\alpha \circ \alpha)^{\otimes p-2} \otimes \Delta \circ \alpha\right) \circ g \\
&+ \left(\alpha^p \otimes \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q\right) \circ \rho_l^q - \left(\left(\left(\Delta \otimes \alpha^{\otimes p-1}\right) \circ g\right) \otimes \alpha^p\right) \circ \rho_r^q.
\end{aligned}$$

The equality (***) is given by the relation in Remark 2.3.10, and $\left(M, \lambda_l^q, \lambda_r^q\right)$ is a B-bimodules.

So $S_0^{p+1,q} \circ S_{p-1}^{p,q}(g) = S_p^{p+1,q} \circ S_0^{p,q}(g)$.

· The case when $1 \leq j \leq p-2$, $1 \leq i \leq p-2$ and $i < j$ we have

$S_j^{p+1,q}(g) = \left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{\otimes(p-j)}\right) \circ g,\ S_i^{p,q}(g) = \left(\alpha^{\otimes i} \otimes \Delta \otimes \alpha^{\otimes(p-i-1)}\right) \circ g$

The left hand, we have

$$\begin{aligned} S_j^{p+1,q} \circ S_i^{p,q}(g) &= \left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{\otimes(p-j)}\right) \circ \left(\alpha^{\otimes i} \otimes \Delta \otimes \alpha^{\otimes(p-i-1)}\right) \circ g \\ &= \left((\alpha \circ \alpha)^{\otimes j} \otimes \Delta(\alpha) \otimes (\alpha \circ \alpha)^{\otimes i-j-1} \otimes \alpha^{\otimes 2} \circ \Delta \otimes (\alpha \circ \alpha)^{\otimes(p-i-1)}\right) \circ g. \end{aligned}$$

The right hand, we have

$S_{i+1}^{p+1,q}(g) = \left(\alpha^{\otimes i+1} \otimes \Delta \otimes \alpha^{\otimes(p-i-1)}\right) \circ g.$

$$\begin{aligned} S_{i+1}^{p+1,q} \circ S_j^{p,q}(g) &= \left(\alpha^{i+1} \otimes \Delta \otimes \alpha^{\otimes(p-i-1)}\right) \circ \left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{\otimes(p-j-1)}\right) \circ g \\ &= \left((\alpha \circ \alpha)^{\otimes j} \otimes \alpha^{\otimes 2} \circ \Delta \otimes (\alpha \circ \alpha)^{\otimes i-j-1} \otimes \Delta \circ \alpha \otimes (\alpha \circ \alpha)^{\otimes(p-i-1)}\right) \circ g, \end{aligned}$$

So $S_j^{p+1,q} \circ S_i^{p,q}(g) = S_{i+1}^{p+1,q} \circ S_j^{p,q}(g)$.

· When the $1 \leq j \leq p-2$. and $i = p-1$,

$S_j^{p+1,q}(g) = \left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{\otimes(p-j)}\right) \circ g,\ S_{p-1}^{p,q}(g) = \left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g - \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q$

$$\begin{aligned} S_j^{p+1,q} \circ S_{p-1}^{p,q}(g) &= \left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{\otimes(p-j)}\right) \circ \left(\left(\alpha^{\otimes p-1} \otimes \Delta\right) \circ g - \left(g \otimes \alpha^{p-1}\right) \circ \rho_r^q\right) \\ &= \left((\alpha \circ \alpha)^{\otimes j} \otimes \Delta \circ \alpha \otimes (\alpha \circ \alpha)^{\otimes p-j-2} \otimes \alpha^2 \circ \Delta\right) \circ g \\ &- \left(\left(\left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{\otimes p-j-1}\right) \circ g\right) \otimes \alpha^p\right) \circ \rho_r^q. \end{aligned}$$

The right hand, we have

$$\begin{aligned} S_p^{p+1,q} \circ S_j^{p,q}(g) &= \left(\left(\alpha^{\otimes p} \otimes \Delta\right) \circ \left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{p-j-1}\right) \circ g - \left(\left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{p-j-1}\right) \circ g \otimes \alpha^p\right) \circ \rho_r^q\right) \\ &= \left((\alpha \circ \alpha)^{\otimes j} \otimes \alpha^{\otimes 2} \circ \Delta \otimes (\alpha \circ \alpha)^{p-j-2} \otimes \Delta \circ \alpha\right) \circ g \\ &- \left(\left(\alpha^{\otimes j} \otimes \Delta \otimes \alpha^{p-j-1}\right) \circ g \otimes \alpha^p\right) \circ \rho_r^q. \end{aligned}$$

So $S_j^{p+1,q} \circ S_{p-1}^{p,q}(g) = S_p^{p+1,q} \circ S_j^{p,q}(g)$.

This finishes the proof. ■

Proposition 4.2.4 *Let* $D_i^{p,q} : \mathcal{C}_{Hom}^{p,q}\left(B^{\otimes q}, B^{\otimes p}\right) \longrightarrow \mathcal{C}_{Hom}^{p,q+1}\left(B^{\otimes q+1}, B^{\otimes p}\right)$ *and* $S_i^{p,q} : \mathcal{C}_{Hom}^{p,q}\left(B^{\otimes q}, B^{\otimes p}\right) \longrightarrow \mathcal{C}_{Hom}^{p+1,q}\left(B^{\otimes q}, B^{\otimes p+1}\right)$. *Then*

$$S_j^{p,q+1} \circ D_i^{p,q} = D_i^{p+1,q} \circ S_j^{p,q} \text{ for all } 0 \leq i \leq q-1, 0 \leq j \leq p-1. \tag{4.7}$$

Theorem 4.2.5 *Let $B=(B,\mu,\eta,\Delta,\varepsilon,\alpha)$ be a Hom-bialgebra and $\delta^{p,q}_{Hom,H}:\mathcal{C}^{p,q}_{Hom}\longrightarrow\mathcal{C}^{p,q+1}_{Hom}$, $\delta^{p,q}_{Hom,C}:\mathcal{C}^{p,q}_{Hom}\longrightarrow\mathcal{C}^{p+1,q}_{Hom}$ the operators defined in (4.1), (4.2) then $\left(\mathcal{C}^{p,q}_{Hom},\delta^{p,q}_{Hom,H},\delta^{p,q}_{Hom,C}\right)$ is a bicomplex (see [45], and [9]),*

i.e

$$\delta^{p,q+1}_{Hom,H}\circ\delta^{p,q}_{Hom,H}=0,\delta^{p,q+1}_{Hom,C}\circ\delta^{p,q}_{Hom,H}=\delta^{p+1,q}_{Hom,H}\circ\delta^{p,q}_{Hom,C},\delta^{p+1,q}_{Hom,C}\circ\delta^{p,q}_{Hom,C}=0 \qquad (4.8)$$

Proof. We prove the first identity.

$$\delta^{p+1,q}_{Hom,C}\circ\delta^{p,q}_{Hom,C}=\left(\sum_{i=0}^{p}(-1)^{i+1}S^{p+1,q}_i\right)\circ\left(\sum_{j=0}^{p-1}(-1)^{j+1}S^{p,q}_j\right)$$

$$=\sum_{i=0}^{p}\sum_{j=0}^{p-1}(-1)^{i+j}S^{p+1,q}_i\circ S^{p,q}_j=\sum_{0\le j<i\le n}(-1)^{i+j}S^{p+1,q}_i\circ S^{p,q}_j+\sum_{0\le i\le j\le n-1}(-1)^{i+j}S^{p+1,q}_i\circ S^{p,q}_j$$

$$\overset{(4.6)}{=}\sum_{0\le j<i\le n}(-1)^{i+j}S^{p+1,q}_iS^{p,q}_j+\sum_{0\le i\le j\le n-1}(-1)^{i+j}S^{p+1,q}_{j+1}S^{p,q}_i$$

$$=\sum_{0\le j<i\le n}(-1)^{i+j}S^{p+1,q}_iS^{p,q}_j+\sum_{0\le i<k\le n}(-1)^{i+k-1}S^{p+1,q}_kS^{p,q}_i=0.$$

The second equality, we have

$$\begin{aligned}\delta^{p,q+1}_{Hom,C}\circ\delta^{p,q}_{Hom,H}&=\left(\sum_{i=0}^{p-1}(-1)^{i+1}S^{p,q+1}_i\right)\circ\left(\sum_{j=0}^{q-1}(-1)^{j+1}D^{p,q}_j\right)\\&=\left(\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}(-1)^{i+j}S^{p,q+1}_i\circ D^{p,q}_j\right)\\&\overset{(??)}{=}\left(\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}(-1)^{i+j}D^{p+1,q}_j\circ S^{p,q}_i\right)\\&=\left(\sum_{j=0}^{q-1}(-1)^{j+1}D^{p+1,q}_j\right)\circ\left(\sum_{i=0}^{p-1}(-1)^{i+1}S^{p,q}_i\right)\\&=\delta^{p+1,q}_{Hom,H}\circ\delta^{p,q}_{Hom,C}\end{aligned}$$

For the therd identity, see ([2]). ■

There is a canonical way to construct a complex from a given bicomplex.

The **cochains** are given by the bicomplex

$$\hat{\mathcal{C}}_{Hom}=\sum_n\oplus\hat{\mathcal{C}}^n_{Hom},\quad \hat{\mathcal{C}}^n_{Hom}=\sum_{p+q=n+1,p,q\ge1}\oplus\mathcal{C}^{p,q}_{Hom},\quad \mathcal{C}^{p,q}_{Hom}=Hom_{\Bbbk}\left(B^{\otimes q},B^{\otimes p}\right)\ n\ge1;$$

The **coboundary** operator is $\delta^n_{Hom} : \hat{\mathcal{C}}^n_{Hom} \longrightarrow \hat{\mathcal{C}}^{n+1}_{Hom}$ defined as

$$\delta^n_{Hom|\mathcal{C}^{n+1-q,q}_{Hom}} = \delta^{p,q}_{Hom,H} \oplus (-1)^q \, \delta^{p,q}_{Hom,C}, \quad 1 \leq q \leq n, \; p = n+1-q.$$

Hence, for each $n \geq 1$, we get a complex

$$0 \longrightarrow \hat{\mathcal{C}}^1_{Hom} \longrightarrow^{\delta^1_{Hom}} \hat{\mathcal{C}}^2_{Hom} \longrightarrow^{\delta^2_{Hom}} \hat{\mathcal{C}}^3_{Hom} \longrightarrow^{\delta^3_{Hom}} \hat{\mathcal{C}}^4_{Hom}$$

Remark 4.2.6 *The composite* $\delta^2_{Hom} \circ \delta^1_{Hom} = 0$*, according to Proposition* 4.2.1.

We define the $n-th$ **cohomology group** of the above complex to be the Hom- bialgebra cohomology of B, which will be denoted by $H^n_{Hom}(B,B)$, $n \geq 1$.

Definition 4.2.7 *The kernel of* δ^n_{Hom} *in* $\hat{\mathcal{C}}^n_{Hom}$ *is the space of* n***-cocycles*** *is defined by:*

$$Z^n_{Hom}(B,B) = \left\{ \varphi \in \hat{\mathcal{C}}^n_{Hom}, \quad \delta^n_{Hom}(\varphi) = 0 \right\} \tag{4.9}$$

The image of δ^n_{Hom} *is the space of* n***-coboundaries*** *is defined by:*

$$B^n_{Hom}(B,B) = \left\{ \varphi \in \hat{\mathcal{C}}^n_{Hom}, \quad \varphi = \delta^{n-1}_{Hom}(\psi), \quad \psi \in \hat{\mathcal{C}}^{n-1}_{Hom} \right\} \tag{4.10}$$

The ***Gerstenhaber-Shack cohomology group*** *of the Hom-bialgebra* $B = (B, \mu, \eta, \Delta, \varepsilon, \alpha)$ *with coefficient in it self is*

$$H^n_{Hom}(B,B) = Z^n_{Hom}(B,B) / B^n_{Hom}(B,B) \tag{4.11}$$

In particular,

$$\bullet H^1_{Hom}(B,B) = \left\{ f : B \longrightarrow B, \; \delta^{1,1}_{Hom,H}(f) = 0 \text{ and } \delta^{1,1}_{Hom,C}(f) = 0 \right\}$$

where

$$\delta^{1,1}_{Hom,H}(f) = \mu \circ (id_B \otimes f) - f \circ \mu + \mu \circ (f \otimes id_B)$$

$$\delta^{1,1}_{Hom,C}(f) = (id_B \otimes f) \circ \Delta - \Delta \circ f + (f \otimes id_B) \circ \Delta$$

It is very useful to write out $H^2_{Hom}(B,B)$ and $H^3_{Hom}(B,B)$ explicitly from the definition.

The cohomology groups $H^2_{Hom}(B,B)$ and $H^3_{Hom}(B,B)$ play an important role in deformation theory.

The cohomology group

$$\bullet H^2_{Hom}(B,B) = Z^2_{Hom}(B,B) / B^2_{Hom}(B,B),$$

where

$$\bullet Z^2_{Hom}(B,B) = \left\{(f,g) \in \hat{\mathcal{C}}^2_{Hom},\ \delta^{1,2}_{Hom,H}(f) = 0,\ \delta^{1,2}_{Hom,C}(f) + \delta^{2,1}_{Hom,H}(g) = 0,\ \delta^{2,1}_{Hom,C}(g) = 0\right\} \tag{4.12}$$

where for $f : B \otimes B \longrightarrow B$ and $g : B \longrightarrow B \otimes B$, we have

$$\delta^{1,2}_{Hom,H}(f) = \lambda^1_l \circ (\alpha \otimes f) - f \circ (\mu \otimes \alpha) + f \circ (\alpha \otimes \mu) - \lambda^1_r \circ (f \otimes \alpha)$$

$$\begin{aligned}\delta^{1,2}_{Hom,C}(f) + \delta^{2,1}_{Hom,H}(g) = &\left((id_B \otimes f) \circ \rho^2_l - \Delta \circ f + (f \otimes id_B) \circ \rho^2_r\right) + \\ &(\lambda^2_l \circ (id_B \otimes g) - g \circ \mu + \lambda^2_r \circ (g \otimes id_B))\end{aligned}$$

$$\delta^{2,1}_{Hom,C}(g) = (\alpha \otimes g) \circ \rho^1_l - (\Delta \otimes \alpha) \circ g + (\alpha \otimes \Delta) \circ g - (f \otimes \alpha) \circ \rho^1_r$$

where $\rho^1_l = \rho^1_r = \Delta$, $\lambda^1_l = \lambda^1_r = \mu$, and

$$\bullet B^2_{Hom}(B,B) = \left\{(f,g) \in \hat{\mathcal{C}}^2_{Hom},\ \exists h : B \longrightarrow B,\ f = \delta^{1,1}_{Hom,H}(h),\ g = \delta^{1,1}_{Hom,C}(h)\right\}$$

where

$$\delta^{1,1}_{Hom,H}(h) = \mu \circ (id_B \otimes h) - h \circ \mu + \mu \circ (h \otimes id_B)$$
$$\delta^{1,1}_{Hom,C}(h) = (id_B \otimes h) \circ \Delta - \Delta \circ h + (h \otimes id_B) \circ \Delta$$

The cohomology group

$$\bullet H^3_{Hom}(B,B) = Z^3_{Hom}(B,B) / B^3_{Hom}(B,B),$$

where

$$\bullet Z^3_{Hom}(B,B) = \left\{ \begin{array}{c} (F,H,G) \in \hat{\mathcal{C}}^3_{Hom},\ \delta^{1,3}_{Hom,H}(F) = 0, \delta^{2,2}_{Hom,H}(H) - \delta^{1,3}_{Hom,C}(F) = 0, \\ \delta^{2,2}_{Hom,C}(H) + \delta^{3,1}_{Hom,H}(G) = 0, \delta^{3,1}_{Hom,C}(G) = 0 \end{array} \right\} \tag{4.13}$$

and

$$\bullet B^3_{Hom}(B,B)=\left\{\begin{array}{c}(F,H,G)\in\hat{\mathcal{C}}^3_{Hom},\ \exists(f,g)\in\hat{\mathcal{C}}^2_{Hom},,\ F=\delta^{1,2}_{Hom,H}(f),\\ H=\delta^{1,2}_{Hom,C}(f)+\delta^{2,1}_{Hom,H}(g),G=\delta^{2,1}_{Hom,C}(g)\end{array}\right\}$$

where we write, $F:B\otimes B\otimes B\longrightarrow B$, $H:B\otimes B\longrightarrow B\otimes B$ and $G:B\longrightarrow B\otimes B\otimes B$

Lemma 4.2.8 $B^2_{Hom}(B,B)\subset Z^2_{Hom}(B,B)$ *because* $\delta^2_{Hom}\circ\delta^1_{Hom}(\varphi)=0.$

Example 4.2.9 *We consider* $(T_2)_\lambda$*, the* 4*-dimensional Taft-Sweedler Hom-bialgebra defined in Example 2.4.7 for which we compute for* $\lambda\neq 1$ *and* $\lambda\neq 0$*, the first cohomology groups.*

The space of 1*-cohomology classes of* $(T_2)_\lambda$

$$H^1_{Hom}((T_2)_\lambda,(T_2)_\lambda)=\left\{f:(T_2)_\lambda\longrightarrow(T_2)_\lambda\text{: }\delta^{1,1}_{Hom,H}(f)=0\text{ and }\delta^{1,1}_{Hom,C}(f)=0\right\}$$

The elements are defined with respect to a basis $\{e_1,e_2,e_3,e_4\}$ *by*

$$f(e_1)=0,\ f(e_2)=0,f(e_3)=ae_3,f(e_4)=ae_4,\text{where } a \text{ is a free parameter.}$$

The 2-cocycles of the Hom-bialgebras $(T_2)_\lambda$

$$Z^2_{Hom}((T_2)_\lambda,(T_2)_\lambda)=\left\{(f,g)\in\hat{\mathcal{C}}^2_{Hom}:\ \delta^{1,2}_{Hom,H}(f)=0,\delta^{2,1}_{Hom,C}(g)=0,\ \delta^{1,2}_{Hom,C}(f)+\delta^{2,1}_{Hom,H}(g)=0\right\}.$$

They are defined with respect to the basis $\{e_1,e_2,e_3,e_4\}$*, by the table which describes multiplying the i th row elements by the j th column elements with respect to the same basis :*

f	e_1	e_2	e_3	e_4
e_1	$a(e_1+e_2)$	$a(e_1+e_2)$	$\lambda a(e_3+e_4)$	$\lambda a(e_3+e_4)$
e_2	$a(e_1+e_2)$	$a(e_1-3e_2)$	$\lambda(ce_4-ae_3)$	$\lambda((2a-c)e_3-ae_4)$
e_3	$\lambda a(e_3-e_4)$	$-\lambda(ae_3+ce_4)$	0	0
e_4	$\lambda a(e_4-e_3)$	$-\lambda((2a-c)e_3+ae_4)$	0	0

and

$$g(e_1)=-a(e_1\otimes e_1+e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2),$$

$$g(e_2)=-a(e_1\otimes e_1-e_1\otimes e_2-e_2\otimes e_1+e_2\otimes e_2),$$

$$g(e_3)=\lambda a(e_1\otimes e_3-e_2\otimes e_3-e_3\otimes e_1-e_3\otimes e_2),$$

$$g(e_4)=-\lambda a(e_1\otimes e_4+e_2\otimes e_4-e_4\otimes e_1+e_4\otimes e_2).$$

The space of 2*-coboundaries of the Hom-bialgebra* $(T_2)_\lambda$ *is defined by*

$$B^2_{Hom}\left((T_2)_\lambda,(T_2)_\lambda\right)=\left\{(f,g)\in\hat{\mathcal{C}}^2_{Hom},\ \exists h:(T_2)_\lambda\longrightarrow(T_2)_\lambda,\ f=\delta^{1,1}_{Hom,H}(h),\ g=\delta^{1,1}_{Hom,C}(h)\right\}$$

such that

f	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	0	λce_4	$-\lambda ce_3$
e_3	0	$-\lambda ce_4$	0	0
e_4	0	λce_3	0	0

and $g(e_i)=0$ *for* $i\in\{1,2,3,4\}$, *where* $\lambda,a,c\in\Bbbk$ *are free parameters.*

The 2^{th} *cohomology group of* $(T_2)_\lambda$ *is the quotient*

$$H^2_{Hom}\left((T_2)_\lambda,(T_2)_\lambda\right)=Z^2_{Hom}\left((T_2)_\lambda,(T_2)_\lambda\right)/B^2_{Hom}\left((T_2)_\lambda,(T_2)_\lambda\right),$$

which is defined, with respect to the basis $\{e_1,e_2,e_3,e_4\}$, *by*

f	e_1	e_2	e_3	e_4
e_1	$a(e_1+e_2)$	$a(e_1+e_2)$	$\lambda a(e_3+e_4)$	$\lambda a(e_3+e_4)$
e_2	$a(e_1+e_2)$	$a(e_1-3e_2)$	$-\lambda ae_3$	$\lambda a(2e_3-e_4)$
e_3	$\lambda a(e_3-e_4)$	$-\lambda ae_3$	0	0
e_4	$\lambda a(e_4-e_3)$	$-\lambda a(2e_3+e_4)$	0	0

and

$$g(e_1)=-a(e_1\otimes e_1+e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2),$$

$$g(e_2)=-a(e_1\otimes e_1-e_1\otimes e_2-e_2\otimes e_1+e_2\otimes e_2),$$

$$g(e_3)=\lambda a(e_1\otimes e_3-e_2\otimes e_3-e_3\otimes e_1-e_3\otimes e_2),$$

$$g(e_4)=-\lambda a(e_1\otimes e_4+e_2\otimes e_4-e_4\otimes e_1+e_4\otimes e_2).$$

Chapter 5

Formal deformations of Hom-bialgebras

We discuss, in this chapter, a deformation theory for Hom-bialgebras following Gerstenhaber's approach. Let $(B, \mu, \eta, \Delta, \varepsilon, \beta)$ be a Hom-bialgebra and $\mathbb{k}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{k}$ and let $B[[t]]$ be the set of formal power series whose coefficients are elements of B (note that $B[[t]]$ is obtained by extending the coefficients domain of B from $\mathbb{k}$ to $\mathbb{k}[[t]]$). Then $B[[t]]$ is a $\mathbb{k}[[t]]$-module and when B is finite dimensional, we have $B\,[[t]] = B \otimes_{\mathbb{k}} \mathbb{k}[[t]]$. Notice that B is a submodule of $B[[t]]$.

5.1 Formal deformations

Throughout this section, let $B = (B, \mu_0, \eta_0, \Delta_0, \varepsilon_0, \alpha_0)$ be an arbitrary but fixed Hom-bialgebra. We define a formal deformation of B to be a formal power series in the indeterminate t,

Definition 5.1.1 *A **formal Hom-bialgebra deformation** of B over $\mathbb{k}[[t]]$ consists of a $\mathbb{k}[[t]]$-bilinears maps*

$$\mu_t = \sum_{i\geq 0}\mu_i t^i \ : B\otimes B \longrightarrow B[[t]], \quad \eta_t = \sum_{i\geq 0}\eta_i t^i \ : \mathbb{k} \longrightarrow B[[t]],$$

$$\Delta_t = \sum_{j\geq 0}\Delta_j t^j \ : B \longrightarrow B[[t]]\otimes B[[t]], \quad \varepsilon_t = \sum_{j\geq 0}\varepsilon_j t^j \ : B \longrightarrow \mathbb{k}[[t]],$$

and

$$\alpha_t = \sum_{k\geq 0}\alpha_k t^k \ : B \longrightarrow B[[t]],$$

where each μ_i is a $\mathbb{k}$-bilinear map $\mu_i : B \otimes B \longrightarrow B$ (extended to be $\mathbb{k}[[t]]$-bilinear) , the maps Δ_j, ε_j, η_i and α_k are a $\mathbb{k}$-linear maps $\Delta_j : B \longrightarrow B \otimes B$, $\varepsilon_j : B \longrightarrow \mathbb{k}$, $\eta_i : \mathbb{k} \longrightarrow B$, and $\alpha_k : B \longrightarrow B$ (extended to be $\mathbb{k}[[t]]$-linear).With respect to which the $B_t = (B[[t]], \mu_t, \eta_t, \Delta_t, \varepsilon_t, \alpha_t)$ is again a Hom-bialgebra.

If we study only deformations of $B = (B, \mu_0, \eta_0, \Delta_0, \varepsilon_0, \alpha_0)$ in which the unit and the counit are conserved, that is $B_t = (B[[t]], \mu_t, \eta_0, \Delta_t, \varepsilon_0, \alpha_t)$, then we as consider a deformation as a triple $(\mu_t, \Delta_t, \alpha_t)$ satisfying

$$\begin{cases} \mu_t \circ (\alpha_t \otimes \mu_t) = \mu_t \circ (\mu_t \otimes \alpha_t) & \text{(formal Hom-associativity).} \\ (\Delta_t \otimes \alpha_t) \circ \Delta_t = (\alpha_t \otimes \Delta_t) \circ \Delta_t & \text{(formal Hom-coassociativity).} \\ \Delta_t \circ \mu_t = \mu_t^{\otimes 2} \circ (id_B \otimes \tau_{B\otimes B} \otimes id_B) \circ \Delta_t^{\otimes 2} & \text{(formal compatibility).} \end{cases} \tag{5.1}$$

5.2 Formal automorphisms

We need a notion of equivalence of formal deformations.

Definition 5.2.1 *Let $B = (B, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra.* ***A formal automorphism*** *on B, we mean a formal power series*

$$\Phi_t = id_B + t\Phi_1 + t^2\Phi_2 + t^3\Phi_3 + ...,$$

where each $\Phi_i \in End(B)$, satisfying multiplicativity and comultiplicativity,

$$\Phi_t \circ \mu = \mu \circ (\Phi_t \otimes \Phi_t) \;\; \textit{and} \;\; \Delta \circ \Phi_t = (\Phi_t \otimes \Phi_t) \circ \Delta. \tag{5.2}$$

The same rules of dealing with power series apply here as well. In particular, multiplicativity and comultiplicativity are equivalent to the equality

$$\Phi_n \circ \mu = \sum_{i=0}^{n} \mu \circ (\Phi_i \otimes \Phi_{n-i}) \;\; \text{and} \;\; \Delta \circ \Phi_n = \sum_{i=0}^{n} (\Phi_i \otimes \Phi_{n-i}) \circ \Delta. \tag{5.3}$$

for all $n \geq 0$, in which $\Phi_0 = id_B$. The conditions when $n = 0$ are trivial, as it only says that the identity map on B is multiplicative and comultiplicative.

When $n=1$, the conditions are

$$\Phi_1 \circ \mu = \mu \circ (\Phi_1 \otimes id_B) + \mu \circ (id_B \otimes \Phi_1)$$

$$\text{and } \; \Delta \circ \Phi_1 = (\Phi_1 \otimes id_B) \circ \Delta + (id_B \otimes \Phi_1) \circ \Delta,$$

which are equivalent to say that Φ_1 is a derivation and coderivation on B. More generally, if $\Phi_1 = \Phi_2 = ... = \Phi_n = 0$, then Φ_{n+1} is a derivation and coderivation on B.

Remark 5.2.2 *A formal automorphism Φ_t has a unique formal inverse*

$$\Phi_t^{-1} = id_B + t\Phi_1 + t^2 \left(\Phi_1^2 - \Phi_2\right) + t^3 \left(-\Phi_1^3 - \Phi_1\Phi_2 - \Phi_2\Phi_1 - \Phi_3\right) + ...$$

for which

$$\Phi_t \circ \Phi_t^{-1} = \Phi_t^{-1} \circ \Phi_t = id_B$$

The coefficient of t^n in Φ_t^{-1} is an integral polynomial in $\Phi_1, \Phi_2, ..., \Phi_n$. Moreover, the multiplicativity and the comultiplicativity of Φ_t implies that of Φ_t^{-1}. Indeed, for elements $x, y \in B$, we have

$$\begin{aligned}
\mu (x \otimes y) &= \mu \left(\Phi_t \circ \Phi_t^{-1} (x) \otimes \Phi_t \circ \Phi_t^{-1} (y)\right) \\
&= \Phi_t \circ \mu \left(\Phi_t^{-1} (x) \otimes \Phi_t^{-1} (y)\right) \\
\Phi_t^{-1} \circ \mu (x \otimes y) &= \mu \left(\Phi_t^{-1} (x) \otimes \Phi_t^{-1} (y)\right)
\end{aligned}$$

which implies that Φ_t^{-1} is multiplicative.

$$\begin{aligned}
\Delta &= \left(\Phi_t^{-1} \circ \Phi_t \otimes \Phi_t^{-1} \circ \Phi_t\right) \circ \Delta \\
&= \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ (\Phi_t \otimes \Phi_t) \circ \Delta \\
&= \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ \Delta \circ \Phi_t \\
\Delta \circ \Phi_t^{-1} &= \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ \Delta
\end{aligned}$$

which implies that Φ_t^{-1} is comultiplicative.

We record these facts as follows.

Lemma 5.2.3 *Let $\Phi_t = id_B + t\Phi_1 + t^2\,\Phi_2 + t^3\Phi_3 + ...,$ be a formal automorphism on B. Then the first non-zero i $(i \geq 1)$ is a derivation and coderivation on B. Moreover, the formal inverse Φ_t^{-1} of Φ_t is also a formal automorphism on B.*

Proposition 5.2.4 *Let B_t and Φ_t be, respectively, a formal deformation and a formal automorphism of B. Then the formal power series*

$$\Phi_t^{-1} \circ B_t \circ \Phi_t = \left(B, \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t), (\Phi_t \otimes \Phi_t) \circ \Delta_t \circ \Phi_t^{-1}, \Phi_t^{-1} \circ \alpha_t \circ \Phi_t^{-1}\right)$$

is also a formal deformation of B.

Proof. We need to check the Hom-associativity condition (2.1) for $\mu_t' = \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t)$ and the Hom-coassociativity condition (2.7) for $\Delta_t' = \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ \Delta_t \circ \Phi_t$, where $\alpha_t' = \Phi_t^{-1} \circ \alpha_t \circ \Phi_t$.

For the Hom-associativity condition, we have

$$\begin{aligned}
\mu_t' \circ \left(\mu_t' \otimes \alpha_t'\right) &= \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t) \circ \left(\left(\Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t) \otimes \Phi_t^{-1} \circ \alpha_t \circ \Phi_t\right)\right) \\
&= \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t) \circ \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ ((\mu_t \otimes \alpha_t)(\Phi_t \otimes \Phi_t \otimes \Phi_t)) = \\
&\overset{(2.1)}{=} \Phi_t^{-1} \circ \mu_t \circ ((\alpha_t \otimes \mu_t)(\Phi_t \otimes \Phi_t \otimes \Phi_t)) \\
&= \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t) \circ \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ (\alpha_t \otimes \mu_t) \circ (\Phi_t \otimes \Phi_t \otimes \Phi_t) \\
&= \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t) \circ \left(\left(\Phi_t^{-1} \circ \alpha_t \circ \Phi_t\right) \otimes \left(\Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t)\right)\right) \\
&= \mu_t' \circ \left(\alpha_t' \otimes \mu_t'\right).
\end{aligned}$$

We have used the Hom-associativity for μ_t and the multiplicativity of both Φ_t and Φ_t^{-1} of μ_0, extended to power series. The Hom-associativity for Δ_t' is equally easy to verify.

For the compatibility condition, we have

$$\begin{aligned}
\Delta_t' \circ \mu_t' &= \left(\left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ \Delta_t \circ \Phi_t\right) \circ \left(\Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t)\right) \\
&= \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ (\mu_t \circ \mu_t) \circ (id_B \otimes \tau \otimes id_B) \circ (\Delta_t \otimes \Delta_t) \circ (\Phi_t \otimes \Phi_t) \\
&= \left(\Phi_t^{-1} \circ \mu_t \otimes \Phi_t^{-1} \circ \mu_t\right) (\Phi_t \otimes \Phi_t \otimes \Phi_t \otimes \Phi_t) \circ (id_B \otimes \tau \otimes id_B) \\
&\circ \left(\Phi_t^{-1} \otimes \Phi_t^{-1} \otimes \Phi_t^{-1} \otimes \Phi_t^{-1}\right) \circ (\Delta_t \circ \Phi_t \otimes \Phi_t \circ \Delta_t) \\
&= \left(\mu_t' \otimes \mu_t'\right) \circ (id_B \otimes \tau \otimes id_B) \circ \left(\Delta_t' \otimes \Delta_t'\right).
\end{aligned}$$

Then $\Phi_t^{-1} \circ B_t \circ \Phi_t$ is a formal deformation of B. ■

5.3 Equivalent and trivial deformations

In this section, we characterize the equivalent and trivial deformations of Hom-bialgebras.

Definition 5.3.1 *Let $B = (B, \mu_0, \eta, \Delta_0, \varepsilon, \alpha)$ be a Hom-bialgebra. Given two deformations of B_0, $B_t = (B, \mu_t, \eta, \Delta_t, \varepsilon, \alpha)$ and $B'_t = (B, \mu'_t, \eta, \Delta'_t, \varepsilon, \alpha)$where $\mu_t = \sum\limits_{i\geq 0}\mu_i t^i$, $\Delta_t = \sum\limits_{j\geq 0}\Delta_j t^j$, $\mu'_t = \sum\limits_{i\geq 0}\mu'_i t^i$, $\Delta'_t = \sum\limits_{j\geq 0}\Delta'_j t^j$ with $\mu'_0 = \mu_0$, and $\Delta'_0 = \Delta_0$.*

*We say that they are **equivalent** if there is a formal automorphism $\Phi_t : B \longrightarrow B[[t]]$ which is a $\Bbbk[[t]]$-linear map that may be written in the form $\Phi_t = \sum\limits_{i\geq 0}\Phi_i t^i$ where $\Phi_i \in End_{\Bbbk}(B)$ and $\Phi_0 = id_B$ such that*

$$\Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \otimes \Phi_t)\,, \tag{5.4}$$

$$(\Phi_t \otimes \Phi_t) \circ \Delta_t = \Delta'_t \circ \Phi_t \tag{5.5}$$

and

$$\Phi_t \circ \alpha = \alpha \circ \Phi_t. \tag{5.6}$$

Definition 5.3.2 *A deformation B_t of B_0 is said to be **trivial** if and only if B_t is equivalent to B_0*

We discuss in the following the equivalence of two deformations.Equation (5.4) is equivalent to

$$\sum_{i,j\geq 0}(\Phi_i \circ \mu_j)\, t^{i+j} = \sum_{i,j,k\geq 0}\mu'_i \circ (\Phi_j \otimes \Phi_k)\, t^{i+j+k}\,. \tag{5.7}$$

By identification of the coefficients, one obtains that the constant coefficients are identical, i.e.

$$\mu_0 = \mu'_0 \quad and \quad \Phi_0 = id_B.$$

For the coefficients of t one finds

$$(\Phi_0 \circ \mu_1) + (\Phi_1 \circ \mu_0) = \mu'_1 \circ (\Phi_0 \otimes \Phi_0) + \mu'_0 \circ (\Phi_1 \otimes \Phi_0) + \mu'_0 \circ (\Phi_0 \otimes \Phi_1)$$

$$\mu_1 + \Phi_1 \circ \mu_0 = \mu'_1 + \mu_0 \circ (\Phi_1 \otimes id_B) + \mu_0 \circ (id_B \otimes \Phi_1)$$

$$\mu_1' = \mu_1 - \left(\mu_0 \circ (\Phi_1 \otimes id_B) - \Phi_1 \circ \mu_0 + \mu_0 \circ (id_B \otimes \Phi_1)\right). \tag{5.8}$$

Equations (5.5) are equivalent to

$$\sum_{i,j\geq 0} \left(\Delta_i' \circ \Phi_j\right) t^{i\ +j}\ - \sum_{i,j,k\geq 0} (\Phi_i \otimes \Phi_j) \circ \Delta_k t^{i\ +j\ +k} = 0.$$

$$\sum_{i+j=n} \left(\Delta_i' \circ \Phi_j\right) - \sum_{i+j+k=n} (\Phi_i \otimes \Phi_j) \circ \Delta_k = 0.\ n = 1, 2, ...$$

Similarly for the comultiplication, setting $\Delta_0 = \Delta_0'$ and $\Phi_0 = id_B$, the coefficients of t leads to

$$\left(\Delta_0' \circ \Phi_1\right) + \left(\Delta_1' \circ \Phi_0\right) = (\Phi_0 \otimes \Phi_0) \circ \Delta_1 + (\Phi_0 \otimes \Phi_1) \circ \Delta_0 + (\Phi_1 \otimes \Phi_0) \circ \Delta_0$$

Hence

$$\Delta_1' = \Delta_1 + (id_B \otimes \Phi_1) \circ \Delta_0 - (\Delta_0 \circ \Phi_1) + (\Phi_1 \otimes id_B) \circ \Delta_0. \tag{5.9}$$

Homomorphisms condition (5.6) is equivalent to $\sum_{i\geq 0} (\Phi_i \circ \alpha)\, t^i = \sum_{i\geq 0} (\alpha \circ \Phi_i)\, t^i$.Therefore

$$\Phi_i \circ \alpha = \alpha \circ \Phi_i,\ \text{ for all } i > 0.$$

The first and second order conditions of the equivalence between two deformations of a Hom-bialgebra are given by (5.8) (5.9) may be written

$$\mu_1' = \mu_1 - \delta^{1,1}_{Hom,H}(\Phi_1) \quad \textit{and} \quad \Delta_1' = \Delta_1 + \delta^{1,1}_{Hom,C}(\Phi_1). \tag{5.10}$$

In general, if the deformations (μ_t, Δ_t) and (μ_t', Δ_t') of (μ_0, Δ_0) are equivalent then

$$\mu_1' = \mu_1 + \delta^{1,1}_{Hom,H}(\Phi_1) \quad \textit{and} \quad \Delta_1' = \Delta_1 + \delta^{1,1}_{Hom,C}(\Phi_1).$$

5.4 Deformations equation and infinitesimals

The general principle here is that every deformation has as infinitesimal which lies in appropriate second cohomology group. In this section we discuss the infinitesimals, beginning with the case of Hom-bialgebras. In each of these (μ_i, Δ_i) is (essentially) the infinitesimal. We can discover the definitions of the cochain groups and coboundary by examining the coefficients of t in the equation above.

Now, we discuss the deformation equation in terms of cohomology. The first problem is to give conditions about μ_i, Δ_j and α_k such that the deformation $(\mu_t, \Delta_t, \alpha_t)$ be Hom-associative, Hom-coassociative and compatibility.

We study the equations (5.1) and thus characterize the deformations of Hom-bialgebras. The coefficients of t^s yields :

$$\begin{cases} \displaystyle\sum_{\substack{i+j+k=s \\ i,j,k\geq 0}} (\mu_i \circ (\mu_j \otimes \alpha_k) - \mu_i \circ (\alpha_k \otimes \mu_j)) = 0 \qquad s = 0,1,2... \\ \displaystyle\sum_{\substack{i+j+k=s \\ i,j,k\geq 0}} (\Delta_j \otimes \alpha_k) \circ \Delta_i - (\alpha_k \otimes \Delta_j) \circ \Delta_i = 0 \qquad s = 0,1,2,... \\ \displaystyle\sum_{\substack{i+j=s \\ i,j\geq 0}} (\Delta_i \circ \mu_j) - \sum_{\substack{i+j+k+r=s \\ i,j,k,r\geq 0}} (\mu_i \otimes \mu_j) \circ \tau_{2,3} \circ (\Delta_k \otimes \Delta_r) = 0 \qquad s = 0,1,2,... \end{cases} \tag{5.11}$$

This infinite system, called the **deformation equation**, gives the necessary and sufficient conditions for B_t to be a Hom-bialgebra. It may be written

$$\begin{cases} \displaystyle\sum_{i=0}^{s}\sum_{j=0}^{s-i} (\mu_i \circ (\alpha_j \otimes \mu_{s-i-j}) - \mu_i \circ (\mu_{s-i-j} \otimes \alpha_j)) = 0 \qquad s = 0,1,2,\cdots \\ \displaystyle\sum_{i=0}^{s}\sum_{j=0}^{s-i} (\Delta_{s-i-j} \otimes \alpha_j) \circ \Delta_i - (\alpha_j \otimes \Delta_{s-i-j}) \circ \Delta_i = 0 \qquad s = 0,1,2,\cdots \\ \displaystyle\sum_{i=0}^{s}\left((\Delta_i \circ \mu_{s-i}) - \sum_{j=0}^{s-i}\sum_{k=0}^{s-i-j} (\mu_i \otimes \mu_j) \circ \tau_{2,3} \circ (\Delta_k \otimes \Delta_{s-i-j-k}) \right) = 0 \qquad s = 0,1,2,\cdots \end{cases}$$

Definition 5.4.1 *We call α_k-**associator** the map*

$$Hom\left(B^{\otimes 2}, B\right) \times Hom\left(B^{\otimes 2}, B\right) \longrightarrow Hom\left(B^{\otimes 3}, B\right), \; (\mu_i, \mu_j) \longmapsto \mu_i \circ_{\alpha_k} \mu_j$$

defined by

$$\mu_i \circ_{\alpha_k} \mu_j = \mu_i \circ (\alpha_k \otimes \mu_j) - \mu_i \circ (\mu_j \otimes \alpha_k).$$

*We call α_k-**coassociator** the map*

$$Hom\left(B, B^{\otimes 2}\right) \times Hom\left(B, B^{\otimes 2}\right) \longrightarrow Hom\left(B, B^{\otimes 3}\right), (\Delta_i, \Delta_j) \longmapsto \Delta_i \circ_{\alpha_k} \Delta_j$$

defined by

$$\Delta_i \circ_{\alpha_k} \Delta_j = (\Delta_j \otimes \alpha_k) \circ \Delta_i - (\alpha_k \otimes \Delta_j) \circ \Delta_i.$$

By using α_j-associators, and α_j-coassociators, the deformation equations may be written as follows

$$\begin{cases} \sum_{i=0}^{s}\sum_{j=0}^{s-i} \mu_i \circ_{\alpha_j} \mu_{s-i-j} = 0 \qquad s = 0,1,2,\cdots \\ \sum_{i=0}^{s}\sum_{j=0}^{s-i} \Delta_i \circ_{\alpha_j} \Delta_{s-i-j} = 0 \qquad s = 0,1,2,\cdots \\ \sum_{i=0}^{s}\left((\Delta_i \circ \mu_{s-i}) - \sum_{j=0}^{s-i}\sum_{k=0}^{s-i-j} (\mu_i \otimes \mu_j) \circ \tau_{2,3} \circ (\Delta_k \otimes \Delta_{s-i-j-k}) \right) = 0 \qquad s = 0,1,2,\cdots \end{cases}$$

The first equations corresponding to $s = 0$, are the Hom-associativity condition for μ_0, the Hom-coassociativity condition for Δ_0 and the compatibility condition of μ_0 and Δ_0.

If the structure map is not deformed, then one gets the following system, where $\alpha_0 = \alpha$,

$$\begin{cases} \sum_{i=1}^{s-1} \mu_i \circ_{\alpha} \mu_{s-i} = -\delta^{1,2}_{Hom,H}(\mu_1) \qquad s = 1,2,\cdots \\ \sum_{i=0}^{s} \Delta_i \circ_{\alpha} \Delta_{s-i} = -\delta^{2,1}_{Hom,C}(\Delta_1) \qquad s = 1,2,\cdots \\ \sum_{i=1}^{s-1}\left((\Delta_i \circ \mu_{s-i}) - \sum_{j=0}^{s-i}\sum_{k=0}^{s-i-j} (\mu_i \otimes \mu_j) \circ \tau_{2,3} \circ (\Delta_k \otimes \Delta_{s-i-j-k}) \right) = 0 \qquad s = 1,2,\cdots \end{cases}$$

In particular, for $s = 1$ we have

•$\mu_0 \circ_\alpha \mu_1 + \mu_1 \circ_\alpha \mu_0 = 0$, which is equivalent $\delta^{1,2}_{Hom,H}(\mu_1) = 0$

•$\Delta_0 \circ_\alpha \Delta_1 + \Delta_1 \circ_\alpha \Delta_0 = 0$, which is equivalent $\delta^{2,1}_{Hom,C}(\Delta_1) = 0$

•the compatibility condition which is equivalent $\delta^{1,2}_{Hom,C}(\mu_1) + \delta^{2,1}_{Hom,H}(\Delta_1) = 0$, Therefore, we have

Proposition 5.4.2 *The first term (μ_1, Δ_1) of a deformation of a Hom-bialgebra, where the structure map is not deformed, is always a 2-cocycle for the Hom-bialgebra Gerstenhaber-Schack cohomology.*

It turns out that (μ_1, Δ_1) is always a 2-cocycle for the Hochschild cohomology (i.e. $(\mu_1, \Delta_1) \in Z^2_{Hom}(B, B)$), whose cohomology class is determined by the equivalence class of the deformation (μ_t, Δ_t).

More generally, suppose that (μ_m, Δ_m) be the first non-zero coefficient after (μ_0, Δ_0) in the deformation (μ_t, Δ_t). Then (μ_m, Δ_m) is a 2-cocycle for the Hochschild cohomology.

Definition 5.4.3 *In every case the 2-cocycle for the Hochschild cohomology (μ_m, Δ_m) is commonly called* ***infinitesimal of the deformation*** *B_t of B_0.*

This terminology would be bener applied to the cohomology class of μ_1 in $H^2(B,B)$ since in each case equivalent deformations have cohomologous infinitesimals. Conversely, a cocycle cohomologous to the infinitesimal of a deformation necessarily appears as that of an equivalent deformation. For instance, if $\mu_1 = \delta^1_{Hom}\varphi_1$ then μ_t is equivalent to a deformation μ'_t for which $\mu'_1 = 0$. (The equivalence $\mu'_t \simeq \mu_t$ is given by the linear isomorphism $id_B - t\varphi_1 : B[[t]] \longrightarrow B[[t]]$). Continuing μ'_2 is then a cocycle and, if it is a coboundary, say $\mu'_2 = \delta^1_{Hom}\varphi_2$, then there is a further equivalent deformation μ''_t having $\mu''_1 = \mu''_2 = 0$, (The equivalence $\mu''_t \simeq \mu'_t$ is given by $id_B - t^2\varphi_2$). These remarks apply in every case.

5.5 Obstructions

A basic task of deformation theory is to construct and catalog the deformation of a given algebra. In later sections we shall consider this problem using some recently minted techniques. But first we must take a more foundational approach: Obstruction theory. This describes the relationships among the cochains (μ_i, Δ_i) in a deformation (μ_t, Δ_t). For example, in the Hom-bialgebra case,

gathering the first and the last terms in the k^{th} equations of the system 5.11, for an arbitrary k, $k > 1$; the equations may be written

$$\delta^2_{Hom}(\mu_k, \Delta_k) = \begin{cases} \delta^{1,2}_{Hom,H}(\mu_k) = \mu_0 \circ_\alpha \mu_k + \mu_k \circ_\alpha \mu_0 = -\sum\limits_{\substack{i+j=k \\ i,j\neq k}} \mu_i \circ_\alpha \mu_j \\ \delta^{2,1}_{Hom,C}(\Delta_k) = \Delta_0 \circ_\alpha \Delta_k + \Delta_k \circ_\alpha \Delta_0 = -\sum\limits_{\substack{i+j=k \\ i,j\neq k}} \Delta_i \circ_\alpha \Delta_j \\ \delta^{1,2}_{Hom,C}(\mu_k) + \delta^{2,1}_{Hom,H}(\Delta_k) = \sum\limits_{\substack{i+j=k \\ i,j\neq k}} (\Delta_i \circ \mu_j) - \sum\limits_{\substack{i+j+t+r=k \\ i,j,t,r\neq k}} (\mu_i \otimes \mu_j) \circ \tau_{2,3} \circ (\Delta_t \otimes \Delta_r) \end{cases}$$

from which one sees, in particular, that $\delta^2_{Hom}(\mu_1, \Delta_1) = 0$, It follows that the integrability of an $(\mu_1, \Delta_1) \in Z^2_{Hom}(B,B)$, i.e., the existence of a deformation with the given (μ_1, Δ_1) as its linear term, depends only on the cohomology class $[(\mu_1, \Delta_1)]$ of (μ_1, Δ_1). We may view $[(\mu_1, \Delta_1)] \in H^2_{Hom}(B,B)$ as the infinitesimal of the equivalence class of the deformation B_t.

Definition 5.5.1 *Let $B = (B, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra, and (μ_1, Δ_1) be an el-*

ement of $Z^2_{Hom}(B,B)$, *the* 2*-cocycle* (μ_1,Δ_1) *is said* ***integrable*** *if there exists a family* $(\mu_t,\Delta_t)_{t\geq 0}$ *such that* $\mu_t = \sum_{i\geq 0}\mu_i t^i$ *and* $\Delta_t = \sum_{i\geq 0}\Delta_i t^i$ *defines a formal deformation* $B_t = (B[[t]], \mu_t, \eta, \Delta_t, \varepsilon, \alpha)$ *of* B.

Therefore, we have the following observation:

Proposition 5.5.2 *The integrability of* (μ_1,Δ_1) *depends only on its cohomology class.*

Proof. Recall that two elements are cohomologous if their difference is a coboundary, when this is the case, $\mu_1' = \mu_1 - \delta^{1,1}_{Hom,H}(f)$, and $\Delta_1' = \Delta_1 + \delta^{1,1}_{Hom,C}(f)$

If the equation $\delta^2_{Hom}(\mu_1,\Delta_1) = 0$ implies that

$$\delta^{1,2}_{Hom,H}(\mu_1) = 0,\ \delta^{2,1}_{Hom,C}(\Delta_1) = 0,\ \delta^{1,2}_{Hom,C}(\mu_1) + \delta^{2,1}_{Hom,H}(\Delta_1) = 0$$

We have

$$\begin{cases} \delta^{1,2}_{Hom,H}(\mu_1') = \delta^{1,2}_{Hom,H}\left(\mu_1 - \delta^{1,1}_{Hom,H}(f)\right) = \delta^{1,2}_{Hom,H}(\mu_1) - \delta^{1,2}_{Hom,H}\circ\delta^{1,1}_{Hom,H}(f) = 0, \\ \delta^{2,1}_{Hom,C}(\Delta_1') = \delta^{2,1}_{Hom,C}\left(\Delta_1 + \delta^{1,1}_{Hom,C}(f)\right) = \delta^{2,1}_{Hom,C}(\Delta_1) + \delta^{2,1}_{Hom,C}\circ\delta^{1,1}_{Hom,C}(f) = 0 \\ \delta^{1,2}_{Hom,C}(\mu_1') + \delta^{2,1}_{Hom,H}(\Delta_1') = \delta^{1,2}_{Hom,C}\left(\mu_1 - \delta^{1,1}_{Hom,H}(f)\right) + \delta^{2,1}_{Hom,H}\left(\Delta_1 + \delta^{1,1}_{Hom,C}(f)\right) \\ \quad = \delta^{1,2}_{Hom,C}(\mu_1) + \delta^{2,1}_{Hom,H}(\Delta_1) - \left(\delta^{1,2}_{Hom,C}\circ\delta^{1,1}_{Hom,H}(f) - \delta^{2,1}_{Hom,H}\circ\delta^{1,1}_{Hom,C}(f)\right) = 0. \end{cases}$$

Is given by Proposition 4.2.1 implies that $\delta^2_{Hom}(\mu_1',\Delta_1') = 0$.

If the equations $(\mu_1,\Delta_1) = \delta^1_{Hom}(f)$ implies that

$$\mu_1 = \delta^{1,1}_{Hom,H}(f), \Delta_1 = \delta^{1,1}_{Hom,C}(f)$$

and

$$\mu_1' = \mu_1 + \delta^{1,1}_{Hom,H}(\Phi_1) \quad \textit{and} \quad \Delta_1' = \Delta_1 + \delta^{1,1}_{Hom,C}(\Phi_1).$$

We have

$$\mu_1' = \delta^{1,1}_{Hom,H}(f) + \delta^{1,1}_{Hom,H}(\Phi_1) = \delta^{1,1}_{Hom,H}(f+\Phi_1)$$
$$\Delta_1' = \delta^{1,1}_{Hom,C}(f) + \delta^{1,1}_{Hom,C}(\Phi_1) = \delta^{1,1}_{Hom,C}(f+\Phi_1).$$

Then if two integrable 2-cocycles are cohomologous, then the corresponding deformations are equivalent. ■

Proposition 5.5.3 *Let $B_0 = (B, \mu_0, \eta, \Delta_0, \varepsilon, \alpha)$ be a Hom-bialgebra. There is, over $\Bbbk[[t]]/t^2$, a one to-one correspondence between the elements of $H^2_{Hom}(B,B)$ and the infinitesimal deformation of B_0 defined by*

$$\mu_t = \mu_0 + \mu_1 t \quad \textit{and} \quad \Delta_t = \Delta_0 + \Delta_1 t. \tag{5.12}$$

Proof. The deformation equation is equivalent to $\delta^2_{Hom}(\mu_1, \Delta_1) = 0$, that is $(\mu_1, \Delta_1) \in Z^2_{Hom}(B,B)$. ■

Suppose that the truncated deformation $B_t^{m-1} = \left(B[[t]]/t^m, \mu_t^{m-1}, \Delta_t^{m-1}\right)$ Write $\mu_t^{m-1} = \mu_0 + t\mu_1 + t^2\mu_2 + ... + t^{m-1}\mu_{m-1}$ and $\Delta_t^{m-1} = \Delta_0 + t\Delta_1 + t^2\Delta_2 + ... + t^{m-1}\Delta_{m-1}$ satisfies the deformation equation. The truncated deformation is extended to a deformation of order m, ie $\mu_t^m = \mu_0 + t\mu_1 + t^2\mu_2 + ... + t^{m-1}\mu_{m-1} + t^m\mu_m$ and $\Delta_t^m = \Delta_0 + t\Delta_1 + t^2\Delta_2 + ... + t^m\Delta_m$ satisfying the deformation equation if,

$$\delta^2_{Hom}(\mu_m, \Delta_m) = \begin{cases} \delta^{1,2}_{Hom,H}(\mu_m) = \mu_0 \circ_\alpha \mu_m + \mu_m \circ_\alpha \mu_0 = -\sum\limits_{\substack{i+j=m \\ i,j\neq m}} \mu_i \circ_\alpha \mu_j \\ \delta^{2,1}_{Hom,C}(\mu_m) = \Delta_0 \circ_\alpha \Delta_m + \Delta_m \circ_\alpha \Delta_0 = -\sum\limits_{\substack{i+j=m \\ i,j\neq m}} \Delta_i \circ_\alpha \Delta_j \\ \delta^{1,2}_{Hom,C}(\mu_m) + \delta^{2,1}_{Hom,H}(\Delta_m) = \sum\limits_{\substack{i+j=m \\ i,j\neq m}} (\Delta_i \circ \mu_j) - \sum\limits_{\substack{i+j+t+r=m \\ i,j,t,r\neq m}} (\mu_i \otimes \mu_j) \circ \tau_{2,3} \circ (\Delta_t \otimes \Delta_r \end{cases}$$

To this end, consider the following cochains

$$Ob(\mu_m) = \sum_{\substack{i+j=m \\ i,j\neq m}} \mu_i \circ_\alpha \mu_j,$$

$$Ob(\Delta_m) = \sum_{\substack{i+j=m \\ i,j\neq m}} \Delta_i \circ_\alpha \Delta_j,$$

$$Ob(\Delta_m \circ \mu_m) = \sum_{\substack{i+j=m \\ i,j\neq m}} (\Delta_i \circ \mu_j) - \sum_{\substack{i+j+t+r=m \\ i,j,t,r\neq m}} (\mu_i \otimes \mu_j) \circ \tau_{2,3} \circ (\Delta_t \otimes \Delta_r)$$

The $Ob(\mu_m)$, $Ob(\Delta_m)$ and $Ob(\Delta_m \circ \mu_m)$ are called the **obstruction** to finding μ_m, Δ_m, $\Delta_m \circ \mu_m$ extending the deformation.

A standard deformation theory argument [19] if $\alpha = id_B$ shows that $Ob(\mu_m), Ob(\Delta_m)$, are a 3-coboundary if and only if μ_t^{m-1} and Δ_t^{m-1} extends to a $\Bbbk[[t]]/t^{m+1}$algebra and coalgebra structures on $B\left[[t]\right]/t^{m+1}$. In this case, any 2-cochain whose coboundary is $\left(Ob\left(\mu_m\right), Ob\left(\Delta_m\right)\right)$ gives an extension. An analogous argument applied to our setting yields the following result.

If one has only $\mu_t^{m-1} = \mu_0 + t\mu_1 + t^2\mu_2 + ... + t^{m-1}\mu_{m-1}$ and $\Delta_t^{m-1} = \Delta_0 + t\Delta_1 + t^2\Delta_2 + ... + t^{m-1}\Delta_{m-1}$ satisfies the deformation equation (5.11) for $k = 1, ..., m-1$ then

$$\left(Ob\left(\mu_m\right), Ob\left(\Delta_m\right), Ob\left(\Delta_m \circ \mu_m\right)\right) \in Z^3_{Hom}\left(B, B\right)$$

and the cohomology class of this cocycle is the obstruction to finding an $\left(\mu_m, \Delta_m, \Delta_m \circ \mu_m\right)$ such that (5.11) is satisfied for $k = m$.

Remark 5.5.4 *Suppose that* $\left(B\left[[t]\right]/t^m, \mu_t^{m-1}, \Delta_t^{m-1}\right)$ *is a deformation of* B_0.

If $B_t = \left(B\left[[t]\right]/t^{m+1}, \mu_t^{m-1} + t^m\mu_m, \Delta_t^{m-1} + t^m\Delta_m\right)$ *is a deformation of* B_0*, then* $B_t' = \left(B\left[[t]\right]/t^{m+1}, \left(\mu_t^m\right)' = \mu_t^{m-1} + t^m\mu_m', \left(\Delta_t^m\right)' = \Delta_t^{m-1} + t^m\Delta_m'\right)$ *is a deformation of* B_0 *if, and only if*

$\left(\mu_m' - \mu_m, \Delta_m' - \Delta_m\right) \in Z^2_{Hom}\left(B, B\right)$. *Note also that if* $\left(\mu_m' - \mu_m, \Delta_m' - \Delta_m\right) \in B^2_{Hom}\left(B, B\right)$, *then deformations* B_t *and* B_t' *are equvalent.*

Hom-bialgebras for which every formal deformation is equivalent to a trivial deformation are said to be analytically **rigid**. The nullity of the second cohomology group $\left(H^2_{Hom}\left(B, B\right) = 0\right)$ gives a sufficient criterion for rigidity.

In the following we assume that $H^2_{Hom}\left(B, B\right) \neq 0$, then one may obtain nontrivial one-parameter formal deformations. We consider the problem of extending a one parameter formal deformation of order $m-1$ to a deformation of order m.

Suppose now that

$$\mu_t = \mu_0 + \mu_1 t + \mu_2 t^2 + ... \quad \textit{and} \quad \Delta_t = \Delta_0 + \Delta_1 t + \Delta_2 t^2 + ...$$

are a one parameter family of deformation of $\left(\mu_0, \Delta_0\right)$ for which

$$\mu_1 = \mu_2 = ... = \mu_{m-1} = 0 \quad \textit{and} \quad \Delta_1 = \Delta_2 = ... = \Delta_{m-1} = 0.$$

The deformation equation implies

$$\delta^2_{Hom}\left(\mu_m, \Delta_m\right) = 0 \quad \left(\left(\mu_m, \Delta_m\right) \in Z^2_{Hom}\left(B, B\right)\right).$$

If further $(\mu_m, \Delta_m) \in B^2_{Hom}(B, B)$ (ie. $(\mu_m, \Delta_m) = \delta^1_{Hom}(f, g)$), then setting the morphism $\Phi_t = id_B + \Phi_m t^m$ we have,

$$\mu'_t = \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t) = \mu_0 + \mu_{m+1} t^{m+1} + ...$$

$$\Delta'_t = (\Phi_t \otimes \Phi_t) \circ \Delta_t \circ \Phi_t^{-1} = \Delta_0 + \Delta_{m+1} t^{m+1} + ...$$

And again $(\mu_{m+1}, \Delta_{m+1}) \in Z^2_{Hom}(B, B)$, we can now prove:

Corollary 5.5.5 *If $H^2_{Hom}(B, B) = 0$ then all deformations of Hom-bialgebra B are equivalent to a trivial deformation.*

5.6 Unital and Counital Hom-bialgebra Deformations

We discuss unitality and counitality of Hom-bialgebra deformations.

Proposition 5.6.1 *The unit (resp. the counit) of Hom-bialgebra B is also the unit (resp. the counit) of the formal deformation B_t of B if and only if*

$$\mu_n(x \otimes 1_B) = \mu_n(1_B \otimes x) = 0 \quad \forall n \geq 1, \quad \forall x \in B, \quad \eta(1_{\Bbbk}) = 1_B$$

$$(resp.\ \ (id_B \otimes \varepsilon) \circ \Delta_n = (\varepsilon \otimes id_B) \circ \Delta_n = 0 \quad \forall n \geq 1).$$

Proof. The element 1_B is a unit for B_t if $\mu_t \circ (\eta \otimes id_B) = \mu_t \circ (id_B \otimes \eta) = \alpha$.

$$\forall x \in B,\ \mu_t(x \otimes 1_B) = \alpha(x), \quad \mu_t(1_B \otimes x) = \alpha(x), \text{ where } \mu_t = \sum_{i \geq 1} t^i \mu_i.$$

We have

$$\mu_t(x \otimes 1_B) = \alpha(x) = \mu_0(x \otimes 1_B) + \sum_{i \geq 1} \mu_i(x \otimes 1_B) t^i$$

$$\alpha(x) = \alpha(x) + \sum_{i \geq 1} \mu_i(x \otimes 1_B) t^i$$

By identification, we obtain $\mu_n(x \otimes 1_B) = 0, \quad \forall n \geq 1$, and similarly $\mu_n(1_B \otimes x) = 0, \quad \forall n \geq 1$. The map ε is a counit for B_t if $(\varepsilon \otimes id_B) \circ \Delta_t = (id_B \otimes \varepsilon) \circ \Delta_t = \alpha$, where

$\Delta_t = \sum\limits_{i\geq 1} t^i \Delta_i.$

$$(\varepsilon \otimes id_B)\left(\sum_{i\geq 0}\Delta_i t^i\right) = \alpha = (\varepsilon \otimes id_V)\circ\Delta_0 + \sum_{i\geq 1}(\varepsilon \otimes id_V)\circ\Delta_i t^i$$

$$\alpha = \alpha + \sum_{i\geq 1}(\varepsilon \otimes id_B)\circ\Delta_i t^i.$$

By identification, we obtain $(\varepsilon \otimes id_B)\circ\Delta_i = 0,\ \forall i \geq 1$. Similarly $(id_B \otimes \varepsilon_j)\circ\Delta_i = 0,\ \forall i \geq 1$.

■

Theorem 5.6.2 *Let $B = (B, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra with a surjective map α. Every nontrivial formal deformation $B_t = (B, \mu_t, \eta_t, \Delta_t, \varepsilon_t, \alpha)$ is equivalent to a unital and counital deformation with the same unit η and counit ε.*

Proof. We show that the unit is conserved by deformation. Assume $\mu_t = \mu + \sum\limits_{i\geq p} t^i\mu_i$. Two deformations are equivalent if there is a formal isomorphism $\Phi_t = Id + t\Phi_1 + t^2\Phi_2 + \cdots$, where $\Phi_i \in End_{\mathbb{K}}(V)$, which leads to

$$\mu_1'(x,y) = \mu_1(x,y) + f_1(\mu_0(x,y)) - \mu_0(f_1(x),y) - \mu_0(x,f_1(y)). \tag{5.13}$$

Since μ_1 is a 2-cocycle, then $\sum_{i+j=1}\ \mu_i\left(\mu_j\left(x,y\right),\alpha(z)\right) - \mu_i\left(\alpha(x),\mu_j\left(y,z\right)\right) = 0$.

We set $y = z = 1$, respectively $x = y = 1$ and $z = x$. Then, we obtain

$$\mu_1(\alpha(x),1) = \mu_0(\alpha(x),\mu_1(1,1)), \quad \mu_1(1,\alpha(x)) = \mu_0(\mu_1(1,1),\alpha(x)).$$

If α is surjective, then we have

$$\mu_1(x,1) = \mu_0(x,\mu_1(1,1)), \quad \mu_1(1,x) = \mu_0(\mu_1(1,1),x). \tag{5.14}$$

We consider the formal isomorphism satisfying $f_1(1) = \mu_1(1,1),\ f_n = 0$ for $n \geq 2$. Using (5.13) and (5.14), the equivalent multiplication leads to a new deformed multiplication satisfying

$$\begin{aligned}\mu_1'(x,1) &= \mu_1(x,1) + f_1(\mu_0(x,1)) - \mu_0(f_1(x),1) - \mu_0(x,\mu_1(1,1))\\ &= \mu_1(x,1) + f_1(\alpha(x)) - \alpha(f_1(x)) - \mu_1(x,1) = 0.\end{aligned}$$

Similarly, we obtain $\mu_1'(1,x)=0$. By induction on n, we show that for all $n\geq 1$, $\mu_n'(1,x)=\mu_n'(x,1)=0$. Indeed, we assume $\mu_k'(1,x)=\mu_k'(x,1)=0$ for $k=1,\cdots,n-1$. We consider the isomorphism f_t satisfying $f_n(1)=\mu_n(1,1)$ and $f_k=0$ $\forall k\neq n$. Then, using (5.13) and (5.14), we obtain $\mu_n'(1,x)=\mu_n'(x,1)=0$.

Observe that the product $(1+f_1t^1)\cdots(1+f_nt^n)$ converge when n tends to infinity. Therefore, according to Proposition 5.6.1, the unit is conserved by deformation. The proof is similar for the counit. ■

5.7 Twistings and Deformations

In this section, we discuss the connection between the twistings of Hom-bialgebras (see Proposition 2.1.8) and their formal deformations.

Proposition 5.7.1 *Let $B_t=(B[[t]],\mu_t,\eta_t,\Delta_t,\varepsilon_t,\alpha)$ be a formal deformation of a Hom-bialgebra $B=(B,\mu_0,\eta_0,\Delta_0,\varepsilon_0,\alpha)$ and $\beta:B\longrightarrow B$ be a Hom-bialgebra morphism of B and B_t. Then $B_{t,\beta}=(B[[t]],\beta\circ\mu_t,\beta\circ\eta_t,\Delta_t\circ\beta,\varepsilon_t\circ\beta,\beta\circ\alpha)$ is a formal deformation of the Hom-bialgebra $B_\beta=(B,\beta\circ\mu_0,\eta_0,\Delta_0\circ\beta,\varepsilon_0,\alpha)$.*

Hence, for any $n\in\mathbb{N}$ $B_{t,\beta^n}=(B[[t]],\beta^n\circ\mu_t,\beta^n\circ\eta_t,\Delta_t\circ\beta^n,\varepsilon_t\circ\beta^n,\beta^n\circ\alpha)$ is a formal deformation of the Hom-bialgebra B_{β^n}.

Proof. The proof is analogous to that of Proposition 2.1.8. ■

Corollary 5.7.2 *Let $B=(B,\mu_0,\eta_0,\Delta_0,\varepsilon_0)$ be a bialgebra and $\alpha:B\longrightarrow B$ be a bialgebra morphism (i.e. $\alpha\circ\mu_0=\mu_0\circ(\alpha\otimes\alpha)$, $\Delta_0\circ\alpha=(\alpha\otimes\alpha)\circ\Delta_0$, $\alpha\circ\eta_0=\eta_0$ and $\varepsilon_0\circ\alpha=\varepsilon_0$). If $B_t=(B[[t]],\mu_t,\eta_t,\Delta_t,\varepsilon_t)$ is a formal deformation of the bialgebra B and α is a bialgebra morphism for B_t (i.e. $\alpha\circ\mu_t=\mu_t\circ(\alpha\otimes\alpha)$, $\Delta_t\circ\alpha=(\alpha\otimes\alpha)\circ\Delta_t$, $\alpha\circ\eta_t=\eta_t$ and $\varepsilon_t\circ\alpha=\varepsilon_t$). Then $B_{t,\alpha}=(B[[t]],\alpha\circ\mu_t,\eta_t,\Delta_t\circ\alpha,\varepsilon_t,\alpha)$ is a formal deformation of the Hom-bialgebra $B_\alpha=(B,\alpha\circ\mu_0,\eta_0,\Delta_0\circ\alpha,\varepsilon_0,\alpha)$.*

Proposition 5.7.3 *Let $B_t=(B,\mu_t,\eta_t,\Delta_t,\varepsilon_t,\alpha)$ and $B_t'=(B,\mu_t',\eta_t',\Delta_t',\varepsilon_t',\alpha)$ be two equivalent deformations of a Hom-bialgebra $B=(B,\mu_0,\eta_0,\Delta_0,\varepsilon_0,\alpha)$. Then, $B_{t,\alpha^n}=$*

$\left(B\left[[t]\right], \alpha^n \circ \mu_t, \eta_t, \Delta_t \circ \alpha^n, \varepsilon_t, \alpha^{n+1}\right)$ and $B'_{t,\alpha^n} = \left(B\left[[t]\right], \alpha^n \circ \mu'_t, \eta'_t, \Delta'_t \circ \alpha^n, \varepsilon'_t, \alpha^{n+1}\right)$ are equivalent deformations of the Hom-bialgebra $B_{\alpha^n} = \left(B, \alpha^n \circ \mu_0, \eta_0, \Delta_0 \circ \alpha^n, \varepsilon_0, \alpha^{n+1}\right)$, for any $n \in \mathbb{N}$.

Proof. We know that there exists a formal automorphism $\Phi_t = \sum\limits_{i\geq 0} \Phi_i t^i$, where $\Phi_i \in End_{\Bbbk}(B)$ and $\Phi_0 = id_B$ such that $\Phi_t \circ \mu'_t = \mu_t \circ \Phi_t^{\otimes 2},\ \Phi_t^{\otimes 2} \circ \Delta'_t = \Delta_t \circ \Phi_t,\ and\ \Phi_t \circ \alpha = \alpha \circ \Phi_t.$ Then, we have

$$\alpha^n \circ \Phi_t \circ \mu'_t = \alpha^n \circ \mu_t \circ \Phi_t^{\otimes 2},\ \Phi_t^{\otimes 2} \circ \Delta'_t \circ \alpha^n = \Delta_t \circ \Phi_t \circ \alpha^n,\ and\ \Phi_t \circ \alpha \circ \alpha^n = \alpha \circ \Phi_t \circ \alpha^n,$$

$$\Phi_t \circ \left(\alpha^n \circ \mu'_t\right) = \left(\alpha^n \circ \mu_t\right) \circ \Phi_t^{\otimes 2},\ \Phi_t^{\otimes 2} \circ \left(\Delta'_t \circ \alpha^n\right) = \left(\Delta_t \circ \alpha^n\right) \circ \Phi_t,\ and\ \Phi_t \circ \alpha^{n+1} = \alpha^{n+1} \circ \Phi_t.$$

Hence B_{t,α^n} is equivalent to B'_{t,α^n}, for any $n \in \mathbb{N}$. ■

Proposition 5.7.4 *Let $B_t = (V, \mu_t, \eta_t, \Delta_t, \varepsilon_t, \alpha)$ be a formal deformation of a Hom-bialgebra $B = (V, \mu_0, \eta_0, \Delta_0, \varepsilon_0, \alpha)$. Then B_{t,α^n} is equivalent to a formal deformation B'_{t,α^n} of a Hom-bialgebra B_{α^n} with the same unit and counit as B_{α^n}.*

Proof. The proof is similar to Theorem 5.6.2. Notice that surjectivity is not required. ■

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Résumé :

Le travail porte sur la cohomologie et les déformations des Hom-bialgèbres et algèbres Hom-Hopf qui sont des versions modifiées par un morphisme des structures classiques de bialgèbre et algèbre de Hopf liées aux groupes quantiques. Les algèbres de type-hom sont apparues dans les déformations quantiques des algèbres de Witt et Virasoro, comme une généralisation des algèbres de Lie. Premièrement on rappelle la théorie des algèbres de type-Hom et les propriétés établies, puis on introduit les A-bimodules et C-bicomodules nécessaire pour la définition de la cohomologie, puis leur dualité. Enfin on établit une théorie des déformations formelles pour les Hom-bialgèbres généralisant la théorie de déformation de Gerstenhaber.

Classification A.M.S 2000 :

16W25, 16W99, 16E40, 16E99, 17B35, 17B37, 17B99.

Mot clés :

Algèbre Hom-associative, Hom-coalgèbre, Hom-bialgèbre, Algèbre Hom-Hopf, Bimodule, Bicomodule, Cohomologie de Hochschild et déformation..

Abstract :

The study is conducted on the cohomology and the deformations of Hom-bialgebras and Hom-Hopf algebras, a generalized version of bialgebras and Hopf algebras obtained by modifying the classical structures by a morphism linked to quantum groups. Hom-type algebras appeared in the quantum deformations of Witt and Virasoro algebras as a generalisation of Lie algebras. First we recall the theory of Hom-type algebras and describe some properties of those structures, and then introduce the needed A-bimodule and C-bicomodule for the definition of the cohomology, and then its duality. In the final we establish a deformation formelle theory of Hom-bialgebras generalised the deformations theory of Gerstenhaber.

2000 A.M.S Subject classification :

16W25, 16W99, 16E40, 16E99, 17B35, 17B37, 17B99.

Key words:

Hom-associative algebra, Hom-coalgebra, Hom-bialgebra, Hom-Hopf algebra, Bimodule, Bicomodule, Hochschild's cohomology and deformation.

ملخص

هذا العمل مكرس لدراسة التماثل الثنوي و تشوهات الجبر المزدوج و جبر هوبف التشاكليين التي تم تعديلهما بإضافة تطبيق تشاكلي على شروط بنيتها الجبرية الكلاسيكية المتعلقة بالزمر الكمية. ظهر هذا النوع في تشوهات الكمية في الجبرين ويت و فيرازورو التي هي تعميم للجبر لي. اولا نذكر بالنظريات الخاصة بأنواع الجبريات التشاكلية و خصائصها الثابتة, وبعد ذلك نشير الى بنية المقياس المزدوج و المقياس المزدوج الثنوي اللازمة لتعريف التماثل الثنوي, وفي النهاية ننشأ نظريات على التشويهات الشكلية من اجل الجبريات المزدوجة المعممة لتشويهات قارستنهابر.

تصنيف الجمعية الامريكية

16W25, 16W99, 16E40, 16E99, 17B35, 17B37, 17B99.

الكلمات الفاتحة

الجبر الجمعي التشاكلي, الجبر المزدوج التشاكلي, جبر هوبف التشاكلي, المقياس المزدوج, التماثل التنوي, تشويه.